Andrzej Pokrzywa

Spectral approximation of positive operators by iteration subspace method

Aplikace matematiky, Vol. 29 (1984), No. 2, 104–113

Persistent URL: http://dml.cz/dmlcz/104074

SPECTRAL APPROXIMATION OF POSITIVE OPERATORS BY ITERATION SUBSPACE METHOD

ANDRZEJ POKRZYWA

(Received March 21, 1983)

Let H denote a real or complex Hilbert space with a norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$ and suppose that A is a bounded linear positive operator acting in H and X_0 an m-dimensional subspace of H. Then the subspaces $X_n = A^n X_0$ are also m-dimensional. Let P_n denote the orthogonal projection on X_n. We shall describe the behaviour of the spectra and the eigenspaces of the operators $A_n = P_n A|_{X_n}$. We shall investigate what happens if instead of X_0 its subspace $\tilde{X}_0$ is taken, and a simple way of approximating the spectra of the operators A_n will be given. The case $\dim X_0 = 1$ was studied in the papers of Kolomý and others (see [2], [3] and references the ire in), the iteration subspace method for matrices was studied in [4] and [5].

Let $\{E(\lambda)\}$ denote the spectral family of A. We shall use the notation $E[a, b] = E(b + 0) - E(a - 0)$, $E(a, b] = E(b + 0) - E(a + 0)$, etc. Since $\dim E(\lambda, \infty) X_0$ is an integer-valued nonincreasing function of λ, the set of its points of discontinuity is finite. Let $\alpha_1 > \alpha_2 > \ldots > \alpha_k$ be all such points, we put in addition $\alpha_{k+1} = 0$. Let λ_j $(j = 1, 2, \ldots, m)$ be such real numbers that

$$(1) \qquad \dim E(\lambda_j, \infty) X_0 < j \text{ and } \dim E(\lambda_j - \varepsilon, \infty) X_0 \geqq j \text{ for any } \varepsilon > 0 .$$

Then $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_m$ and $\{\alpha_j\}_1^k = \{\lambda_j\}_1^m \subset \sigma(A)$ (the spectrum of A), since $E(\lambda)$ is constant in some neighbourhood of any $\tilde{\lambda} \notin \sigma(A)$.

Lemma 1. *Suppose that Y is a finite-dimensional subspace of H, $0 < \alpha < \lambda$ and $E(\lambda, \infty) y \neq 0$ for all nonzero $y \in Y$. Then there exists a positive number c such that $\langle A^n E(0, \alpha] y, y\rangle \leqq c(\alpha/\lambda)^n \langle A^n y, y\rangle$ $(\forall y \in Y)$.*

Proof. Since the unit ball in Y is compact we can find a positive number c_1 such that $\|E(\lambda, \infty) y\| \geqq c_1 \|y\|$ $(\forall y \in Y)$. This implies that for all $y \in Y$,

$$(2) \qquad \langle A^n y, y\rangle = \int_{(0,\infty)} \xi^n \,\mathrm{d}\langle E(\xi) y, y\rangle \geqq \lambda^n \int_{(\lambda,\infty)} \mathrm{d}\,\langle E(\xi) y, y\rangle =$$

$$= \lambda^n \|E(\lambda, \infty)\, y\|^2 \geqq c_1^2\, \lambda^n \|y\|^2 .$$

In a similar way one can show that

$$(3) \qquad \langle E(0, \alpha]\, A^n y, y\rangle \leqq \alpha^n \|E(0, \alpha]\, y\|^2 \leqq \alpha^n \|y\|^2 \quad (\forall y \in Y) .$$

Dividing (2) by (3) we obtain the assertion.

Theorem 1. *Let $\lambda_{1,n} \geqq \lambda_{2,n} \geqq \ldots \geqq \lambda_{m,n}$ be the eigenvalues of A_n. Then $\lambda_{j,n} \nearrow \lambda_j$ with $n \to \infty$ $(j = 1, 2, \ldots, m)$.*

Proof. The operator A_n is a selfadjoint operator acting in the m-dimensional space X_n, therefore its eigenvalues satisfy the max-min principle (see e.g. [1], p. 60)

$$(4) \qquad \lambda_{j,n} = \max_{\substack{X \subset X_n \\ \dim X = j}} \min_{\substack{x \in X \\ \|x\| = 1}} \langle Ax, x\rangle = \max_{\substack{X \subset X_0 \\ \dim X = j}} \min_{\substack{x \in X \\ x \neq 0}} \frac{\langle A^{n+1}x, A^n x\rangle}{\|A^n x\|^2} .$$

Since $\langle A^n x, x\rangle^2 = \langle A^{(n-1)/2}x, A^{(n+1)/2}x\rangle^2 \leqq \|A^{(n-1)/2}x\|^2 \, \|A^{(n+1)/2}x\|^2 = \langle A^{n-1}x, x\rangle \langle A^{n+1}x, x\rangle$ for any $x \in H$, we have, for all nonzero $x \in H$, $\langle A^{n+1}x, A^n x\rangle / \|A^n x\|^2 = \langle A^{2n+1}x, x\rangle / \langle A^{2n}x, x\rangle \geqq \langle A^{2n}x, x\rangle / \langle A^{2n-1}x, x\rangle \geqq \langle A^{2n-1}x, x\rangle / \langle A^{2n-2}x, x\rangle = \langle A^n x, A^{n-1}x\rangle / \|A^{n-1}x\|^2$. This equality and (4) imply that

$$(5) \qquad \lambda_{j,n} \geqq \lambda_{j,n-1} \quad j = 1, 2, \ldots, m , \quad n = 1, 2, \ldots .$$

