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CONVERGENCE OF EXTRAPOLATION COEFFICIENTS

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1. INTRODUCTION

Let X be a Hilbert space and let $T, H \in [X]$. We consider an operator equation

$$x = Tx + b \tag{1}$$

and an iterative process

$$x_{n+1} = Tx_n + b\,, \tag{2}$$

where b is a given element from X. Let for some $x_0 \in X$ the sequence $\{x_n\}_{n=0}^{\infty}$ determined by (2) converge to $x^* \in X$. Let $l > 0$, k, $m_0, m_1, \ldots, m_l$ be integers such that the inequalities

$$m_l > m_{l-1} > \ldots > m_1 > m_0 = 0\,, \tag{3}$$

$$k > m_l \tag{4}$$

hold.

In the paper [1] we solved the problem of finding complex numbers $\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots$ $\ldots, \alpha_l^{(k)}$ such that

$$\sum_{i=0}^{l} \alpha_i^{(k)} = 1\,, \tag{5}$$

$$\|H(x^* - \sum_{i=0}^{l} \alpha_i^{(k)} x_{k-m_i})\| = \min_{\beta_0 + \ldots + \beta_l = 1} \|H(x^* - \sum_{i=0}^{l} \beta_i x_{k-m_i})\|\,. \tag{6}$$

The norm is defined by using the scalar product $(\cdot, \cdot)$ in X. In order to summarize shortly the results from [1] we recall some notations and assumptions from that paper which will be adopted throughout the present paper. If

$$\boldsymbol{M}_k = (\mu_0, \mu_1, \ldots, \mu_t)\,, \quad \boldsymbol{N}_k = (\nu_0, \nu_1, \ldots, \nu_s)$$

are two row vectors with components in X, then $\boldsymbol{N}_k \circledast \boldsymbol{M}_k$ is a complex $(s + 1) \times$ $\times (t + 1)$ matrix and $(\boldsymbol{N}_k \circledast \boldsymbol{M}_k)_{i,j} = (\mu_j, \nu_i)$.

We put

$$\varepsilon_k = x^* - x_k\,, \quad \eta_k = H\varepsilon_k\,, \tag{7}$$

$$\boldsymbol{H}_k = (\eta_k, \eta_{k-m_1}, \ldots, \eta_{k-m_l})\,, \tag{7'}$$

$$\mathbf{Q}_k = \boldsymbol{H}_k \circledast \boldsymbol{H}_k\,. \tag{7''}$$

Further, we assume that the resolvent operator $R(\lambda, T)$ has r poles $\lambda_1, \lambda_2, \ldots, \lambda_r$ with multiplicities $i_1, i_2, \ldots, i_r$, respectively, and satisfying the inequalities

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_r| > 0\,. \tag{8}$$

Moreover, $|\lambda_r| > |\lambda|$ for every $\lambda \in \sigma(T)$, $\lambda \neq \lambda_j$, $j = 1, \ldots, r$, and $\lambda_i \neq \lambda_j$ for $i \neq j$.

For a given $j \in \langle 1, r \rangle$ let $C_j = \{\lambda \in C \mid |\lambda - \lambda_j| = \varrho_j\}$, where ϱ_j is assumed to fulfil

$$\{\lambda \in C \mid |\lambda - \lambda_j| \leqq \varrho_j\} \cap \sigma(T) = \{\lambda_j\}\,.$$

The symbol C denotes the set of complex numbers. Let

$$K_0 = \{\lambda \in C \mid |\lambda| = \varrho_0\} \tag{9}$$

with ϱ_0 such that

$$\{\lambda \in C \mid |\lambda| \leqq \varrho_0\} \cap \sigma(T) = \sigma(T) \dot{-} \{\lambda_1, \ldots, \lambda_r\}\,.$$

Denote

$$B_{ji} = \frac{1}{2\pi \mathrm{i}} \int_{C_j} (\lambda - \lambda_j)^{i-1} R(\lambda, T)\, \mathrm{d}\lambda\,. \tag{10}$$

Without any loss of generality we can assume that (see [1])

$$l < \sum_{j=1}^{r} i_j \equiv t \quad \text{and} \quad B_{ji_j}\varepsilon_0 \neq 0 \quad \text{for all} \quad j = 1, \ldots, r\,. \tag{11}$$

On the basis of the just presented conditions we have proved (see Theorems 2 and 4 in [1]) that there exists an integer $k_0 > \max_{j=1,\ldots,r} (i_j) + m_l$ such that for every $k \geqq k_0$ only one vector

$$\boldsymbol{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_l^{(k)})^{\mathsf{T}}$$

can be found which solves (5) and (6). This vector is given by the formula

$$\boldsymbol{\alpha}^{(k)} = (\mathbf{e}^{\mathsf{T}}(n)\, \mathbf{Q}_k^{-1}\, \mathbf{e}(n))^{-1}\, \mathbf{Q}_k^{-1}\, \mathbf{e}(n)\,.$$

Let us remark that $\mathbf{e}_i(n)$ is the i-th column of the $n \times n$ identity matrix and $\mathbf{e}(n) = \sum_{i=1}^{n} \mathbf{e}_i(n)$.

Given a sequence $\{u_k\}_{k=0}^{\infty} \subset X$ and two integers $i, j \in \langle 1, l \rangle$ we denote for $k > m_l$

(12) $$\delta_{ij} u_k = u_{k-m_{i-1}} - u_{k-m_j}$$

and

(12') $$\delta_i u_k = \delta_{ii} u_k \,.$$

Define

(13) $$\boldsymbol{L}_k = (\delta_1 \eta_k, \delta_2 \eta_k, \ldots, \delta_l \eta_k)\,,$$

(14) $$\boldsymbol{S}_k = \begin{pmatrix} \boldsymbol{L}_k \otimes \boldsymbol{H}_k \\ \mathbf{e}^{\mathrm{T}}(l+1) \end{pmatrix}.$$

The matrix $\boldsymbol{S}_k$ is nonsingular and the vector $\boldsymbol{\alpha}^{(k)}$ is the solution of the system

(15) $$\boldsymbol{S}_k \boldsymbol{\alpha}^{(k)} = \mathbf{e}_{l+1}(l+1)\,.$$

(See [1], Theorem 2). We call the components of the vector $\boldsymbol{\alpha}^{(k)}$ the coefficients of extrapolation.

In this paper we shall study the convergence of the coefficients $\alpha_i^{(k)}$ for $k \to \infty$ and construct a polynomial

$$P(z) = \sigma_0 z^{m_l} + \sigma_1 z^{m_l - m_1} + \ldots + \sigma_{l-1} z^{m_l - m_{l-1}} + \sigma_l$$

such that the $\alpha_i^{(k)}$'s converge to the coefficients of this polynomial, i.e. $\lim_{k\to\infty} \alpha_i^{(k)} = \sigma_i$. In the special cases $m_i = i$ or $m_i = in$ $(i = 0, 1, \ldots, l)$ where n is a given integer, it is shown that it is possible to express the coefficients σ_i as functions of some poles of the resolvent operator $R(\lambda, T)$. Extrapolation by means of polynomials with coefficients σ_i in the case $m_i = in$ for $i = 0, \ldots, l$ was studied in the paper [5].

In Sections 2 and 3 auxiliary assertions are proved, which are used in Sections 4 and 5. In Section 4 we study the convergence of $\alpha_i^{(k)}$ for $k \to \infty$. On the basis of the asymptotic behaviour of $\alpha_i^{(k)}$ for $k \to \infty$ it is shown in Section 5 that if $\{y_k\}_{k=m_l}^{\infty} \subset X$ is defined by

(16) $$y_k = \alpha_0^{(k)} x_k + \alpha_1^{(k)} x_{k-m_1} + \ldots + \alpha_l^{(k)} x_{k-m_l}$$

then

$$\lim_{k\to\infty} (\|x^* - y_k\| / \|x^* - x_k\|^p) = 0$$

for some $p \geqq 1$.

Let all notations and assumptions concerning the integers $l, m_0, m_1, \ldots, m_l, t$ and the poles of $R(\lambda, T)$ as well as the operators B_{ji} be valid throughout all this paper.

