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INTERNAL FINITE ELEMENT APPROXIMATIONS IN THE DUAL VARIATIONAL METHOD FOR SECOND ORDER ELLIPTIC PROBLEMS WITH CURVED BOUNDARIES

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1. INTRODUCTION

Internal finite element approximations of the dual variational formulation for second order elliptic boundary value problems in R^2 have been restricted, to the authors' knowledge, to domains with polygonal boundaries. It is the aim of the the present paper to extend the results to domains with piecewise smooth (C^2) curved boundaries.

The space of divergence-free vector functions with vanishing normal flux on some part of the boundary is approximated by (internal) subspaces of finite elements, having the same properties. We also satisfy the requirement to save the order of approximation which belongs to polygonal domains. Thus we construct the so-called conforming dual finite element approximations. If also a conforming primal approximation is available, one obtains a posteriori error estimates and two-sided bounds of energy [4, 7, 12, 14].

Using the concept of stream functions [5], some a priori L^2-error estimates are deduced, provided the exact solution is regular enough. We also prove the convergence of the proposed method to a non-regular solution.

Let us introduce some notations. Let $\Omega \subset R^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$ (see [12]). The outward unit normal to $\partial\Omega$ will be denoted by $\boldsymbol{\nu}$. The usual norm and semi-norm in the Cartesian product of the Sobolev spaces $(W^{k,p}(\Omega))^r$, $r = 1, 2, \ldots$, are denoted by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$, respectively. We shall omit the subscript p in the case $p = 2$ and we put $H^k(\Omega) = W^{k,2}(\Omega)$. Further, let $L^\infty(\Omega) = W^{0,\infty}(\Omega)$ and let the notation $(\cdot, \cdot)_{0,\Omega}$ be used for the usual scalar product in the space $(L^2(\Omega))^r = (H^0(\Omega))^r$. By $P_j(\Omega)$ we denote the space of polynomials of order at most j defined on Ω. Let $C^k(\bar{\Omega})$ denote the space of continuous functions, the derivatives of which up to the order k are continuous in $\bar{\Omega}$.

Next, we define the operator curl : $H^1(\Omega) \to (L^2(\Omega))^2$ by the relation

$$\operatorname{curl} v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1} \right)^{\mathrm{T}}, \quad v \in H^1(\Omega).$$

Let us emphasize that all statements will alwys hold only for a sufficiently small discretization parameter h. Moreover, notations $C, C_1, C_2, \dots$ are reserved for the so-called generic constants.

Many boundary value problems of mathematical physics can be formulated in the following classical way: Find u such that

$$(1.1) \qquad \begin{aligned} -\operatorname{div}(A \operatorname{grad} u) &= f \quad \text{in} \quad \Omega, \\ u &= \bar{u} \quad \text{on} \quad \Gamma_u, \\ (A \operatorname{grad} u)^{\mathrm{T}} \boldsymbol{\nu} &= g \quad \text{on} \quad \Gamma_g, \end{aligned}$$

where Γ_u, Γ_g are disjoint and open in $\partial\Omega$ (one of them can be empty),

$$(1.2) \qquad M_1 \cup \Gamma_u \cup \Gamma_g = \partial\Omega,$$

and M_1 is a finite set of those points, where one type of the boundary condition changes into another. Further, $f \in L^2(\Omega)$, $\bar{u} \in H^1(\Omega)$, $g \in L^2(\Gamma_g)$ and $A \in (L^\infty(\Omega))^4$ is supposed to be a symmetric and uniformly positive definite 2×2 matrix. In the case $\Gamma_u = \emptyset$, we moreover assume that

$$\int_\Omega f \,\mathrm{d}x + \int_{\partial\Omega} g \,\mathrm{d}s = 0.$$

Let us recall that the *dual variational formulation* of the problem (1.1) consists in finding $\hat{\boldsymbol{p}}$ which minimizes the functional