It follows from (1) that if X is a j-dimensional subspace of X_0 $(1 \leqq j \leqq m)$ then there is a nonzero $x \in X$ such that $E(\lambda_j, \infty)\, x = 0$, and then $\langle A^{n+1}x, A^n x\rangle = \|A^{1/2}E(0, \lambda_j]\, A^n x\|^2 \leqq \|A^{1/2}E(0, \lambda_j]\|^2 \, \|A^n x\|^2 = \lambda_j \|A^n x\|^2$. This inequality and (4) imply that

$$(6) \qquad \lambda_{j,n} \leqq \lambda_j , \quad j = 1, 2, \ldots, m , \quad n = 1, 2, \ldots .$$

It follows from (1) that for each $\varepsilon \in (0, \lambda_j/2)$ there exists a j-dimensional subspace X of X_0 such that $E(\lambda_j - \varepsilon, \infty)\, x \neq 0$ for all nonzero $x \in X$. By Lemma 1 we can find a positive number c such that $\|A^n E(0, \lambda_j - 2\varepsilon]\, x\| \leqq c((\lambda_j - 2\varepsilon)/(\lambda_j - \varepsilon))^n \, \|A^n x\|$ $(\forall x \in X)$. Thus for any nonzero $x \in X$ we have

$$\langle A^{n+1}x, A^n x\rangle = \int_{(0,\infty)} \xi^{2n+1} \, \mathrm{d}\langle E(\xi)\, x, x\rangle \geqq$$

$$\geqq (\lambda_j - 2\varepsilon) \int_{(\lambda_j - 2\varepsilon, \infty)} \xi^{2n} \, \mathrm{d}\langle E(\xi)\, x, x\rangle = (\lambda_j - 2\varepsilon) \, \|A^n E(\lambda_j - 2\varepsilon, \infty)\, x\|^2 =$$

$$= (\lambda_j - 2\varepsilon) \left(\|A^n x\|^2 - \|E(0, \lambda_j - 2\varepsilon]\, A^n x\|^2\right) \geqq (\lambda_j - 2\varepsilon) \left(1 - c\left(\frac{\lambda_j - 2\varepsilon}{\lambda_j - \varepsilon}\right)^{2n}\right) \times$$

$$\times \|A^n x\|^2 ,$$

and using the max-min principle we get

$$\lambda_{j,n} \geqq (\lambda_j - 2\varepsilon) \times \left(1 - c\left(\frac{\lambda_j - 2\varepsilon}{\lambda_j - \varepsilon}\right)^{2n}\right).$$

This inequality together with (5) and (6) implies that $\lambda_j \geqq \lim_n \lambda_{j,n} \geqq \lambda_j - 2\varepsilon$ for any $\varepsilon > 0$, and this completes the proof.

Let $V_{j,n}$ be the subspace of X_n spanned by those eigenvectors of A_n which correspond to the eigenvalues of A_n lying in the interval $(\alpha_{j+1}, \alpha_j]$. In the case $\dim X_0 = 1$ we obviously have $V_{1,n} = A^n X_0$. In general we cannot find a subspace Z_j such that $V_{j,n} = A^n Z_j$, nevertheless, we shall show that there are subspaces Z_j which satisfy this identity approximately.

For any two subspace M, N of H we set (cf. [1], § IV.2)

$$\delta(M, N) = \sup_{\substack{x \in M \\ \|x\| = 1}} \inf_{y \in N} \|x - y\|, \quad \hat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}.$$

$\hat{\delta}(M, N)$ is called the gap between the subspaces M, N and if P, Q are orthogonal projections on M, N, respectively, then $\hat{\delta}(M, N) = \|P - Q\|$, $\delta(M, N) = \|(1 - Q) P\|$. Thus $\hat{\delta}$ is a distance function. It is known that

$$\text{(7)} \qquad \text{if} \quad \delta(M, N) < 1 \quad \text{then} \quad \dim M \leqq \dim N$$

(see [1], Corollary IV. 2.6.) and (cf. [1], Th. I.6.34)

$$\text{(8)} \qquad \text{if} \quad \dim M = \dim N \quad \text{then} \quad \delta(M, N) = \delta(N, M) = \hat{\delta}(M, N).$$

We put

$$\text{(9)} \qquad Y_j = X_0 \cap \ker E(\alpha_j, \infty) = X_0 \cap \operatorname{ran} E[0, \alpha_j] \quad (j = 1, 2, \ldots, k + 1).$$

Then $\{0\} = Y_{k+1} \subsetneqq Y_k \subsetneqq \ldots \subsetneqq Y_1 = X_0$, and let Z_j be a subspace complementary to Y_{j+1} in Y_j, i.e. $Z_j \cap Y_{j+1} = \{0\}$ and $Z_j + Y_{j+1} = Y_j$. We also set $Z_{j,n} = A^n Z_j$; then we have $Z_{1,n} + Z_{2,n} + \ldots + Z_{k,n} = X_n$.