2. AUXILIARY THEOREMS

Let $\mathscr{K}$ denote the set of all pairs (j, i) for $j = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, i_j$ for every j. Order this set in the following sequence:

$$(17) \qquad \begin{array}{l} (1, i_1),\ (1, i_1 - 1),\ \ldots,\ (1, 1)\,, \\ (2, i_2),\ (2, i_2 - 1),\ \ldots,\ (2, 1)\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (r, i_r),\ (r, i_{r-1}),\ \ldots,\ (r, 1)\,. \end{array}$$

Put

$$H(B_{ji}\varepsilon_0/\lambda_j^{i-1}) = v_{ji}$$

and

$$H\left(\frac{1}{2\pi i}\int_{K_0} \lambda^k R(\lambda, T)\, \varepsilon_0\, d\lambda\right) = v(k)\,.$$

The symbol $\mathbf{c}(k)$ denotes a vector from C^t whose p-th component is $\binom{k}{i-1}\lambda_j^k$, where (j, i) lies at the p-th place in the sequence (17). Analogously, $\boldsymbol{V}$ denotes a t-dimensional "vector" with components v_{ji}. For a given positive integer $v < k$ let $\mathscr{L}_{k,v} \subset X$ be a subspace generated by the vectors $v(k), v(k-1), \ldots, v(k-v)$. The vector η_k defined by (7) can be expressed in the form (see [1])

$$(18) \qquad \eta_k = \sum_{j=1}^{r} \sum_{i=1}^{i_j} \binom{k}{i-1} \lambda_j^k v_{ji} + v(k) = \boldsymbol{V}^{\mathsf{T}}\mathbf{c}(k) + v(k)\,.$$

The operation $\boldsymbol{V}^{\mathsf{T}}\mathbf{c}(k)$ is performed in the same way as for vectors with complex components. Further, let $l > 1$.

When proving the convergence of $\alpha_i^{(k)}$ for $k \to \infty$ we shall work with matrices

$$(19) \qquad \mathbf{R}_k = \boldsymbol{L}_k \circledS \boldsymbol{L}_k\,, \quad \mathbf{R}_k(j, i) = \boldsymbol{L}_{1,k}(j) \circledS \boldsymbol{L}_{2,k}(i)\,,$$

where $\boldsymbol{L}_k$ is defined by (13),

$$(19') \qquad \boldsymbol{L}_{1,k}(j) = (\delta_1\eta_k, \ldots, \delta_{j-1}\eta_k, \delta_{j+1}\eta_k, \ldots, \delta_l\eta_k)$$

for $j = 1, 2, \ldots, l$ and

$$(19'') \qquad \boldsymbol{L}_{2,k}(i) = (\delta_1\eta_k, \ldots, \delta_{i-2}\eta_k, \delta_{i-1,i}\, \eta_k, \delta_{i+1}\eta_k, \ldots, \delta_l\eta_k)$$

for $i = 1, 2, \ldots, l + 1$.

Put

$$y(k) = \eta_k - v(k)\,,$$

$$\boldsymbol{L}_k^{(-)} = (\delta_1\, y(k), \delta_2\, y(k), \ldots, \delta_l\, y(k))\,,$$

$$\boldsymbol{L}_{1,k}^{(-)}(j) = (\delta_1\, y(k), \ldots, \delta_{j-1}\, y(k), \delta_{j+1}\, y(k), \ldots, \delta_l\, y(k))\,,$$

$$\boldsymbol{L}_{2,k}^{(-)}(i) = (\delta_1\, y(k), \ldots, \delta_{i-2}\, y(k), \delta_{i-1,i}\, y(k), \delta_{i+1}\, y(k), \ldots, \delta_l\, y(k))$$

for $j = 1, \ldots, l$; $i = 1, \ldots, l + 1$. If we extend the validity of the operators δ_{ij} and δ_i also for sequences $\{\boldsymbol{u}_k\}_{k=0}^{\infty} \subset C^t$ according to the relations (12) and (12') then (18) and the definitions of $y(k)$ and $\mathbf{c}(k)$ immediately imply that

$$L_k^{(-)} = V^{\mathsf{T}} J_k\,, \quad L_{1,k}^{(-)}(j) = V^{\mathsf{T}} J_{1,k}(j)\,, \quad L_{2,k}^{(-)}(i) = V^{\mathsf{T}} J_{2,k}(i)\,,$$

where we have put

$$\begin{aligned} J_k \quad &= (\delta_1\, \mathbf{c}(k), \delta_2\, \mathbf{c}(k), \ldots, \delta_l\, \mathbf{c}(k))\,, \\ J_{1,k}(j) &= (\delta_1\, \mathbf{c}(k), \ldots, \delta_{j-1}\, \mathbf{c}(k), \delta_{j+1}\, \mathbf{c}(k), \ldots, \delta_l\, \mathbf{c}(k))\,, \\ J_{2,k}(i) &= (\delta_1\, \mathbf{c}(k), \ldots, \delta_{i-2}\, \mathbf{c}(k), \delta_{i-1,i}\, \mathbf{c}(k), \delta_{i+1}\, \mathbf{c}(k), \ldots, \delta_l\, \mathbf{c}(k))\,. \end{aligned}$$

Let us remark that for vectors $\boldsymbol{u}_i \in C^t$, $i = 1, 2, \ldots, s$ the symbol $(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_s)$ denotes the matrix with columns $\boldsymbol{u}_i$. In order to express the vectors L_k, $L_{1,k}(j)$, $L_{2,k}(i)$ which we use for the construction of the matrices $\mathbf{R}_k$ and $\mathbf{R}_k(j, i)$ it is necessary first to calculate $\delta_{ij}\, \eta_k$.

Lemma 1. *Let $k_0 > 0$, $\varkappa$ be integers, $\{\gamma_p\}_{p=\varkappa}^{\infty} \subset C$ with $\gamma_\varkappa \neq 0$. Let the series $\sum_{p=\varkappa}^{\infty} \gamma_p / k_0^p$ be absolutely convergent. Then there exist an integer k' and a sequence of numbers $\{\gamma'_p\}_{p=-\varkappa}^{\infty} \subset C$ such that $\sum_{p=-\varkappa}^{\infty} \gamma'_p / k^p$ is absolutely convergent for $k > k'$, $\sum_{p=-\varkappa}^{\infty} \gamma_p / k^p \neq 0$ for $k > k'$ and*

$$\left(\sum_{p=\varkappa}^{\infty} \frac{\gamma_p}{k^p}\right)^{-1} = \sum_{p=-\varkappa}^{\infty} \frac{\gamma'_p}{k^p}\,.$$

The proof is given in [2].

Lemma 2. *Let k_0, n_1, n_2, q be nonnegative integers, $k_0 > \max\{n_1, n_2\} + q$. Then there exists a sequence of real numbers $\{v_p\}_{p=0}^{\infty}$ such that the series $\sum_{p=0}^{\infty} v_p / k_0^p$ is absolutely convergent and the equality*

$$\binom{k - n_1}{q} \binom{k - n_2}{q}^{-1} = \sum_{p=0}^{\infty} \frac{v_p}{k^p}\,.$$

holds for all $k \geqq k_0$.

The proof is obvious.

It is easy to see that for integers $k > \max_{j=1,\ldots,r} (i_j) + m_l$ and $p, q \in \langle 1, l \rangle$ we have

$$\delta_{p,q} \eta_k = \delta_{p,q}\, y(k) + v(k - m_{p-1}) - v(k - m_q)$$

and

$$\delta_{p,q}\, y(k) = V^{\mathsf{T}}[\mathbf{c}(k - m_{p-1}) - \mathbf{c}(k - m_q)]\,.$$

For the first component of the vector in brackets we have

$$\mathbf{e}_1^{\mathsf{T}}(t)\, [\mathbf{c}(k - m_{p-1}) - \mathbf{c}(k - m_q)] = b_1(k)\, \lambda_1^{k - m_q}\,,$$

where

$$b_1(k) = \binom{k - m_q}{i_1 - 1}\left[\binom{k - m_{p-1}}{i_1 - 1}\binom{k - m_q}{i_1 - 1}^{-1} \lambda^{m_q - m_{p-1}} - 1\right].$$

Lemmas 1 and 2 imply that there exist an integer μ and a sequence $\{\varphi_n\}_{n=\mu}^{\infty} \subset C$ such that

$$b_1(k) = \sum_{n=\mu}^{\infty} \frac{\varphi_n}{k^n}$$

and the series is absolutely convergent for all $k > \max(i_j) + m_l$. The same can be said for the other components of $\mathbf{c}(k - m_{p-1}) - \mathbf{c}(k - m_q)$. For all $k > 0$ let us define a vector $\mathbf{g}(k)$ by the relation

(20) $$\mathbf{g}(k) = (\underbrace{\lambda_1^k, \ldots, \lambda_1^k}_{i_1\text{-times}}, \underbrace{\lambda_2^k, \ldots, \lambda_2^k}_{i_2\text{-times}}, \ldots, \underbrace{\lambda_r^k, \ldots, \lambda_r^k}_{i_r\text{-times}})^{\mathsf{T}}.$$