$$(1.3) \qquad J(\boldsymbol{q}) = \tfrac{1}{2} b(\boldsymbol{q}, \boldsymbol{q}) - l(\boldsymbol{q})$$

over the space

$$(1.4) \qquad Q = \{\boldsymbol{q} \in (L^2(\Omega))^2 \mid (\boldsymbol{q}, \operatorname{grad} v)_{0,\Omega} = 0 \;\; \forall v \in V\},$$

where

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_u\},$$

$$(1.5) \qquad b(\boldsymbol{q}, \boldsymbol{q}') = (A^{-1}\boldsymbol{q}, \boldsymbol{q}')_{0,\Omega}$$

is a symmetric and Q-elliptic bilinear form and

$$(1.6) \qquad l(\boldsymbol{q}) = b(\bar{\boldsymbol{p}}, \boldsymbol{q}) - (\boldsymbol{q}, \operatorname{grad} \bar{u})_{0,\Omega},$$

where $\bar{\boldsymbol{p}} \in (L^2(\Omega))^2$ satisfies the equation

$$(\bar{\boldsymbol{p}}, \operatorname{grad} v)_{0,\Omega} = (f, v)_{0,\Omega} + \int_{\Gamma_g} g v \,\mathrm{d}s \quad \forall v \in V.$$

For other details see [7, 9, 16]. Let us note that if $\boldsymbol{q} \in Q \cap (H^1(\Omega))^2$, then $\operatorname{div} \boldsymbol{q} = 0$ in Ω and $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{\nu} = 0$ on Γ_g. The space Q can be characterized also as follows. If the sets Γ_u

and Γ_g are connected, then

$$Q = \operatorname{curl} W, \tag{1.7}$$

where

$$W = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_g\} . \tag{1.8}$$

The general case is described in the following theorem (see [9], p. 59).

Theorem 1.1. *Let $\partial\Omega_0, \ldots, \partial\Omega_H$ be the components of $\partial\Omega$ and let m be the number of those components for which $\partial\Omega_i \cap \Gamma_u \neq \emptyset$. Let n be the number of the components of Γ_g. Then, if $m \geqq 2$ or $n \geqq 2$, there exist functions $\alpha^1, \ldots, \alpha^{m-1}, \beta^1, \ldots, \beta^{n-1} \in \in (L^2(\Omega))^2 - \operatorname{curl} W$ such that*

$$Q = \mathscr{L}(\operatorname{curl} W \cup \{\alpha^1, \ldots, \alpha^{m-1}, \beta^1, \ldots, \beta^{n-1}\}),$$

where Q and W are defined by (1.4) and (1.8), respectively, and $\mathscr{L}$ denotes the linear hull. ∎

The details on the functions α^i, β^j will be presented in Remark 2.1.

In Chapters 2 and 4 some subspaces Q_h of the space Q will be constructed. A function $\boldsymbol{p}_h$, minimizing the functional (1.3) over Q_h, will be called internal approximation of the solution of the dual problem. An algorithm for finding $\boldsymbol{p}_h$ will be presented in Chapter 5.

2. PIECEWISE CONSTANT EQUILIBRIUM FINITE ELEMENT SPACES

In this chapter we introduce subspaces $Q_h \subset Q$ consisting of constant elements and we derive their approximation properties. We shall deal only with the problems from the class $\mathscr{C}^{(2)}$ according to the following definition.

Definition. *A couple (Ω, Γ_g) is said to be from the class $\mathscr{C}^{(2)}$, if*

(i) *$\Omega \subset R^2$ is a bounded domain with a Lipschitz boundary, which consists of a finite number of arcs from the class $C^{(2)}$. The set of the end points of these arcs will be denoted by M_2.*

(ii) *the part Γ_g of the boundary $\partial\Omega$ consists of a finite number of convex and concave arcs. The set of the end points of these arcs will be denoted by M_3.*

An arc $\Gamma \subset \partial\Omega$ is said to be convex (concave), if there exists a convex domain $\Omega_0 \subset \Omega$ ($\Omega_0 \subset R^2 - \bar{\Omega}$) such that $\Gamma \subset \partial\Omega_0$.