Lemma 2. *For any $\varepsilon > 0$ there exists a positive number c such that*

$$\text{(10)} \qquad \hat{\delta}(Z_{j,n}, E(\alpha_j - \varepsilon, \alpha_j]\, Z_{j,n}) \leqq c(1 - \varepsilon/\alpha_j)^n.$$

Furthermore,

$$\text{(11)} \qquad \|(A - \alpha_j) \mid Z_{j,n}\| \to 0 \quad \text{with} \quad n \to \infty.$$

Proof. Lemma 1 applied to $Y = Z_j$, $\alpha = \alpha_j - \varepsilon$, $\lambda = \alpha_j$ implies the existence of a positive number c such that $\|A^n E(0, \alpha_j - \varepsilon]\, z\| \leqq c(1 - \varepsilon/\alpha_j)^n \|A^n z\|$ $(\forall z \in Z_j)$, which gives $\|z - E(\alpha_j - \varepsilon, \alpha_j]\, z\| = \|E(0, \alpha_j - \varepsilon]\, z\| \leqq c(1 - \varepsilon/\alpha_j)^n$ for all $z \in Z_{j,n}$. Consequently, for large n, $\delta(Z_{j,n}, E(\alpha_j - \varepsilon, \alpha_j]\, Z_{j,n}) \leqq c(1 - \varepsilon/\alpha_j)^n < 1$, hence, by (7), $\dim Z_{j,n} \leqq \dim E(\alpha_j - \varepsilon, \alpha_j]\, Z_{j,n} \leqq \dim Z_{j,n}$ and applying (8) we obtain (10). In virtue of the inequalities

$$\|(A - \alpha_j)|_{\operatorname{ran} E(\alpha_j - \varepsilon, \alpha_j]}\| \leqq \varepsilon \quad \text{and} \quad \|(A - \alpha_j)|_{Z_{j,n}}\| \leqq$$
$$\leqq \|(A - \alpha_j)|_{E(\alpha_j - \varepsilon, \alpha_j] Z_{j,n}}\| + \|A - \alpha_j\| \, \delta(Z_{j,n}, E(\alpha_j - \varepsilon, \alpha_j] \, Z_{j,n}) \leqq$$
$$\leqq \varepsilon + c(1 - \varepsilon/\alpha_j)^n$$

we have $\|(A - \alpha_j)|_{Z_{j,n}}\| \leqq 2\varepsilon$ for sufficiently large n, which proves (11).

Theorem 2.

$$\hat{\delta}(V_{j,n}, Z_{j,n}) \underset{n}{\rightarrow} 0 \quad \textit{and} \quad \|(A - \alpha_j)|_{V_{j,n}}\| \underset{n}{\rightarrow} 0$$

for $j = 1, 2, \ldots, k$.

Proof. We put $\eta = \min\limits_{1 \leqq j \leqq k} (\alpha_j - \alpha_{j+1})$ and $\mu_n = \max\limits_{1 \leqq j \leqq m} (\lambda_j - \lambda_{j,n})$, where $\lambda_{j,n}$ are the eigenvalues of A_n (cf. Th. 1). Assume that $z \in Z_{j,n}$, $\|z\| = 1$ and v_i are orthogonal eigenvectors of A_n such that $A_n v_i = \lambda_{i,n} v_i$, $z = v_1 + v_2 + \ldots + v_m$. Then

$$\|(A_n - \alpha_j) z\|^2 = \|(A_n - \alpha_j) \sum_{i=1}^{m} v_i\|^2 = \|\sum_{i=1}^{m} (\lambda_{i,n} - \alpha_j) v_i\|^2 = \sum_{i=1}^{m} (\lambda_{i,n} - \alpha_j)^2 \|v_i\|^2 .$$

Thus, if n is so large that $\mu_n < \eta/2$, then

$$\|(A_n - \alpha_j)|_{Z_{j,n}}\|^2 \geqq \|(A_n - \alpha_j) z\|^2 \geqq (\eta^2/4) \sum_{v_i \notin V_{j,n}} \|v_i\|^2 = (\eta^2/4) \|z - \sum_{v_i \in V_{j,n}} v_i\|^2 .$$

In this way we have shown that

$$\delta(Z_{j,n}, V_{j,n}) \leqq 2\|(A_n - \alpha_j)|_{Z_{j,n}}\|/\eta \leqq 2\|(A - \alpha_j)|_{Z_{j,n}}\|/\eta$$

for sufficiently large n. Lemma 2 implies that $\delta(Z_{j,n}, V_{j,n}) < 1$ for n large enough. Then, by (7), $\dim Z_{j,n} \leqq \dim V_{j,n}$ and since

$$\sum_{j=1}^{k} \dim Z_{j,n} = \sum_{j=1}^{k} \dim V_{j,n}$$

we have in fact $\dim Z_{j,n} = \dim V_{j,n}$ and in virtue of (8),

$$\hat{\delta}(Z_{j,n}, V_{j,n}) = \delta(Z_{j,n}, V_{j,n}) \leqq 2\|(A - \alpha_j)|_{Z_{j,n}}\|/\eta . \tag{12}$$

This together with Lemma 2 shows that $\hat{\delta}(Z_{j,n}, V_{j,n}) \underset{n}{\rightarrow} 0$. To complete the proof it suffices to note that

$$\|(A - \alpha_j)|_{V_{j,n}}\| \leqq \|(A - \alpha_j)|_{Z_{j,n}}\| + \|A - \alpha_j\| \, \hat{\delta}(Z_{j,n}, V_{j,n}) \underset{n}{\rightarrow} 0 .$$

We shall study now what happens if instead of the initial subspace X_0 some larger or smaller space is taken. Suppose that $\tilde{X}_0$ is an m-dimensional subspace of X_0 and let $\tilde{P}_n$, $\tilde{A}_n$, $\tilde{k}$, $\tilde{\alpha}_j$, $\tilde{\lambda}_j$, $\tilde{\lambda}_{j,n}$ mean the same for $\tilde{X}_0$ as the non-waved symbols mean with respect to X_0.