Since every component of the vector $\mathbf{c}(k - m_{p-1}) - \mathbf{c}(k - m_q)$ can be expressed as a product of $\lambda_1^{k-m_q}$ and an absolutely convergent series of the above described form, it is possible to construct integers μ, $\mu(j)$, $\mu(i)$, sequences $\{\boldsymbol{\Phi}_n\}_{n=\mu}^{\infty}$, $\{\boldsymbol{\Phi}_{1,n}(j)\}_{n=\mu(j)}^{\infty}$ and $\{\boldsymbol{\Phi}_{2,n}(i)\}_{n=\mu(i)}^{\infty}$ of rectangular matrices of order $t \times l$, $t \times (l-1)$ and $t \times (l-1)$, respectively, such that the series

$$\sum_{n=\mu}^{\infty} \frac{\boldsymbol{\Phi}_n}{k^n}, \quad \sum_{n=\mu(j)}^{\infty} \frac{\boldsymbol{\Phi}_{1,n}(j)}{k^n} \quad \text{and} \quad \sum_{n=\mu(i)}^{\infty} \frac{\boldsymbol{\Phi}_{2,n}(i)}{k^n}$$

are absolutely convergent for all $k \geqq \max(i_j) + m_l$; if we denote their sums by $\mathbf{B}_k$, $\mathbf{B}_{1,k}(j)$ and $\mathbf{B}_{2,k}(i)$, respectively, then the elements of the matrices $\mathbf{J}_k$, $\mathbf{J}_{1,k}(j)$ and $\mathbf{J}_{2,k}(i)$ have the following form:

(21) $$\mathbf{J}_k\, \mathbf{e}_s(l) = [\operatorname{diag}(\mathbf{B}_k\, \mathbf{e}_s(l))]\, \mathbf{g}(k - m_s)$$

for $s = 1, \ldots, l$,

(21′) $$\mathbf{J}_{1,k}(j)\, \mathbf{e}_s(l-1) = [\operatorname{diag}(\mathbf{B}_{1,k}(j)\, \mathbf{e}_s(l-1))]\, \mathbf{g}(k - m_{s+v})$$

for $s = 1, \ldots, l-1$

$(v = 0 \text{ for } s \in \langle 1, j),\ v = 1 \text{ for } s \in \langle j, l-1\rangle)$,

(21″) $$\mathbf{J}_{2,k}(i)\, \mathbf{e}_s(l-1) = [\operatorname{diag}(\mathbf{B}_{2,k}(i)\, \mathbf{e}_s(l-1))]\, \mathbf{g}(k - m_{s+v})$$

for $s = 1, \ldots, l-1$

$(v = 0 \text{ for } s \in \langle 1, i-1),\ v = 1 \text{ for } s \in \langle i-1, l-1\rangle)$.

Let us remark that for a vector $\mathbf{w} \in C^t$ the symbol $\operatorname{diag}(\mathbf{w})$ denotes the diagonal

$t \times t$ matrix whose diagonal elements are the components of $\boldsymbol{w}$ in their natural order.

Since

$$\boldsymbol{L}_k^{(-)} = \boldsymbol{V}^{\mathsf{T}}\boldsymbol{J}_k, \quad \boldsymbol{L}_{1,k}^{(-)}(j) = \boldsymbol{V}^{\mathsf{T}}\boldsymbol{J}_{1,k}(j) \quad \text{and} \quad \boldsymbol{L}_{2,k}^{(-)}(i) = \boldsymbol{V}^{\mathsf{T}}\boldsymbol{J}_{2,k}(i),$$

we have

$$(21''') \qquad \boldsymbol{L}_k = \boldsymbol{L}_k^{(-)} + \boldsymbol{q}_k,$$

$$\boldsymbol{L}_{1,k}(j) = \boldsymbol{L}_{1,k}^{(-)}(j) + \boldsymbol{q}_{1,k}(j) \quad \text{and} \quad \boldsymbol{L}_{2,k}(i) = \boldsymbol{L}_{2,k}^{(-)}(i) + \boldsymbol{q}_{2,k}(i)$$

where all components of the vectors $\boldsymbol{q}_k$, $\boldsymbol{q}_{1,k}(j)$ and $\boldsymbol{q}_{2,k}(i)$ lie in the space $\mathscr{L}_{k,m_l}$.

Lemma 3. *Let $k > \max(i_j) + m_l$ and $m_l < t$ $(t = \sum_{j=1}^{r} i_j)$. Then the matrices $\boldsymbol{J}_k, \boldsymbol{J}_{1,k}(j)$ and $\boldsymbol{J}_{2,k}(i)$ have maximal ranks.*

Proof. We have proved in [1] (Lemma 4) that the vectors $y(k)$, $y(k - m_1), \ldots$ $\ldots, y(k - m_l)$ as well as $\delta_1\, y(k), \ldots, \delta_l\, y(k)$ (Lemma 1 in [1]) are linearly independent.

Let for some $\beta_1, \beta_2, \ldots, \beta_l$

$$(22) \qquad \beta_1(\boldsymbol{J}_k\, \mathbf{e}_1(t)) + \beta_2(\boldsymbol{J}_k\, \mathbf{e}_2(t)) + \ldots + \beta_l(\boldsymbol{J}_k\, \mathbf{e}_l(t)) = 0\,.$$

If

$$(23) \qquad \sum_{i=1}^{l} |\beta_i|^2 > 0$$

then (22) yields

$$\boldsymbol{V}^{\mathsf{T}}[\beta_1(\boldsymbol{J}_k\, \mathbf{e}_1(t)) + \beta_2(\boldsymbol{J}_k\, \mathbf{e}(t)) + \ldots + \beta_l(\boldsymbol{J}_k\, \mathbf{e}_l(t))] = 0\,,$$

i.e.

$$\sum_{i=1}^{l} \beta_i \delta_i\, y(k) = 0\,,$$

which contradicts (23). Analogously we can prove that $\boldsymbol{J}_{1,k}(j)$ and $\boldsymbol{J}_{2,k}(i)$ have maximal ranks. □

We have defined the vectors (13), (19′), (19″) and the matrices (19). As we shall study the properties of all matrices (19) together we introduce the following generalization.

Let $\varrho > 0$, $\mu_1, \mu_2, m, n_1, \ldots, n_\varrho, \nu_1, \nu_2, \ldots, \nu_\varrho$ be integers,

$$(24) \qquad 0 \leqq n_1 < n_2 < \ldots < n_\varrho < t = \sum_{j=1}^{r} i_j\,,$$

$$(25) \qquad n_i > \nu_i \quad \forall i \quad \text{and} \quad m > \max_{j=1,\ldots,r}(i_j) + n_\varrho\,.$$

Let $\{\boldsymbol{\Omega}_j^{(1)}\}_{j=\mu_1}^{\infty}$, $\{\boldsymbol{\Omega}_j^{(2)}\}_{j=\mu_2}^{\infty}$ be two sequences of $t \times \varrho$ matrices such that the series

$$(26) \qquad \sum_{j=\mu_1}^{\infty} \frac{\boldsymbol{\Omega}_j^{(1)}}{k^j} \quad \text{and} \quad \sum_{j=\mu_2}^{\infty} \frac{\boldsymbol{\Omega}_j^{(2)}}{k^j} \quad \text{are}$$

absolutely convergent for all $k \geqq m$. We denote

$$\mathbf{A}_k^{(s)} = \sum_{j=\mu_s}^{\infty} \frac{\mathbf{\Omega}_j^{(s)}}{k^j} \quad \text{for} \quad s = 1, 2 .$$

Let $\mathbf{F}_k^{(1)}, \mathbf{F}_k^{(2)}$ be two $t \times \varrho$ matrices defined by

$$\mathbf{F}_k^{(s)}\, \mathbf{e}_i(\varrho) = \operatorname{diag}\left(\mathbf{A}_k^{(s)}\, \mathbf{e}_i(\varrho)\right) \cdot \mathbf{g}(k - n_i) \tag{27}$$

for $i = 1, \ldots, \varrho$ and $s = 1, 2$. Let $\vartheta_{k,i}^{(s)}$, $i = 1, \ldots, \varrho$; $s = 1, 2$, be elements of X having the following form:

$$\vartheta_{k,i}^{(s)} = \mathbf{V}^{\mathsf{T}}[\mathbf{F}_k^{(s)}\, \mathbf{e}_i(\varrho)] + \zeta_i^{(s)}(k, v_i)\,, \tag{28}$$

where $\zeta_i^{(s)}(k, v_i) \in \mathscr{L}_{k,v_i}$. Put

$$\mathbf{M}_k^{(s)} = \left(\vartheta_{k,1}^{(s)}, \vartheta_{k,2}^{(s)}, \ldots, \vartheta_{k,\varrho}^{(s)}\right) \tag{29}$$

and

$$\mathbf{U}_k = \mathbf{M}_k^{(2)} \otimes \mathbf{M}_k^{(1)}\,. \tag{30}$$

It is easy to see that

$$\mathbf{M}_k^{(s)} = \mathbf{V}^{\mathsf{T}}\mathbf{F}_k^{(s)} + \mathbf{w}_k^{(s)} \tag{31}$$

where all ϱ components of $\mathbf{w}_k^{(s)}$ lie in $\mathscr{L}_{k,v_\varrho}$.