Note that for $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$ the set

$$M = M_1 \cup M_2 \cup M_3 \tag{2.1}$$

is finite.

Let us describe the way of triangulation of a domain from the class $\mathscr{C}^{(2)}$. First we establish an approximation Γ_{gh} of the part Γ_g, such that $\Gamma_{gh} \subset \bar{\Omega}$. Denote by

$\Gamma_g^0, \ldots, \Gamma_g^{n-1}$ all components of Γ_g. Every component (curve) Γ_g^i will be now approximated by a polygonal curve $\Gamma_{gh}^i \subset \bar{\Omega}$ consisting of a finite number of line segments the length of which does not exceed h. Each of those segments is a chord

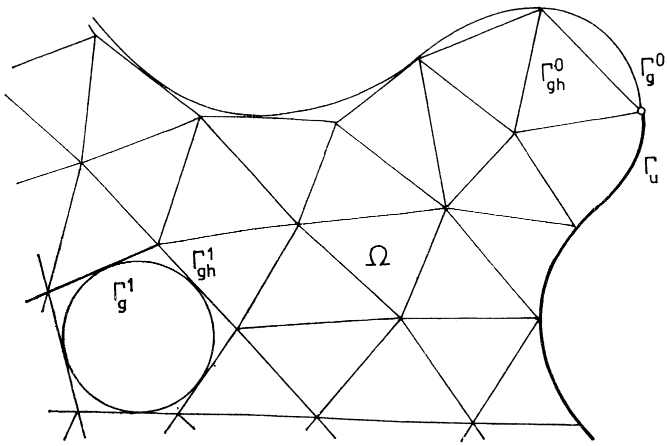

Fig. 1.

or a tangent of a convex or of a concave arc, respectively, which is contained in Γ_g^i (see Figs. 1 and 3). If Γ_g^i is a closed curve, we require Γ_{gh}^i to be also a closed curve. Moreover, we require that $M_1 \cup M_3 \subset \bar{\Gamma}_{gh} \cap \bar{\Gamma}_g$, where

$$\Gamma_{gh} = \bigcup_{i=0}^{n-1} \Gamma_{gh}^i .$$

The subdomain of Ω, bounded by Γ_u and Γ_{gh}, will be denoted by Ω_h, and we define

$$D_h = \Omega - \bar{\Omega}_h .$$

Now $\mathscr{T}_h$ will denote the triangulation of the domain Ω_h generated in a standard way, assuming that the triangles adjacent to Γ_u may have at most one curved side (i.e. the inner triangles are "straight ones" only).

Furthermore, we shall always assume the validity of the so-called consistence condition of a triangulation, i.e. the interior of any side of any triangle $K \in \mathscr{T}_h$ is disjoint with the set M (see (2.1)). Each segment from $\Gamma_{gh} - \Gamma_g$ coincides with a side of one triangle K.

Moreover, we assume that all triangulations belong to a regular family of triangulations $\mathfrak{M}$. (A family of triangulations $\mathfrak{M}$ is said to be regular if

(i) there exists a constant $\varkappa > 0$ such that for any $\mathscr{T}_h \in \mathfrak{M}$ and any $K \in \mathscr{T}_h$ there exists a circle B_K with a radius ϱ_K such that $B_K \subset K$ and

$$(2.2) \qquad \varkappa h_K \leqq \varrho_K ,$$

where $h_K = \operatorname{diam} K$,

(ii) for any $\varepsilon > 0$ there exists $\mathscr{T}_h \in \mathfrak{M}$ such that

$$h = \max_{K \in \mathscr{T}_h} h_K \leqq \varepsilon .)$$

Finally, we can define the space of equilibrium finite elements as follows:

$$(2.3) \qquad Q_h = \{\boldsymbol{q} \in Q \mid \boldsymbol{q}|_{D_h} = 0,\ \boldsymbol{q}|_K \in (P_0(K))^2\ \forall K \in \mathscr{T}_h\} .$$