Theorem 3. *Under the above assumptions the following statements hold*:

i) $\{\tilde{\alpha}_j\}_{j=1}^{\tilde{k}} \subset \{\alpha_j\}_{j=1}^{k}$,

ii) *the sequence* $\{\tilde{\lambda}_j\}_1^{\tilde{m}}$ *is a subsequence of* $\{\lambda_j\}_1^m$,

iii) $\lambda_j \geqq \tilde{\lambda}_j \geqq \lambda_{j+m-\tilde{m}}$, $j = 1, 2, \ldots, \tilde{m}$,

iv) $\lambda_{j,n} \geqq \tilde{\lambda}_{j,n} \geqq \lambda_{j+m-\tilde{m},n}$, $j = 1, 2, \ldots, \tilde{m}$, $n = 1, 2, \ldots$,

v) *if* $\tilde{V}_{j,n}$ *is a subspace spanned by those eigenvectors of* $\tilde{A}_n$ *which correspond to the eigenvalues lying in the interval* $(\alpha_{j+1}, \alpha_j]$ *(non-waved!) then*

$$\delta(\tilde{V}_{j,n}, V_{j,n}) \underset{n}{\to} 0 .$$

Proof. Note that if $\alpha' \notin \{\alpha_j\}_1^k$ then $X_0 \cap \ker E(\alpha, \infty)$ does not depend on α in some neighbourhood of α', hence $X_0 \cap \ker E(\alpha, \infty)$ does not change in this neighbourhood as well – this shows i).

Let $\tilde{Y}_j = Y_j \cap \tilde{X}_0$ $(j = 1, 2, \ldots, k+1)$ (cf. (9)), then $\{0\} = \tilde{Y}_{k+1} \subseteq \tilde{Y}_k \subseteq \ldots \ldots \subseteq \tilde{Y}_1 = \tilde{X}_0$. Setting $\tilde{Z}_j = \tilde{Y}_{j+1}^{\perp} \cap \tilde{Y}_j$ we see that $\tilde{Z}_j \cap Y_{j+1} = \tilde{Y}_{j+1}^{\perp} \cap \tilde{Y}_j \cap \cap Y_{j+1} = \tilde{Y}_{j+1}^{\perp} \cap \tilde{Y}_{j+1} = \{0\}$. Since $\tilde{Z}_j \cap Y_{j+1} = \{0\}$ and $\tilde{Z}_j + Y_{j+1} \subset Y_j$ there exists a subspace Z_j complementary to Y_{j+1} in Y_j and containing $\tilde{Z}_j$, i.e. $\tilde{Z}_j \subset Z_j$, $Z_j + Y_{j+1} = Y_j$ and $Z_j \cap Y_{j+1} = \{0\}$. It follows from Theorem 2 that $\hat{\delta}(V_{j,n}, A^n Z_j) \underset{n}{\to} 0$ and $\hat{\delta}(\tilde{V}_{j,n}, A^n \tilde{Z}_j) \underset{n}{\to} 0$. This convergence and the inequality $\delta(\tilde{V}_{j,n}, V_{j,n}) \leqq \hat{\delta}(\tilde{V}_{j,n}, A^n \tilde{Z}_j) + \delta(A^n \tilde{Z}_j, A^n Z_j) + \hat{\delta}(A^n Z_j, V_{j,n})$ imply v) since $\delta(A^n \tilde{Z}_j, A^n Z_j) = 0$.

The inequalities $\hat{\delta}(V_{j,n}, A^n Z_j) < 1$, $\hat{\delta}(\tilde{V}_{j,n}, A^n \tilde{Z}_j) < 1$, which hold for sufficiently large n, imply together with (7) that $\dim \tilde{V}_{j,n} = \dim \tilde{Z}_j \leqq \dim Z_j = \dim V_{j,n}$. Thus if we put

$$\text{(13)} \qquad \tilde{m}_1 = m_1 = 0\,, \quad \tilde{m}_j = \sum_{i=1}^{j-1} \dim \tilde{Z}_i\,, \quad m_j = \sum_{i=1}^{j-1} \dim Z_i\,,$$

then it is a consequence of the definition of $\tilde{V}_{j,n}$, $V_{j,n}$ and Theorem 1 that $\lambda_{m_j+i,n} \nearrow \nearrow \lambda_{m_j+i} = \alpha_j$ and $\tilde{\lambda}_{\tilde{m}_j+i,n} \nearrow \tilde{\lambda}_{\tilde{m}_j+i} = \alpha_j$, $i = 1, 2, \ldots, \dim \tilde{Z}_j$. These relations imply ii). iv) is an asertion of Th. II.6.46 [1] aplied to the operators A_n and $\tilde{A}_n = \tilde{P}_n A_n|_{\tilde{X}_n}$ and iii) may be obtained from iv) by going to infinity with n.

With the notation (13) it follows from the above theorem that for $j = 1$ one has $\tilde{\lambda}_{\tilde{m}_j+i,n} \leqq \lambda_{m_j+i,n} \leqq \alpha_j$ $(i = 1, 2, \ldots, \dim Z_j)$; these inequalities do not hold for $j > 1$ in general. Nevertheless, one might expect that the convergence $\lambda_{m_j+i,n} \underset{n}{\to} \alpha_j$ is not worse than $\tilde{\lambda}_{\tilde{m}_j+i,n} \underset{n}{\to} \alpha_j$, i.e. that

$$\text{(14)} \qquad \limsup_{n\to\infty}\, (\alpha_j - \lambda_{m_j+i,n})/(\alpha_j - \tilde{\lambda}_{m_j+i,n}) < \infty\,, \quad i = 1, 2, \ldots, \dim Z_j\,.$$

The next theorem shows that (14) holds if $\dim X_0 = 2$, however, no general solution has been found.