Lemma 4. *Let $s = 1$ or $s = 2$. Let the matrices $F_k^{(s)}$ have a rank ϱ for all $k \geqq m$. Then there exists an integer $k_0 \geqq m$ such that the elements $\vartheta_{k,i}^{(s)}$ for $i = 1, 2, \ldots, \varrho$ are linearly independent for all $k \geqq k_0$.*

The proof is analogous to that of Lemma 4 or Theorem 3 in [1].

3. CALCULATION OF det $\mathbf{U}_k$

Let $\varphi_1, \varphi_2, \ldots, \varphi_\varrho \in X$ and $\mathbf{A} = (a_{ij})_{i,j=1,\ldots,\varrho}$, $a_{ij} \in C$. We define

$$(\varphi_1, \varphi_2, \ldots, \varphi_\varrho)\, \mathbf{A} = \left(\sum_{i=1}^{\varrho} a_{i1}\varphi_i, \sum_{i=1}^{\varrho} a_{i2}\varphi_i, \ldots, \sum_{i=1}^{\varrho} a_{i\varrho}\varphi_i\right).$$

Our aim in this section is to show an explicit form for det $\mathbf{U}_k$. If we succeed in finding, for $s = 1, 2$, nonsingular transformations $\mathbf{Z}_k^{(s)}$ and permutations $\mathbf{P}_k^{(s)}$ such that the relations

$$\mathbf{e}_i^{\mathsf{T}}(t)\,(\mathbf{P}_k^{(s)}\mathbf{F}_k^{(s)}\mathbf{Z}_k^{(s)})\, \mathbf{e}_j(\varrho) = 0 \tag{32}$$

hold for $i, j = 1, 2, \ldots, \varrho$; $i \neq j$, then we can easily express det $\mathbf{U}_k$ by using (28), (29), (30) and the following assertion.

Lemma 5. *If $\mathbf{A}_1$ and $\mathbf{A}_2$ are complex $\varrho \times \varrho$ matrices, then*

$$\mathbf{U}_k \mathbf{A}_1 = M_k^{(2)} \otimes N_k^{(1)}, \tag{33}$$

$$\mathbf{A}_2^{\mathrm{H}} \mathbf{U}_k = N_k^{(2)} \otimes M_k^{(1)} \tag{33'}$$

and

$$\mathbf{A}_2^{\mathrm{H}} \mathbf{U}_k \mathbf{A}_1 = N_k^{(2)} \otimes N_k^{(1)}, \tag{33''}$$

where

$$N_k^{(1)} = M_k^{(1)} \mathbf{A}_1 \quad \text{and} \quad N_k^{(2)} = M_k^{(2)} \mathbf{A}_2 .$$

Proof. The formulas (33), (33′), (33″) can be obtained by a straightforward calculation. □

Lemma 6. *Let $s = 1$ or $s = 2$. Let $s_1, s_2, \ldots, s_\varrho$ be mutually different integers from the interval $\langle 0, t \rangle$ and $\mathbf{G}_k^{(s)}(s_1, \ldots, s_\varrho)$ the $\varrho \times \varrho$ matrix the i-th row of which is identical with the s_i-th row of $\mathbf{F}_k^{(s)}$.*

Then either $\det \mathbf{G}_k(s_1, \ldots, s_\varrho) = 0$ *for all k or there exists an integer k_0 such that* $\det \mathbf{G}_k(s_1, \ldots, s_\varrho) \neq 0$ *for all $k \geqq k_0$.*

The proof is obvious.

In the following **we shall assume that there exists an integer m such that the matrices $\mathbf{F}_k^{(1)}$ and $\mathbf{F}_k^{(2)}$ have a rank ϱ for all** $k \geqq m$. The matrix $\mathbf{F}_k^{(1)}$ has a rank ϱ for all $k \geqq m$; therefore for a given $k \geqq m$ there exist integers $s_1, \ldots, s_\varrho$ such that

$$\det \mathbf{G}_k^{(1)}(s_1, \ldots, s_\varrho) \neq 0 , \tag{34}$$

and an analogous assertion for $\mathbf{F}_k^{(2)}$ holds.

Assumption 1. *Let for $s = 1, 2$.*

$$\det \mathbf{G}_k^{(s)}(1, 2, \ldots, \varrho) \neq 0 \tag{35}$$

for all $k \geqq m$. We shall write $\mathbf{G}_k^{(s)}$ instead of $\mathbf{G}_k^{(s)}(1, 2, \ldots, \varrho)$. □

In the sequel we shall study only the matrices $\mathbf{F}_k^{(1)}$. It is easy to see that the same assertion will be valid for $\mathbf{F}_k^{(2)}$.

Since (35) holds, it is possible by using the Gauss-Jordan elimination to construct permutation matrices

$$\mathbf{P}_{1,k}^{(1)}, \mathbf{P}_{1,k}^{(2)}, \ldots, \mathbf{P}_{1,k}^{(\varrho-1)}, \mathbf{P}_k^{(1)}, \mathbf{P}_k^{(2)}, \ldots, \mathbf{P}_k^{(\varrho-1)},$$

upper triangular matrices $\mathbf{W}_k^{(1)}, \mathbf{W}_k^{(2)}, \ldots, \mathbf{W}_k^{(\varrho-1)}$ and lower triangular matrices $\mathbf{L}_k^{(1)}, \mathbf{L}_k^{(2)}, \ldots, \mathbf{L}_k^{(\varrho-1)}$ such that

$$\mathbf{P}_{1,k}^{(\varrho-1)} \ldots \mathbf{P}_{1,k}^{(2)} \mathbf{P}_{1,k}^{(1)} \mathbf{G}_k^{(1)} \mathbf{P}_k^{(1)} \mathbf{W}_k^{(1)} \mathbf{P}_k^{(2)} \mathbf{W}_k^{(2)} \ldots \mathbf{P}_k^{(\varrho-1)} \mathbf{W}_k^{(\varrho-1)} \mathbf{L}_k^{(1)} \mathbf{L}_k^{(2)} \ldots \mathbf{L}_k^{(\varrho-1)} \tag{36}$$

is a diagonal matrix with non-zero diagonal elements. All investigated matrices are $\varrho \times \varrho$. The elimination is made in the following way. If the matrix

$$\mathbf{P}_{1,k}^{(i-1)} \ldots \mathbf{P}_{1,k}^{(2)} \mathbf{P}_{1,k}^{(1)} \mathbf{G}_k^{(1)} \mathbf{P}_k^{(1)} \mathbf{W}_k^{(1)} \ldots \mathbf{P}_k^{(i-1)} \mathbf{W}_k^{(i-1)}$$

has zero in the positions (l_1, l_2), where $l_1 = 1, \ldots, i-1$ and $l_2 = l_1 + 1, \ldots, \varrho$, then, moreover,

$$\mathbf{P}^{(i)}_{1,k}\mathbf{P}^{(i-1)}_{1,k} \ldots \mathbf{P}^{(2)}_{1,k}\mathbf{P}^{(1)}_{1,k}\mathbf{G}^{(1)}_k\mathbf{P}^{(1)}_k\mathbf{W}^{(1)}_k \ldots \mathbf{P}^{(i-1)}_k\mathbf{W}^{(i-1)}_k\mathbf{P}^{(i)}_k\mathbf{W}^{(i)}_k$$

has zero in the positions $(i, i+1), (i, i+2), \ldots, (i, \varrho)$. Analogously, after multiplying the matrix

$$\mathbf{P}^{(\varrho-1)}_{1,k} \ldots \mathbf{P}^{(2)}_{1,k}\mathbf{P}^{(1)}_{1,k}\mathbf{G}^{(1)}_k\mathbf{P}^{(1)}_k\mathbf{W}^{(1)}_k \ldots \mathbf{P}^{(\varrho-1)}_k\mathbf{W}^{(\varrho-1)}_k\mathbf{L}^{(1)}_k \ldots \mathbf{L}^{(i-1)}_k$$

by $\mathbf{L}^{(i)}_k$ we obtain zero in the positions $(\varrho - i + 1, 1), (\varrho - i + 1, 2), \ldots, (\varrho - i + 1, \varrho - i)$.