We shall now examine the approximation properties of the space $Q_h \subset Q$. If we introduce the space

$$W_h = \{v \in W \mid v|_{D_h} = 0,\ v|_K \in P_1(K)\ \forall K \in \mathscr{T}_h\} ,$$

then a linear approximation operator $r_h : W \cap H^2(\Omega) \to W_h$ will be determined by the relations

$$(r_h v)(x) = v(x)$$

for all nodal points x of the triangulation $\mathscr{T}_h$ such that $x \notin \Gamma_{gh}$. Note that $r_h v = 0$ on Γ_{gh} and the function $r_h v \in W_h$ is therefore uniquely determined. We have the well-known lemma for $\Gamma_g = \emptyset$ (see e.g. [6], p. 41):

Lemma 2.1. *Let* $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$ *and let* $\Gamma_u = \partial\Omega$. *Then*

$$\|v - r_h v\|_{1,\Omega} \leqq Ch\|v\|_{2,\Omega} \quad \forall v \in H^2(\Omega) .$$

Proof. There exists (see [11], p. 80) a linear continuous extension operator $E : H^2(\Omega) \to H^2(R^2)$ such that $Ev|_\Omega = v$ and

$$(2.4) \qquad \|Ev\|_{2,R^2} \leqq C_1\|v\|_{2,\Omega} .$$

Let $\tilde{\mathscr{T}}_h$ be a subset of all curved triangles from $\mathscr{T}_h$. Consider $K \in \tilde{\mathscr{T}}_h$ with vertices A_1, A_2, A_3, where $\widehat{A_2A_3}$ lies on $\partial\Omega$ (see Fig. 2), and let $F(K)$ denote the straight

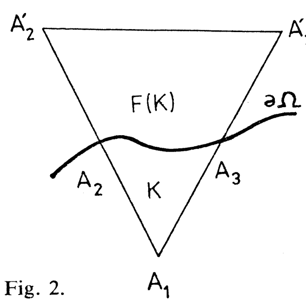

Fig. 2.

triangle with vertices A_1, A'_2, A'_3 such that A_i, $i = 2, 3$, are midpoints of the segments $A_1A'_i$. Since $\partial\Omega$ is piecewise from $C^{(2)}$, we get

$$K \subset F(K) \quad \forall K \in \tilde{\mathcal{T}}_h \tag{2.5}$$

for h small enough. Let us put $F(K) = K$ for $K \in \mathcal{T}_h - \tilde{\mathcal{T}}_h$ and for any triangle let us define the function $\tilde{r}_K Ev \in P_1(F(K))$ by the relation

$$\tilde{r}_K Ev|_K = r_h v|_K .$$

Then (see [1])

$$\|Ev - \tilde{r}_K Ev\|_{1,F(K)} \leqq C_2 h |Ev|_{2,F(K)} \quad \forall K \in \mathcal{T}_h .$$

Hence, by (2.4) and (2.5), we have

$$\begin{aligned} \|v - r_h v\|_{1,\Omega}^2 &= \sum_{K \in \mathcal{T}_h} \|v - r_h v\|_{1,K}^2 \leqq \sum_{K \in \mathcal{T}_h} \|Ev - \tilde{r}_K Ev\|_{1,F(K)}^2 \leqq \\ &\leqq C_2^2 h^2 \sum_{K \in \mathcal{T}_h} |Ev|_{2,F(K)}^2 \leqq C_3 h^2 \|Ev\|_{2,R^2}^2 \leqq C_4 h^2 \|v\|_{2,\Omega}^2 . \quad \blacksquare \end{aligned} \tag{2.6}$$