Theorem 4. *In addition to the assumptions of Theorem* 3 *assume that* $\dim \tilde{X}_0 = 1$, $\dim X_0 = 2$ and $\tilde{\lambda}_1 = \lambda_2 < \lambda_1$. *Then*

$$\limsup_{n\to\infty} (\lambda_2 - \lambda_{2,n})/(\lambda_2 - \tilde{\lambda}_{1,n}) \leqq \lambda_1/(\lambda_1 - \lambda_2) .$$

Proof. Let v_n, w_n be the orthonormal eigenvectors of A_n, x_n — the unit vector in $\tilde{X}_n$ and $y_n \in X_n$ a unit vector orthogonal to x_n. Then it follows from (4) that $\lambda_{2,n} = \langle Av_n, v_n\rangle \leqq \tilde{\lambda}_{1,n} = \langle Ax_n, x_n\rangle \leqq \lambda_{1,n} = \langle Aw_n, w_n\rangle$, and we put $\mu_n = \langle Ay_n, y_n\rangle$, $\gamma_n = \langle Ax_n, y_n\rangle$. $\lambda_{1,n}, \lambda_{2,n}$ are the eigenvalues of the matrix

$$\mathscr{A}_n = \begin{bmatrix} \tilde{\lambda}_{1,n} & \bar{\gamma}_n \\ \bar{\gamma}_n & \mu_n \end{bmatrix},$$

thus solving the quadratic equation $\det(\mathscr{A}_n - \lambda) = 0$ we have

$$\lambda_{2,n} = (\tilde{\lambda}_{1,n} + \mu_n - ((\tilde{\lambda}_{1,n} - \mu_n)^2 + 4|\gamma_n|^2)^{1/2})/2 . \tag{15}$$

We shall estimate $|\gamma_n|$. Since $\tilde{\lambda}_{1,n} \to \tilde{\lambda}_1$ it follows from Theorem 1 that $E(\tilde{\lambda}_1, \infty)\, x_n = 0$ for all n, which implies $\|(A - \tilde{\lambda}_1/2)\, x_n\| \leqq \tilde{\lambda}_1/2$ and consequently, $\|(A - \tilde{\lambda}_1)\, x_n\|^2 = \|(A - \tilde{\lambda}_1/2)\, x_n - \tilde{\lambda}_1 x_n/2\|^2 = \|(A - \tilde{\lambda}_1/2)\, x_n\|^2 - 2\langle(A - \tilde{\lambda}_1/2)\, x_n, \tilde{\lambda}_1 x_n/2\rangle + \|\tilde{\lambda}_1 x_n/2\|^2 \leqq (\tilde{\lambda}_1/2)^2 - 2(\tilde{\lambda}_{1,n} - \tilde{\lambda}_1/2)\, \tilde{\lambda}_1/2 + \tilde{\lambda}_1^2/4 = \tilde{\lambda}_1(\tilde{\lambda}_1 - \tilde{\lambda}_{1,n})$. This inequality and the eigenvalue expansion of A_n further gives $\tilde{\lambda}_1(\tilde{\lambda}_1 - \tilde{\lambda}_{1,n}) \geqq \|(A_n - \tilde{\lambda}_1)\, x_n\|^2 = \|(A_n - \lambda_2)(\langle x_n, v_n\rangle v_n + \langle x_n, w_n\rangle w_n)\|^2 = \|\langle x_n, v_n\rangle (\lambda_{2,n} - \lambda_2) . v_n + \langle x_n, w_n\rangle (\lambda_{1,n} - \lambda_2)\, w_n\|^2 \geqq |\langle x_n, w_n\rangle|^2 (\lambda_{1,n} - \lambda_2)^2$. This inequality implies

$$|\langle x_n, w_n\rangle|^2 \leqq \lambda_2(\tilde{\lambda}_1 - \tilde{\lambda}_{1,n})/(\lambda_{1,n} - \lambda_2)^2 . \tag{16}$$

The identities $w_n = \langle w_n, y_n\rangle\, y_n + \langle w_n, x_n\rangle\, x_n$ and $|\langle w_n, y_n\rangle|^2 + |\langle w_n, x_n\rangle|^2 = \|w_n\|^2 = 1$ imply

$$|\gamma_n|^2 = |\langle Ax_n, y_n\rangle|^2 = \left|\frac{\langle Ax_n, w_n - \langle w_n, x_n\rangle\, x_n\rangle}{\langle w_n, y_n\rangle}\right|^2 = \tag{17}$$

$$= \frac{|\langle x_n, Aw_n\rangle - \langle x_n, w_n\rangle \langle Ax_n, x_n\rangle|^2}{1 - |\langle w_n, x_n\rangle|^2} = (\lambda_{1,n} - \tilde{\lambda}_{1,n})^2 \frac{|\langle x_n, w_n\rangle|^2}{1 - |\langle w_n, x_n\rangle|^2}$$

Note also that

$$\mu_n = \operatorname{trace} \mathscr{A}_n - \tilde{\lambda}_{1,n} = \lambda_{1,n} + \lambda_{2,n} - \tilde{\lambda}_{1,n} \underset{n}{\to} \lambda_1 . \tag{18}$$