Putting

$$\mathbf{P}_{1,k} = \mathbf{P}^{(\varrho-1)}_{1,k} \ldots \mathbf{P}^{(2)}_{1,k}\mathbf{P}^{(1)}_{1,k},$$

$$\bar{\mathbf{P}}_k = \begin{pmatrix} \mathbf{P}_{1,k}, & \mathbf{\Theta} \\ \mathbf{\Theta}, & \mathbf{I}_{t-\varrho} \end{pmatrix}$$

we have

$$\mathbf{e}^{\mathrm{T}}_i(t)\left(\bar{\mathbf{P}}_k\mathbf{F}^{(1)}_k\mathbf{P}^{(1)}_k\mathbf{W}^{(1)}_k \ldots \mathbf{P}^{(\varrho-1)}_k\mathbf{W}^{(\varrho-1)}_k\mathbf{L}^{(1)}_k \ldots \mathbf{L}^{(\varrho-1)}_k\right)\mathbf{e}_j(\varrho) = 0 \tag{37}$$

for $i \neq j$; $i, j = 1, 2, \ldots, \varrho$.

Without any loss of generality let all permutations in the following considerations be identity matrices.

The matrices $\mathbf{W}^{(i)}_k$ and $\mathbf{L}^{(i)}_k$ from the Gauss-Jordan elimination have the form

$$\mathbf{W}^{(i)}_k = \mathbf{I}_\varrho + \mathbf{W}^{(i)}_{1,k} \quad \text{and} \quad \mathbf{L}^{(i)}_k = \mathbf{I}_\varrho + \mathbf{L}^{(i)}_{1,k},$$

where $\mathbf{W}^{(i)}_{1,k}$ and $\mathbf{L}^{(i)}_{1,k}$ are strictly upper and lower triangular matrices, respectively. From the formulas for the elements of $\mathbf{G}^{(1)}_k$ it follows that the nonzero elements of $\mathbf{W}^{(i)}_{1,k}$ or $\mathbf{L}^{(i)}_{1,k}$ have the following form: if $z \neq 0$ is an element of $\mathbf{W}^{(i)}_{1,k}$ or $\mathbf{L}^{(i)}_{1,k}$ then there exists a sequence $\{\varphi_n(z)\}^{\infty}_{n=\mu(z)} \subset C$ such that the series $\sum\limits_{k=\mu(z)}^{\infty} \varphi_n(z)/k^n$ is absolutely convergent with the sum z.

Let the symbol $\mathbf{D}(s_1, s_2, s_3)$ denote the diagonal matrix defined by

$$\mathbf{e}^{\mathrm{T}}_i(t)\,\mathbf{D}(s_1, s_2, s_3)\,\mathbf{e}_i(t) = \begin{cases} 0 & \text{for } 1 \leqq i < s_1, \\ 1 & \text{for } s_1 \leqq i \leqq s_2, \\ 0 & \text{for } s_2 < i < s_3, \\ 1 & \text{for } i \geqq s_3 \end{cases}$$

for integers $1 \leqq s_1 \leqq s_2 \leqq s_3 \leqq t$.

For $\mathbf{a} \in C^t$ we put

$$\mathbf{b}^{(n_i)}(s_1, s_2, s_3, \mathbf{a}) = \mathbf{D}(s_1, s_2, s_3)\operatorname{diag}(\mathbf{a})\,\mathbf{g}(k - n_i).$$

Theorem 1. *Let* (35) *hold for all* $k \geqq m$. *Then there exist integers* $\mu(1), k_0(1)$, *a sequence of nonsingular* $\varrho \times \varrho$ *matrices* $\{\mathbf{Z}^{(1)}_k\}^{\infty}_{k=k_0(1)}$ *and a sequence of* $t \times \varrho$

rectangular matrices $\{\boldsymbol{\Phi}_j^{(1)}\}_{j=\mu(1)}^{\infty}$ *such that the series* $\sum_{j=\mu(1)}^{\infty} \boldsymbol{\Phi}_j^{(1)}/k^j$ *is absolutely convergent for* $k \geqq k_0(1)$ *and if we put* $\mathbf{B}_k^{(1)} = \sum_{j=\mu(1)}^{\infty} \boldsymbol{\Phi}_j^{(1)}/k^j$, *then for the sequence of matrices* $\{\mathbf{E}_k^{(1)}\}_{k=k_0}^{\infty}$ *defined by*

$$\mathbf{E}_k^{(1)} = \boldsymbol{F}_k^{(1)}\boldsymbol{Z}_k^{(1)} \tag{38}$$

we have

$$\mathbf{E}_k^{(1)}\, \mathbf{e}_i(\varrho) = \boldsymbol{b}^{(n_i)}(i, i, \varrho + 1, \mathbf{B}_k^{(1)}\, \mathbf{e}_i(\varrho)) \quad \text{for} \quad i = 1, \ldots, \varrho\,. \tag{39}$$

Moreover, the equality

$$\det \boldsymbol{Z}_k^{(1)} = 1 \tag{40}$$

holds for all $k \geqq k_0$.

An analogous theorem with the matrices $\{\boldsymbol{Z}_k^{(2)}\}_{k=k_0(2)}^{\infty}$, $\{\boldsymbol{\Phi}_j^{(2)}\}_{j=\mu(2)}^{\infty}$, $\mathbf{B}_k^{(2)}$ $\mathbf{E}_k^{(2)}$ could be formulated for a transformation of the matrices $\boldsymbol{F}_k^{(2)}$.

Remark. If the permutations in (36) are not identity matrices then instead of (40) we have $|\det \boldsymbol{Z}_k^{(1)}| = 1$.

Proof. The matrix $\boldsymbol{Z}_k^{(1)}$ is the product of the matrices

$$\boldsymbol{W}_k^{(1)} \ldots \boldsymbol{W}_k^{(\varrho-1)}\boldsymbol{L}_k^{(1)} \ldots \boldsymbol{L}_k^{(\varrho-1)}$$

defined by (36). Since the matrix $\boldsymbol{G}_k^{(1)}$ was formed from the first rows of $\boldsymbol{F}_k^{(1)}$, we obtain from (36) immediately the assertion of Theorem 1.

By using Lemma 5 we obtain

$$(\boldsymbol{Z}_k^{(2)})^{\mathsf{H}}\, \boldsymbol{U}_k \boldsymbol{Z}_k^{(1)} = \mathcal{N}_k^{(2)} \otimes \mathcal{N}_k^{(1)}\,,$$

where for $s = 1, 2$

$$\mathcal{N}_k^{(s)} = \boldsymbol{V}^{\mathsf{T}}\boldsymbol{F}_k^{(s)}\boldsymbol{Z}_k^{(s)} + w_k^{(s)}\boldsymbol{Z}_k^{(s)} =$$

$$= \boldsymbol{V}^{\mathsf{T}}(\boldsymbol{b}^{(n_1)}(1, 1, \varrho + 1, \mathbf{B}_k^{(s)}\, \mathbf{e}_1(\varrho)),\ \boldsymbol{b}^{(n_2)}(2, 2, \varrho + 1,\ \mathbf{B}_k^{(s)}\, \mathbf{e}_2(\varrho)), \ldots$$

$$\ldots, \boldsymbol{b}^{(n_\varrho)}(\varrho, \varrho, \varrho + 1,\ \mathbf{B}_k^{(s)}\, \mathbf{e}_\varrho(\varrho))) + (\chi_{k,1}^{(s)}, \chi_{k,2}^{(s)}, \ldots, \chi_{k,\varrho}^{(s)})\,,$$

where

$$\chi_{k,i}^{(s)} = \sum_{j=0}^{\varrho} \beta_{i,j}^{(s)}(k)\, v(k - v_j)\,,$$

$v(k - v_j) \in \mathscr{L}_{k,v_\varrho}$ and it is possible to write every $\beta_{i,j}^{(s)}(k)$ in the form $\beta_{i,j}^{(s)}(k) = \sum_{j=\varkappa}^{\infty} \varphi_{i,j}^{(s)}/k^j$, where this series is absolutely convergent, $\varkappa$ is an integer and $\varphi_{i,j}^{(s)} \in C$.

Let (p_i, q_i) be the pair at the i-th place in (17).