Theorem 2.1. *Let $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$, $\Gamma_u = \partial\Omega$ and let the domain Ω be simply connected. Then there exists a linear operator $R_h : Q \cap (H^1(\Omega))^2 \to Q_h$ such that*

$$\|\boldsymbol{q} - R_h \boldsymbol{q}\|_{0,\Omega} \leqq Ch \|\boldsymbol{q}\|_{1,\Omega} .$$

Proof. Let $\boldsymbol{q} \in Q \cap (H^1(\Omega))^2$ be arbitrary. Since Ω is simply connected, there exists $v \in H^1(\Omega)$ (see [5], p. 25, the so-called stream function) such that $(v, 1)_{0,\Omega} = 0$ and

$$\boldsymbol{q} = \operatorname{curl} v . \tag{2.7}$$

In our case, however, we have even $v \in H^2(\Omega)$. Setting

$$R_h \boldsymbol{q} = \operatorname{curl} r_h v , \tag{2.8}$$

we see that $R_h \boldsymbol{q} \in Q_h$ and by Lemma 2.1, (2.7) and (2.8) we obtain

$$\|\boldsymbol{q} - R_h \boldsymbol{q}\|_{0,\Omega} = \|\operatorname{curl}(v - r_h v)\|_{0,\Omega} = |v - r_h v|_{1,\Omega} \leqq C_1 h \|v\|_{2,\Omega} . \tag{2.9}$$

Using the Poincaré inequality

$$\|v\|_{1,\Omega} \leqq C_2 |v|_{1,\Omega} , \tag{2.10}$$

we get

$$\|v\|_{2,\Omega}^2 = \|v\|_{1,\Omega}^2 + |v|_{2,\Omega}^2 \leqq (C_2^2 + 1)(|v|_{1,\Omega}^2 + |v|_{2,\Omega}^2) = (C_2^2 + 1) \|\boldsymbol{q}\|_{1,\Omega}^2 .$$

Combining this relation and (2.9), we arrive at the assertion of the lemma. $\blacksquare$

The case $\Gamma_u = \partial\Omega$ for a multiply connected domain will be considered later in Theorem 2.3.

Now we shall deal with the case $\Gamma_g \neq \emptyset$. Then the Friedrichs inequality holds:

$$\|v\|_{1,\Omega} \leqq C|v|_{1,\Omega} \quad \forall v \in W . \tag{2.11}$$

Let $\varepsilon > 0$ be fixed. By $G \subset \Omega$ we shall denote the ε-strip of that part of Γ_g which is curved (see Fig. 3), i.e.

$$G = \{y \in \Omega \mid \exists x \in \Gamma_g - \Gamma'_g : \operatorname{dist}(x, y) < \varepsilon\} ,$$

where

$$\Gamma'_g = \{x \in \Gamma_g \mid \exists \text{ a straight segment } S \subset \Gamma_g : x \in S\} .$$

(For instance, $G = \emptyset$ if Γ_g is polygonal.) First of all we prove an auxiliary lemma.

Lemma 2.2. *Let $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$ and let $\Gamma_g \neq \emptyset$. Then*

$$\|v - r_h v\|_{1,\Omega} \leqq Ch(\|v\|_{2,\Omega} + |v|_{1,\infty,G}) \quad \forall v \in \tilde{W},$$

where

$$\tilde{W} = \{v \in W \cap H^2(\Omega) \mid v|_G \in W^{1,\infty}(G)\} .$$

Proof. Let E_h denote the union of all triangles $K \in \mathscr{T}_h$ at least one vertex of which lies on $\Gamma_{gh} - \Gamma_g$ (i.e. at the points of intersection of the tangents – see Fig. 3). Consequently, $E_h \subset G$ holds for sufficiently small h. Further, for $v \in \tilde{W}$ we have

$$(2.12) \qquad |v - r_h v|^2_{1,\Omega} = |v - r_h v|^2_{1,D_h} + |v - r_h v|^2_{1,\Omega_h - E_h} + |v - r_h v|^2_{1,E_h} .$$

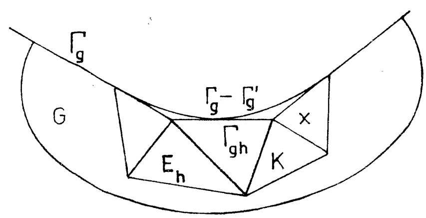

Fig. 3.