Now the identity

$$\lambda_2 - \lambda_{2,n} = [((\tilde{\lambda}_{1,n} - \mu_n)^2 + 4|\gamma_n|^2)^{1/2} - (\tilde{\lambda}_{1,n} + \mu_n - 2\lambda_2)]/2 =$$

$$= \frac{(\tilde{\lambda}_{1,n} - \mu_n)^2 - (\tilde{\lambda}_{1,n} + \mu_n - 2\lambda_2)^2 + 4|\gamma_n|^2}{2(((\tilde{\lambda}_{1,n} - \mu_n)^2 + 4|\gamma_n|^2)^{1/2} + \tilde{\lambda}_{1,n} + \mu_n - 2\lambda_2)} =$$

$$= (\lambda_2 - \tilde{\lambda}_{1,n}) \frac{2(\mu_n - \lambda_2) + 2(\gamma_n)^2/(\lambda_2 - \tilde{\lambda}_{1,n})}{((\mu_n - \tilde{\lambda}_{1,n})^2 + 4|\gamma_n|^2)^{1/2} + \tilde{\lambda}_{1,n} + \mu_n - 2\lambda_2}$$

implies in virtue of (16), (17) and (18) that

$$\limsup_{n\to\infty} \frac{\lambda_2 - \lambda_{2,n}}{\lambda_2 - \tilde{\lambda}_{1,n}} \leqq 2\left(\lambda_1 - \lambda_2 + \lim_n \frac{(\lambda_{1,n} - \tilde{\lambda}_{1,n})^2}{1 - |\langle x_n, w_n \rangle|^2} \frac{2\lambda_2}{(\lambda_{1,n} - \lambda_2)^2}\right) \times$$

$$\times \left[((\lambda_1 - \lambda_2)^2 + 0)^{1/2} + \lambda_2 + \lambda_1 - 2\lambda_2\right]^{-1} = \lambda_1/(\lambda_1 - \lambda_2),$$

which completes the proof.

It is shown in the above proof that $\mu_n \underset{n}{\rightarrow} \lambda_1$ and $\|w_n - y_n\| \underset{n}{\rightarrow} 0$, thus instead of looking for the exact solution of the eigenvalue problem for the operator A_n one may be satisfied by taking $\tilde{\lambda}_{1,n}, \mu_n, x_n, y_n$ as the approximate solution. This procedure may be generalized in the following way.

Suppose that $\{0\} = M_0 \subset M_1 \subset \ldots \subset M_m = X_0$ are subspace with $\dim M_j = j$. Let $x_{j,n} \in A^n M_j$ be a vector orthogonal to $A^n M_{j-1}$ with unit norm and put $\mu_{j,n} = \langle A x_{j,n}, x_{j,n} \rangle$ $(j = 1, 2, \ldots, m)$. Note that the vectors $x_{1,n}, x_{2,n}, \ldots, x_{m,n}$ may be obtained from the vectors $Ax_{1,n-1}, Ax_{2,n-1}, \ldots, Ax_{m,n-1}$ by the Schmidt orthogonalization process (see e.g. [1] p. 50).

Theorem 5. *The sequences* $\{\mu_{j,n}\}_{n=0}^{\infty}$ $(j = 1, 2, \ldots, m)$ *are convergent and* $\{\lim_n \mu_{j,n}\}_{j=1}^{m} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. *If* $W_{l,n}$ *denotes the subspace spanned by the vectors* $x_{j,n}$ *with indices* j *such that* $\lim_s \mu_{j,s} = \alpha_l$ *then* $\hat{\delta}(W_{l,n}, V_{l,n}) \underset{n}{\rightarrow} 0$.

Proof. Let $P_n(j)$ denote the orthogonal projection on $A^n M_j$ and $F_{l,n}(j)$ – the orthogonal projection on the subspace $V_{l,n}(j)$ spanned by those eigenvectors of the operator $P_n(j)\, A|_{A^n M_j}$ which correspond to the eigenvalues lying in the interval $(\alpha_{l+1}, \alpha_l]$. Theorems 1 and 2 imply that there exists a number n_0 such that $\dim V_{l,n}(j)$ is independent of n for $n > n_0$. Theorem 3 implies that

$$(19) \qquad \delta(V_{l,n}(j-1), V_{l,n}(j)) \underset{n}{\rightarrow} 0,$$

thus, by (7), $\dim V_{l,n}(j-1) \leqq \dim V_{l,n}(j)$ for $n > n_0$. Since $j = \sum_{l=1}^{k} \dim V_{l,n}(j) = 1 + \sum_{l=1}^{k} \dim V_{l,n}(j-1)$ we in fact have $\dim V_{l,n}(j-1) = \dim V_{l,n}(j)$ for all $l = 1, 2, \ldots, k$ except one, denoted by l_j, which together with (19) and (8) implies that for $l \neq l_j$,

$$(20) \qquad \|F_{l,n}(j) - F_{l,n}(j-1)\| = \hat{\delta}(V_{l,n}(j), V_{l,n}(j-1)) \underset{n}{\rightarrow} 0.$$

Since

$$P_n(j) = \sum_{l=1}^{K} F_{l,n}(j),$$

thus setting

$$G_{j,n} = P_n(j) - P_n(j-1) = \langle \cdot, x_{j,n} \rangle\, x_{j,n}$$

we have

$$\|G_{j,n} - (F_{l_j,n}(j) - F_{l_j,n}(j-1))\| = \|\sum_{\substack{l=1\\ l\neq l_j}}^{k} (F_{l,n}(j) - F_{l,n}(j-1))\| \leqq$$

$$\leqq \sum_{\substack{l=1\\ l\neq l_j}}^{k} \hat{\delta}(V_{l,n}(j), V_{l,n}(j-1)).$$

Let

$$Q_{r,n} = \sum_{\substack{j=1\\ l_j=r}}^{m} G_{j,n} \quad \text{and} \quad W_{r,n} = \operatorname{ran} Q_{r,n},$$