Assumption 2. *Let* $p_\varrho > p_{\varrho+1}$ *and* $|\lambda_{p_\varrho}| > |\lambda_{p_{\varrho+1}}|$. □

Put for $j = 1, 2, 3$; $s = 1, 2$

$$\boldsymbol{Y}_k^{(s)}(j) = (y_{k,1}^{(s)}(j), y_{k,2}^{(s)}(j), \ldots, y_{k,\varrho}^{(s)}(j))\,,$$

where $y_{k,i}^{(s)}(j) \in X$ have the form

$$y_{k,i}^{(s)}(1) = \lambda_{p_i}^{k-n_i} \left(\sum_{j=\mu(s)}^{\infty} \frac{\mathbf{e}_i^{\mathsf{T}}(t)\, \boldsymbol{\Phi}_j^{(s)}\, \mathbf{e}_i(\varrho)}{k^j} \right) v_{p_i, q_i}, \tag{40}$$

$$y_{k,i}^{(s)}(2) = \sum_{n=\varrho+1}^{t} \left\{ \lambda_{p_n}^{k-n_i} \left(\sum_{j=\mu(s)}^{\infty} \frac{\mathbf{e}_n^{\mathsf{T}}(t)\, \boldsymbol{\Phi}_j^{(s)}\, \mathbf{e}_i(\varrho)}{k^j} \right) v_{p_n, q_n} \right\}, \tag{40'}$$

$$y_{k,i}^{(s)}(3) = \chi_{k,i}^{(s)}. \tag{40''}$$

Therefore, if we put

$$\boldsymbol{N}_k^{(s)} = (N_{k,1}^{(s)}, N_{k,2}^{(s)}, \ldots, N_{k,\varrho}^{(s)})$$

then

$$N_{k,i}^{(s)} = y_{k,i}^{(s)}(1) + y_{k,i}^{(s)}(2) + y_{k,i}^{(s)}(3).$$

Lemma 7. *Let the assumptions* 1 *and* 2 *be fulfilled and let* k_0 *be the integer from Theorem* 1. *Then for every pair* s, i, *where* $s = 1, 2;\ i = 1, 2, \ldots, \varrho$ *there exist a constant* $\xi_i^{(s)} \neq 0$, *an integer* $\gamma_i^{(s)}$, *a vector* $v_i^{(s)}$ *and a sequence* $\{z_i^{(s)}(k)\}_{k=k_0}^{\infty} \subset X$ *such that for all* $k \geqq k_0$

$$N_{k,i}^{(s)} = \xi_i^{(s)} k^{\gamma_i^{(s)}} \lambda_{p_i}^k v_i^{(s)} + z_i^{(s)}(k) \tag{41}$$

and the equality

$$\lim_{k\to\infty} z_i^{(s)}(k)/(\lambda_{p_i}^k k^{\gamma_i^{(s)}}) = 0 \tag{42}$$

holds.

The vectors $v_1^{(s)}, \ldots, v_\varrho^{(s)}$ *are linearly independent.* □

The proof follows immediately from (40)–(40″) and from the structure of the spectrum of the operator T.

Theorem 2. *Let assumptions* 1 *and* 2 *be valid. Then there exist a complex number* C_ϱ, *an integer* $\varkappa$ *and a function* φ *such that*

$$\det \boldsymbol{U}_k = k^{\varkappa} \prod_{i=1}^{\varrho} |\lambda_{p_i}|^{2k} (C_\varrho + \varphi(k)) \tag{43}$$

and

$$\lim_{k\to\infty} \varphi(k) = 0.$$

If $\boldsymbol{M}_k^{(1)} = \boldsymbol{M}_k^{(2)}$, *then* $C_\varrho > 0$.

Proof. Lemma 5 implies that $\det \boldsymbol{U}_k = \det(\boldsymbol{N}_k^{(2)} \otimes \boldsymbol{N}_k^{(1)})$. From Lemma 7 we obtain

$$(\boldsymbol{N}_k^{(2)} \otimes \boldsymbol{N}_k^{(1)})_{i,j} =$$
$$= (\xi_j^{(1)} \lambda_{p_j}^k k^{\gamma_j^{(1)}} v_j^{(1)} + z_j^{(1)}(k),\ \xi_i^{(2)} \lambda_{p_i}^k k^{\gamma_i^{(2)}} v_i^{(2)} + z_i^{(2)}(k)) =$$
$$= \xi_i^{(2)} \xi_j^{(1)} \bar{\lambda}_{p_i}^k \lambda_{p_j}^k k^{\gamma_i^{(2)}} k^{\gamma_j^{(1)}} [(v_j^{(1)}, v_i^{(2)}) + \omega_{i,j}(k)],$$

where $\lim_{k\to\infty} \omega_{i,j}(k) = 0$. The rest is obvious. □

Remark. If the permutations in (36) are not identity matrices then in (43) $C_\varrho = C_\varrho(k)$ and $|C_\varrho(k)|$ is a constant.

4. CONVERGENCE OF $\alpha_i^{(k)}$

In [1] we have shown that the vector $\boldsymbol{\alpha}^k = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_l^{(k)})^\mathsf{T}$ is a solution of (15). The matrix $\boldsymbol{S}_k$ is defined by (14).

Assumption 3. *Let*

$$\sum_{j=1}^{\tau} i_j = l \tag{44}$$

hold for some integer $\tau \in \langle 1, r \rangle$. □

Let us remark that use the notation described in Section 1. Let $\boldsymbol{G}_k$ be the matrix formed by the first l rows of the matrix

$$(\mathbf{c}(k - m_1), \mathbf{c}(k - m_2), \ldots, \mathbf{c}(k - m_l))$$

and

let there exist an integer k_0 such that

$$\det \boldsymbol{G}_k \neq 0 \tag{45}$$

for all $k \geqq k_0$.

The assumption (45) is fulfilled for a special choice of integers $m_0, m_1, \ldots, m_l$ which will be shown in Theorems 3 and 3′. In the other cases, analogously to Lemma 6, either $\det \boldsymbol{G}_k = 0$ for all k or there exists an integer k_0 such that $\det \boldsymbol{G}_k \neq 0$ for all $k \geqq k_0$.

In the following investigation let $k \geqq k_0$ hold.

Put

$$g_2(z, z_1, \ldots, z_l) = z^{m_l} + z_1 z^{m_l - m_1} + \ldots + z_l\,, \tag{46}$$

$$g_1(z, z_1, \ldots, z_l) = z^{k - m_l} g_2(z, z_1, \ldots, z_l)\,. \tag{47}$$

For $j = 1, 2, \ldots, \tau$ and $i = 1, 2$ define mappings $A_j^{(i)} : C^{l+1} \to C^{i_j}$ in the following way:

$$A_j^{(i)}(z, z_1, \ldots, z_l) = \begin{bmatrix} \dfrac{\partial^{(i_j - 1)} g_i(z, z_1, \ldots, z_l)}{\partial z^{(i_j - 1)}} \\ \dfrac{\partial^{(i_j - 2)} g_i(z, z_1, \ldots, z_l)}{\partial z^{(i_j - 2)}} \\ \text{-------------} \\ \dfrac{\partial g_i(z, z_1, \ldots, z_l)}{\partial z} \\ g_i(z, z_1, \ldots, z_l) \end{bmatrix}.$$

Lemma 8. *If* (45) *holds, then the system of l linear algebraic equations*

$$(48) \qquad A_s^{(2)}(\lambda_s, z_1, z_2, \ldots, z_l) = \mathbf{\Theta}(i_s)\,; \quad s = 1, 2, \ldots, \tau$$

has exactly one solution for the unknowns $z_1, z_2, \ldots, z_l$.

Proof. The set of all solutions of the system (48) coincides with the set of solutions of the system

$$(49) \qquad A_s^{(1)}(\lambda_s, z_1, z_2, \ldots, z_l) = \mathbf{\Theta}(i_s)\,; \quad s = 1, \ldots, \tau\,.$$

But the system (49) is equivalent to

$$(50) \qquad \mathbf{G}_k \,.\, (z_1, \ldots, z_l)^{\mathsf{T}} = -(\mathbf{w}_1^{\mathsf{T}}(k), \ldots, \mathbf{w}_\tau^{\mathsf{T}}(k))^{\mathsf{T}} + \mathbf{\Theta}\,,$$

where

$$(51) \qquad \mathbf{w}_s(k) = \left(\binom{k}{i_s - 1}\lambda_s^k, \binom{k}{i_s - 2}\lambda_s^k, \ldots, \lambda_s^k\right)^{\mathsf{T}}$$

for $s = 1, 2, \ldots, \tau$. The rest is obvious. □

Let us denote the solution of (48) by $(b_1, b_2, \ldots, b_l)^{\mathsf{T}}$. It is independent of k.

Theorem 3. *If* $m_i = i$ *for all* $i = 1, \ldots, l$ *then* $\det \mathbf{G}_k \neq 0$ *for all* $k \geqq \max(i_j) + m_l$ *and the equality*

$$(z - \lambda_1)^{i_1}(z - \lambda_2)^{i_2} \ldots (z - \lambda_\tau)^{i_\tau} = z^l + b_1 z^{l-1} + \ldots + b_{l-1}z + b_l$$

holds for all $z \in C$, *i.e.* $b_1, b_2, \ldots, b_l$ *are the coefficients of the polynomial*

$$(z - \lambda_1)^{i_1}(z - \lambda_2)^{i_2} \ldots (z - \lambda_\tau)^{i_\tau}.$$

Proof. Similarly as in the proof of Lemma 4 in [1] we could show that $\det \mathbf{G}_k \neq 0$ and therefore the system (50) has exactly one solution $(b_1, b_2, \ldots, b_l)^{\mathsf{T}}$. If we put

$$U(z) = z^k + b_1 z^{k-1} + \ldots + b_l z^{k-l}$$

then Lemma 8 yields $U^{(q-1)}(\lambda_p) = 0$ for all pairs (p, q) which lie at the first l places in the sequence (17), and therefore the polynomial $(z - \lambda_1)^{i_1}(z - \lambda_3)^{i_2} \ldots (z - \lambda_\tau)^{i_\tau}$ divides the polynomial $U(z)$. The assertion of Theorem 3 is now clear.

Analogously it is possible to prove the following theorem.

Theorem 3.' *If* $m_i = in\ \forall i = 1, \ldots, l$, *where n is a positive integer and* $\lambda_1^n, \lambda_2^n, \ldots, \lambda_r^n$ *are mutually different then there exists an integer* k' *such that* $\det \mathbf{G}_k \neq 0$ *for all* $k \geqq k'$ *and*

$$(z - \lambda_1^n)^{i_1}(z - \lambda_2^n)^{i_2} \ldots (z - \lambda_\tau^n)^{i_\tau} = z^l + b_1 z^{l-1} + \ldots + b_l$$

holds for all $z \in C$.

For the only solution $(b_1, b_2, \ldots, b_l)^{\mathsf{T}}$ of the system (48) we have that the projection of the vector

$$\eta_k + b_1\eta_{k-m_1} + b_2\eta_{k-m_2} + \ldots + b_l\eta_{k-m_l}$$

on the subspace generated by the vectors $\{v_{ji}\}_{\substack{j=1,\ldots,\tau \\ i=1,\ldots,i_j}}$ is the nullvector. Analogously to what was proved in [5], we may expect that the coefficients of the polynomial $P(z) = P_1(z)/P_1(1)$, where

$$P_1(z) = z^{m_l} + b_1 z^{m_l - m_1} + \ldots + b_{l-1} z^{m_l - m_{l-1}} + b_l\,,$$

will be the desired limits of $\alpha_i^{(k)}$ for $k \to \infty$, which we prove in the sequel.

Assumption 4. *Let* $P_1(1) \neq 0$. □

Let us define

(53) $$P(z) = P_1(z)/P_1(1) = \sigma_0 z^{m_l} + \sigma_1 z^{m_l - m_1} + \ldots + \sigma_{l-1} z^{m_l - m_{l-1}} + \sigma\ ,$$

(54) $$\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_l)^{\mathsf{T}}\,,$$

(55) $$\mathbf{S}_k\boldsymbol{\sigma} = (\gamma_{k,0}, \gamma_{k,1}, \ldots, \gamma_{k,l-1}, 1)^{\mathsf{T}} = \gamma^{(1)}(k)\,,$$

(56) $$\gamma(k) = \gamma^{(1)}(k) - \mathbf{e}_{l+1}(l+1)\,.$$

From (55) and (56) we have

$$\mathbf{S}_k\boldsymbol{\sigma} = \gamma(k) + \mathbf{e}_{l+1}(l+1)$$

or

$$\boldsymbol{\sigma} = \mathbf{S}_k^{-1}\,\mathbf{e}_{l+1}(l+1) + \mathbf{S}_k^{-1}\,\gamma(k)$$

and hence (see 15))

(57) $$\boldsymbol{\alpha}^{(k)} = \boldsymbol{\sigma} - \mathbf{S}_k^{-1}\,\gamma(k)\,.$$

Lemma 9. *Let* (45) *hold for all* $k \geqq k_0$. *Then for every integer* $s \in \langle 0, l-1\rangle$ *there exist an integer* $\varkappa_s$ *and sequences of functions* $\{\Gamma_s(k)\}_{k=k_0}^{\infty}$ *such that*

$$\limsup_{k\to\infty} |\Gamma_s(k)| < +\infty$$

and

(58) $$\gamma_{k,s} = \Gamma_s(k)\,k^{\varkappa_s}\lambda_1^k\lambda_{\tau+1}^k$$

for all $k \geqq k_0$.

Proof. From the form of $\delta_s\eta_k$ and the inequalities (8) we obtain

(59) $$\delta_s\eta_k = k^{\nu_s}\lambda_1^k\,x_s(k)\,,$$

where ν_s is an integer and $\limsup_{k\to\infty} \|x_s k)\| < \infty$. Now we calculate

$$\sum_{i=0}^{l}\sigma_i\eta_{k-m_i} = \boldsymbol{V}^{\mathsf{T}}(\mathbf{c}(k), \mathbf{c}(k-m_1), \ldots, \mathbf{c}(k-m_l))\,\boldsymbol{\sigma} + w(k)\,,$$

where $w(k) \in \mathscr{L}_{k,m_l}$. The first l components of the vector

$$(\mathbf{c}(k), \mathbf{c}(k - m_1), \ldots, \mathbf{c}(k - m_l))\,\boldsymbol{\sigma}$$

equal zero. Therefore

$$\sum_{i=0}^{l} \sigma_i \eta_{k-m_i} = k^{\nu} \lambda_{\tau+1}^{k}\, y(k)\,, \tag{60}$$

where for vectors $y(k)$ we analogously have

$$\limsup_{k\to\infty} \|y(k)\| < \infty\,.$$

The rest is obvious. □

Let $\mathbf{S}_k^{\mathsf{A}}$ denote the adjoint of $\mathbf{S}_k$ and let $\mathbf{S}_k^{\mathsf{A}} = (\mathbf{S}_k^{\mathsf{A}}(i,j))_{i,j=1}^{l+1}$. It is easy to see from (13), (14), (19″), (19‴) by using (19) and (19′) that

$$\det \mathbf{S}_k^{\mathsf{A}}(i,j) = \det(\boldsymbol{L}_{1,k}(j) \circledS \boldsymbol{L}_{2,k}(i)) = \det \mathbf{R}_k(j,i) \tag{61}$$

and

$$\det \mathbf{S}_k = \det \mathbf{R}_k\,. \tag{61'}$$

In the next part we shall express the elements of the matrix $\mathbf{S}_k^{-1}$ in a form that will enable us to easily obtain an estimate for the components of the vector $\mathbf{S}_k^{-1}\,\gamma(k)$. All our considerations are based on the statement of Theorem 2. We shall write the formulas for $\det \mathbf{S}_k$ and $\det \mathbf{S}_k^{\mathsf{A}}(i,j)$ using Theorem 2, thus easily obtaining an expression for the elements of the inverse matrix $\mathbf{S}_k^{-1}$. The proofs of Lemma 10 and Lemma 11 immediately follow from Theorem 2; in the proof of Lemma 10 we, moreover, use the relation (61′).

Lemma 10. *Let $|\lambda_\tau| > |\lambda_{\tau+1}|$ and let the matrix formed by the first l rows of $\mathbf{J}_k$ be nonsingular for all $k \geqq k_0$.*

Then there exist an integer $\varkappa$, a positive constant D nad a sequence of real functions $\{\varphi(k)\}_{k=k_0}^{\infty}$ such that $\lim_{k\to\infty} \varphi(k) = 0$ and

$$\det \mathbf{S}_k = k^{\varkappa} \prod_{s=1}^{\tau} |\lambda_s|^{2ki_s} (D + \varphi(k))$$

for all $k \geqq k_0$. □

We have defined a vector $\mathbf{g}(k) \in C^t$ by the formula (20). Let $(\mathbf{g}(1))_i$ denote the i-th component of $\mathbf{g}(1)$. Let $\mathscr{T}$ be the set of all integers $i \leqq l$ satisfying

$$|(\mathbf{g}(1))_i| = |(\mathbf{g}(1))_l| = |\lambda_\tau|\,.$$

For every pair i, j, $i = 1, \ldots l$; $j = 1, \ldots, l + 1$ the following assertion is valid.

Lemma 11. *Let the assumptions from Lemma 10 be valid and let the matrix formed by the first l rows of $\mathbf{J}_{1,k}(j)$ and $\mathbf{J}_{2,k}(i)$ except the $i_1(j)$-th and $i_2(i)$-th row, respectively, where $i_1(j) \in \mathscr{T}$ and $i_2(i) \in \mathscr{T}$ be nonsigular for all $k \geqq k_0$. Then* *there exist an integer* $\varkappa_{ij}$, *a complex number* D_{ij} *and a function* $\varphi_{ij}(k)$ *such that*

$$\lim_{k\to\infty} \varphi_{ij}(k) = 0$$

and

$$\det \mathbf{S}_k^{\mathsf{A}}(i,j) = k^{\varkappa_{ij}} \frac{\prod_{s=1}^{\tau} |\lambda_s|^{2ki_s}}{|\lambda_\tau|^{2k}} (D_{ij} + \varphi_{ij}(k)) . \quad \square$$

Lemma 12. *Let the assumptions from Lemma* 10 *and Lemma* 11 *be fulfilled. Then the element of the matrix* $\mathbf{S}_k^{-1}$ *in an* (i,j)*-position has the form*

$$k^{\chi_{ij}} \Lambda_{ij}(k)/|\lambda_\tau|^{2k} , \tag{63}$$

where χ_{ij} *is an integer and* $\lim_{k\to\infty} \Lambda_{ij}(k) = D_{ij}/D$, *D and* D_{ij} *being the constants from Lemma* 10 *and Lemma* 11.