$Q_{r,n}$ is the orthogonal projection on $W_{r,n}$. The previous inequality implies that

$$(21) \qquad \hat{\delta}(W_{r,n}, V_{r,n}) = \|Q_{r,n} - F_{r,n}(m)\| =$$

$$= \|\sum_{\substack{j=1\\ l_j=r}}^{m} (G_{j,n} - (F_{r,n}(j) - F_{r,n}(j-1))) - \sum_{\substack{j=1\\ l_j\neq r}}^{m} (F_{r,n}(j) - F_{r,n}(j-1))\| \leqq$$

$$\leqq \sum_{\substack{j=1\\ l_j=r}}^{m} \sum_{\substack{l=1\\ l\neq l_j}}^{k} \hat{\delta}(V_{l,n}(j), V_{l,n}(j-1)) + \sum_{\substack{j=1\\ l_j\neq r}}^{m} \hat{\delta}(V_{r,n}(j), V_{r,n}(j-1)).$$

This in virtue of (20) shows that $\hat{\delta}(W_{r,n}, V_{r,n}) \underset{n}{\to} 0$, $(r = 1, 2, \ldots, k)$. This convergence and Theorem 2 imply that $\|(A - \alpha_r)|_{W_{r,n}}\| \underset{n}{\to} 0$, and to complete the proof it suffices to note that if $l_j = r$ then $x_{j,n} \in W_{r,n}$ and then

$$|\mu_{j,n} - \alpha_r| = |\langle (A - \alpha_r)\, x_{j,n}, x_{j,n}\rangle| \leqq$$

$$\leqq \|(A - \alpha_r)\, x_{j,n}\| \leqq \|(A - \alpha_r)|_{W_{n,r}}\| \underset{n}{\to} 0.$$

ERROR ESTIMATIONS

Suppose that the eigenvalue problem for the operator A_n has been solved. Then, using the formula

$$\mathrm{d}(\lambda, \sigma(A)) \overset{\mathrm{df}}{=} \inf_{\mu\in A} |\lambda - \mu| = \inf_{\substack{x\in H\\ \|x\|=1}} \|(A - \lambda)\, x\|$$

(cf. [1], p. 277), we can estimate how far the eigenvalues of A_n are from the spectrum of A. Namely, if $(A_n - \lambda_{j,n})\, v_{j,n} = 0$ and $\|v_{j,n}\| = 1$, then $\mathrm{d}(\lambda_{j,n}, \sigma(A)) \leqq \|(A - \lambda_{j,n})\, v_{j,n}\| = (\|Av_{j,n}\|^2 - 2\lambda_{j,n}\langle Av_{j,n}, v_{j,n}\rangle + \lambda_{j,n}^2)^{1/2} = (\|Av_{j,n}\|^2 - \lambda_{j,n}^2)^{1/2}$. In the same way we can find an a posteriori estimate $\mathrm{d}(\mu_{j,n}, \sigma(A)) \leqq (\|Ax_{j,n}\|^2 - \mu_{j,n}^2)^{1/2}$, where $x_{j,n}$, $\mu_{j,n}$ have the same meaning as in Theorem 5.

If α_j is an isolated eigenvalue in $\sigma(A)$ then the following theorem gives an a priori estimate of $\mathrm{d}(\lambda_{j,n}, \sigma(A))$ and provides fast convergence of eigenvalues and eigenvectors of operators A_n.

Theorem 6. *Suppose that for some* $\varepsilon > 0$ $(\alpha_j - \varepsilon, \alpha_j] \cap \sigma(A) = \{\alpha_j\}$. *Then there is a positive number* c *such that* $\hat{\delta}(V_{j,n}, E_j Z_j) \leqq c(1 - \varepsilon/\alpha_j)^n$, *where* $E_j = E(\alpha_j - \varepsilon, \alpha_j] = E[\alpha_j, \alpha_j]$ *and* $\alpha_j - \lambda_{i,n} \leqq c(1 - \varepsilon/\alpha_j)^{2n}$, *for all* i *such that* $\lambda_i = \alpha_j$.

Proof. It follows from (10) and (11) that there is a positive number c_0 such that for sufficiently large n, $\hat{\delta}(V_{j,n}, Z_{j,n}) \leqq c_0 \|(A - \alpha_j)|_{Z_{j,n}}\|$ and $\hat{\delta}(Z_{j,n}, E_j Z_{j,n}) \leqq c_0(1 - \varepsilon/\alpha_j)^n$. Note that $(A - \alpha_j)|_{E_j Z_j} = 0$ and $E_j Z_{j,n} = E_j A^n Z_j = E_j Z_j$; therefore $\|(A - \alpha_j)|_{Z_{j,n}}\| \leqq \|A - \alpha_j\| \times \hat{\delta}(Z_{j,n}, E_j Z_j)$ and $\hat{\delta}(V_{j,n}, E_j Z_j) \leqq \hat{\delta}(V_{j,n}, Z_{j,n}) + \hat{\delta}(Z_{j,n}, E_j Z_j) \leqq c_0(\|A - \alpha_j\| + 1)(1 - \varepsilon/\alpha_j)^n$. Suppose now that $(A_n - \lambda_{i,n})\, v = 0$, $v \in V_{j,n}$, $\|v\| = 1$. Then $\lambda_i = \alpha_j$ and $\lambda_{i,n} = \langle Av, v\rangle = \langle A(1 - E_j)\, v, v\rangle + \langle AE_j v, v\rangle = \langle A(1 - E_j)\, v, (1 - E_j)\, v\rangle + \alpha_j \|E_j v\|^2$. Thus $\alpha_j - \lambda_{i,n} = \alpha_j(1 - \|E_j v\|^2) - \langle A(1 - E_j)\, v, (1 - E_j)\, v\rangle \leqq \alpha_j \|(1 - E_j)\, v\|^2 \leqq \alpha_j(\hat{\delta}(V_{j,n}, E_j V_{j,n}))^2$. It is easy to verify that $\hat{\delta}(E_j Z_j, E_j V_{j,n}) \leqq \hat{\delta}(E_j Z_j, V_{j,n})$, hence $\hat{\delta}(V_{j,n}, E_j V_{j,n}) \leqq \hat{\delta}(V_{j,n}, E_j Z_{j,n}) + \hat{\delta}(E_j Z_j, E_j V_{j,n}) \leqq 2\hat{\delta}(V_{j,n}, E_j Z_j) \leqq 2c_1(1 - \varepsilon/\alpha_j)^n$. The above shows that $\alpha_j - \lambda_{i,n} \leqq c_2(1 - \varepsilon/\alpha_j)^{2n}$, where c_2 is a new constant independent of n.