Moreover, the m-th component of the vector $\mathbf{S}_k^{-1} \gamma(k)$ *has the form*

$$\sum_{s=1}^{l-1} k^{\varkappa_s + \chi_{ms}} \Omega_{m,s}(k) \left(\frac{\lambda_1 \lambda_{\tau+1}}{|\lambda_\tau|^2} \right)^k , \tag{63'}$$

where the integer $\varkappa_s$ *has been defined by* (58) *and*

$$\limsup_{k\to\infty} |\Omega_{m,s}(k)| < \infty$$

for all $s = 1, \ldots, l - 1$. $\square$

Proof. From the form of $\det \mathbf{S}_k$ and $\det \mathbf{S}_k^{\mathsf{A}}(i,j)$ it is easy to see that the quotient $\det \mathbf{S}_k^{\mathsf{A}}(i,j)/\det \mathbf{S}_k$ has the form (63). Together,

$$\Lambda_{ij} = \frac{D_{ij} + \varphi_{ij}(k)}{D + \varphi(k)} ,$$

$$D > 0 \quad \text{and} \quad \lim_{k\to\infty} \varphi_{ij}(k) = \lim_{k\to\infty} \varphi(k) = 0 .$$

This implies that there exists an integer m such that $D + \varphi(k) \neq 0$ for all $k \geqq m$ and, for this k, $\Lambda_{ij}(k)$ has the above described form. For $k < m$ we define $\Lambda_{ij}(k)$ so that the expression (63) gives us the element of the matrix $\mathbf{S}_k^{-1}$ in the position (i,j). From the form of the elements of $\mathbf{S}_k^{-1}$ and $\gamma(k)$ we immediately conclude that $\mathbf{S}_k^{-1} \gamma(k)$ has the form (63'). The rest is obvious.

Theorem 4. *Let the assumptions from Lemma* 11 *be fulfilled. Let P be the polynomial defined by* (53). *If*

$$|\lambda_1 \lambda_{\tau+1}| < |\lambda_\tau|^2$$

then

$$\lim_{k\to\infty} \alpha_i^{(k)} = \sigma_i$$

for $i = 0, 1, \ldots, l$, *where* σ_i *are the coefficients of the polynomial* P.

Proof. For $l > 1$ the result follows from the previous lemma, for $l = 1$ we obtain it by a straightforward calculation.

5. RATE OF CONVERGENCE OF THE EXTRAPOLATED METHOD

From (2) we have obtained a convergent sequence $\{x_k\}_{k=0}^{\infty}$. Let us define a sequence $\{y_k\}_{k=m_l}^{\infty}$ by

$$y_k = \alpha_0^{(k)} x_k + \alpha_1^{(k)} x_{k-m_1} + \ldots + \alpha_l^{(k)} x_{k-m_l}\,.$$

Theorem 5. *Let the assumptions from the previous section, i.e.* (44), (45), *as well as those from Lemma* 10 *and Lemma* 11 *be fulfilled. We suppose that for some* $r_1 \in \langle 1, r)$ *the inequality* $|\lambda_{r_1}| > |\lambda_{r_1+1}|$ *holds. Further, if* $|\lambda_s| = |\lambda_1|$ *for* $s \in \langle 1, r_1 \rangle$ *then let* $i_1 > i_s$. *Moreover, let*

$$\frac{|\lambda_1^{2-p} \lambda_{\tau+1}|}{|\lambda_\tau|^2} < 1 \quad \textit{for some} \quad p \geqq 1\,. \tag{64}$$

Then there exists an integer k_0 *such that* $\varepsilon_k \neq 0$ *for all* $k \geqq k_0$ *and*

$$\lim_{k\to\infty} \frac{\|x^* - y_k\|}{\|x^* - x_k\|^p} = 0\,. \tag{65}$$

Proof. According to (18) we have

$$\|x^* - x_k\| = \varepsilon_k = H^{-1}\eta_k = \tag{66}$$

$$= \sum_{j=1}^{r} \sum_{i=1}^{i_j} \binom{k}{i-1} \lambda_j^k H^{-1} v_{ji} + H^{-1} v(k) = \binom{k}{i_1 - 1} \lambda_1^k (H^{-1} v_{1 i_1} + w(k))\,,$$

where the assumptions of Theorem 5 imply that $\lim_{k\to\infty} w(k) = 0$ and there exists k_0 such that $\varepsilon_k \neq 0$ for all $k \geqq k_0$.

Let us calculate

$$x^* - y_k = x^* - \sum_{i=0}^{l} \alpha_i^{(k)} x_{k-m_i} = \sum_{i=0}^{l} \alpha_i^{(k)} (x^* - x_{k-m_i}) =$$

$$= H^{-1} \sum_{i=0}^{l} \alpha_i^{(k)} \eta_{k-m_i} = H^{-1} \{ \sum_{i=0}^{l} \sigma_i \eta_{k-m_i} + \sum_{i=0}^{l} (\alpha_i^k - \sigma_i)\, \eta_{k-m_i} \}\,.$$

From (61), (63) and (18) we have

$$\|x^* - y_k\| \leqq k^\nu |\lambda_{\tau+1}|^k \cdot \|y(k)\| \cdot \|H^{-1}\| +$$

$$+ \left(\frac{\lambda_1 \lambda_{\tau+1}}{|\lambda_\tau|^2}\right)^k \lambda_1^k \sum_{i=0}^{l} \left\{\left[\sum_{s=1}^{l-1} k^{\varkappa_s + \chi_{is}} \Omega_{i,s}(k)\right] \left[\binom{k - m_i}{i_1 - 1} \lambda_1^{-m_i} \|H^{-1} v_{1 i_1} + w(k - m_i)\|\right]\right\},$$

where $\limsup_{k\to\infty} \|y(k)\| < \infty$ and $\limsup_{k\to\infty} \|\Omega_{i,s}(k)\| < \infty$ for all i, s.

This estimate together with (64) and (65) immediately yields (65).

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Souhrn

KONVERGENCE EXTRAPOLAČNÍCH KOEFICIENTŮ

JAN ZÍTKO

Nechť

$$x_{k+1} = Tx_k + b \tag{1}$$

je iterační proces na řešení operátorové rovnice $x = Tx + b$ v Hilbertově prostoru X, kde b je daný prvek z X a $T \in [X]$. Budiž $x_0 \in X$ a sestrojme posloupnost $\{x_k\}_{k=0}^{\infty}$ podle (1) a předpokládejme, že tato posloupnost konverguje k $x^* = Tx^* + b$. Nechť $l > 1$, $k, m_0, m_1, \ldots, m_l$ jsou celá čísla splňující nerovnosti

$$m_l > m_{l-1} > \ldots > m_1 > m_0 = 0\,, \quad k > m_l\,.$$

V práci [1] jsme sestrojili čísla $\alpha_i^{(k)}$, $i = 0, 1, \ldots, l$ taková, že pro vektor

$$y_k = \alpha_0^{(k)} x_k + \alpha_1^{(k)} x_{k-m_1} + \ldots + \alpha_l^{(k)} x_{k-m_l}$$

se minimalizovala vhodně zvolená norma rozdílu $x^* - y_k$. Normu je možné volit tak, aby konstrukci čísel $\alpha_i^{(k)}$, které nazveme extrapolačními koeficienty, bylo možno realizovat.

V této práci je spočítána limita čísel $\alpha_i^{(k)}$ v obecném případě. Pro ilustraci uveďme speciální případ. Nechť $|\lambda_1| \geqq \ldots \geqq |\lambda_\tau|$, $\lambda_i \neq 1$, přičemž $\lambda_1, \ldots, \lambda_\tau$ jsou póly rezolventy $R(\lambda, T)$ s násobnostmi postupně $i_1, \ldots, i_\tau$, kde $\sum_{j=1}^{\tau} i_j = l$. Položme $m_i = i \; \forall_i$

$$p(z) = (z - \lambda_1)^{i_1} (z - \lambda_2)^{i_2} \ldots (z - \lambda_\tau)^{i_\tau},$$

$$P(z) = p(z)/p(1) \equiv \sigma_0 z^l + \sigma_1 z^{l-1} + \ldots + \sigma_l.$$

Pak $\lim_{k \to \infty} \alpha_i^{(k)} = \sigma_i \; \forall i$. (Podrobněji viz Theorem 5). Na základě toho je ukázáno, že existuje $p \geqq 1$ tak, že

$$\lim_{k \to \infty} (\|x^* - y_k\| / \|x^* - x_k\|^p) = 0.$$

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