A similar theorem may be proved for the approximation process considered in Theorem 5.

Theorem 7. *In addition to the assumptions of Theorem* 5 *suppose that* β_j *are such positive numbers that* $(\alpha_j - \beta_j, \alpha_j] \cap \sigma(A) = \{\alpha_j\}$, *and put* $\gamma = \max_{1 \leqq j \leqq k} (1 - \beta_j/\alpha_j)$. *Then there is a positive number* c *such that for* $j = 1, 2, \ldots, k$ *and* $n = 1, 2, \ldots$ *we have*

$$\hat{\delta}(W_{j,n}, E_j Z_j) \leqq c\gamma^n \quad \text{and} \quad |\alpha_j - \mu_{i,n}| \leqq c\gamma^{2n} \quad \text{for all} \quad i$$

such that $\mu_{i,n} \underset{n}{\rightarrow} \alpha_j$.

Proof. Applying Theorem 6 we have

$$\hat{\delta}(W_{j,n}, E_j Z_j) \leqq \hat{\delta}(W_{j,n}, V_{j,n}) + \hat{\delta}(V_{j,n}, E_j Z_j) \leqq \hat{\delta}(W_{j,n}, V_{j,n}) + c\gamma^n . \tag{22}$$

We keep the notation from the proof of Theorem 5 and put $Z_l(j) = (Y_{l+1} \cap M_j)^\perp \cap Y_l \cap M_j$ (cf. the definition of $\tilde{Z}_j$ in the proof of Theorem 3). There are two possibilities:

i) $Z_l(j) = \{0\}$ – then for sufficiently large n, $V_{l,n}(j) = \{0\}$,

ii) $Z_l(j) \neq \{0\}$ – then we may apply Theorem 6 with M_j instead of X_0. Thus in both cases there exists a positive number c such that

$$\hat{\delta}(V_{l,n}(j), E_l Z_l(j)) \leqq c(1 - \beta_l/\alpha_l)^n \leqq c\gamma^n \underset{n}{\rightarrow} 0 . \tag{23}$$

In the inequality $\hat{\delta}(E_l Z_l(j), E_l Z_l(j-1)) \leqq \hat{\delta}(E_l Z_l(j), V_{l,n}(j)) + \hat{\delta}(V_{l,n}(j), V_{l,n}(j-1)) + \hat{\delta}(V_{l,n}(j-1), E_l Z_l(j-1))$ the righthand side converges to zero for $l \neq l_j$ in virtue of (23) and (20). This implies that $E_l Z_l(j) = E_l Z_l(j-1)$ for $l \neq l_j$, and by (23), $\hat{\delta}(V_{l,n}(j), V_{l,n}(j-1)) \leqq \hat{\delta}(V_{l,n}(j), E_l Z_l(j)) + \hat{\delta}(V_{l,n}(j-1),$

$E_l Z_l(j-1)) \leqq 2c\gamma^n$. This inequality together with (22) and (21) gives

$$\hat{\delta}(W_{j,n}, E_j Z_j) \leqq c\gamma^n + \sum_{\substack{1 \leqq i \leqq m \\ 1 \leqq l \leqq k \\ l_i \neq l}} \hat{\delta}(V_{l,n}(i), V_{l,n}(i-1)) \leqq c_1 \gamma^n ,$$

with a constant c_1 independent of n. The desired estimate of $|\alpha_j - \mu_{i,n}|$ may by obtained in nearly the same way as that of $|\lambda_{i,n} - \alpha_j|$ in the proof of Theorem 6.

References

[1] *T. Kato:* Perturbation Theory for Linear Operators. Springer-Verlag, Berlin—Heidelberg—New York, 1966.

[2] *J. Kolomý:* Determination of eigenvalues and eigenvectors of self-adjoint operators. Mathematica — Revue d'analyse numerique et de theorie de l'approximation. 22 (45), No 1, 1980, pp. 53—58.

[3] *J. Kolomý:* On determination of eigenvalues and eigenvectors of self-adjoint operators. Apl. Mat. 26 (1981), pp. 161—170.

[4] *B. N. Parlett:* The Symmetric Eigenvalue Problem. Prentice-Hall, Inc., Englewood Cliffs, 1980.

[5] *J. H. Wilkinson:* The Algebraic Eigenvalue Problem. Clarendon Press, Oxford, 1965.

Souhrn

SPEKTRÁLNÍ APROXIMACE POSITIVNÍCH OPERÁTORŮ METODOU ITERACE PODPROSTORŮ

ANDRZEJ POKRZYWA

Vyšetřuje se metoda iterace podprostorů pro aproximaci bodů spektra positivního lineárního omezeného operátoru. Je popsáno chování vlastních hodnot a vlastních vektorů A_n, vznikajících při užití této metody, a jejich závislost na počátečním podprostoru. Vyšetřuje se rovněž užití Schmidtova ortogonalizačního procesu k přibližnému výpočtu vlastních prvků operátorů A_n.

Author's address: Dr. *Andrzej Pokrzywa,* Institute of Mathematics Polish Academy of Sciences, Śniadeckich 8, P.O.B.137, 00-950 Warsaw, Poland.