The first term on the right-hand side of (2.12) can be estimated as follows:

$$(2.13) \qquad |v - r_h v|^2_{1,D_h} = |v|^2_{1,D_h} \leqq 2|v|^2_{1,\infty,D_h} \operatorname{mes} D_h \leqq Ch^2 |v|^2_{1,\infty,G} ,$$

since (see [15], Chap. 1.6)

$$\operatorname{mes} D_h \leqq C_1 h^2 .$$

The second term can be estimated in the same manner as in (2.6), i.e.

$$(2.14) \qquad |v - r_h v|^2_{1,\Omega_h - E_h} \leqq \sum_{\substack{K \in \mathscr{T}_h \\ K \not\subset E_h}} \|v - r_h v\|^2_{1,K} \leqq Ch^2 \|v\|^2_{2,\Omega} .$$

Thus, it remains to deal with the third term. Let $K \subset E_h$ be an arbitrary triangle and let $v_K \in P_1(K)$ be the linear interpolation of the function $v|_K$. Then we have

$$(2.15) \qquad |v - v_K|_{1,K} \leqq Ch_K \|v\|_{2,K} .$$

Next we have to estimate the difference

$$w_K = v_K - (r_h v)|_K \, .$$

Let x be an arbitrary vertex of the triangle K. If $x \in \Gamma_{gh}$ (see Fig. 3), then

$$\text{dist}\,(x, \Gamma_g) \leqq C h_K^2 \, , \tag{2.16}$$

and therefore

$$|w_K(x)| \leqq C h_K^2 |v|_{1,\infty,G} \, , \tag{2.17}$$

since $w_K(x) = v_K(x) = v(x)$. On the other hand, when the vertex $x \notin \Gamma_{gh}$, we get even $w_K(x) = 0$. As $w_K \in P_1(K)$, we see that (2.17) holds for all $x \in K$ and using (2.2), we obtain the estimate

$$|w_K|_{1,K}^2 \leqq 2 \text{ mes } K \left(\frac{C\, h_K^2 |v|_{1,\infty,G}}{\varkappa h_K} \right)^2 .$$

From this and (2.15) it follows that

$$|v - r_h v|_{1,K}^2 \leqq |v - v_K|_{1,K}^2 + |w_K|_{1,K}^2 \leqq C h_K^2 (\|v\|_{2,K}^2 + \text{mes } K |v|_{1,\infty,G}^2) \, .$$

Hence

$$|v - r_h v|_{1,E_h}^2 = \sum_{\substack{K \in \mathscr{T}_h \\ K \subset E_h}} |v - r_h v|_{1,K}^2 \leqq C h^2 (\|v\|_{2,\Omega}^2 + \text{mes } E_h |v|_{1,\infty,G}^2) \, .$$

From here, from (2.12), (2.13), (2.14) and the Friedrichs inequality (2.11) we obtain the assertion of the lemma. ∎

Theorem 2.2. *Let $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$, let $\Gamma_g \neq \emptyset$ and let Γ_u, Γ_g be connected sets. Then there exists a linear operator $R_h : \tilde{Q} \to Q_h$ such that*

$$\|\boldsymbol{q} - R_h \boldsymbol{q}\|_{0,\Omega} \leqq C h (\|\boldsymbol{q}\|_{1,\Omega} + \|\boldsymbol{q}\|_{0,\infty,G}) \, ,$$

where

$$\tilde{Q} = \{\boldsymbol{q} \in Q \cap (H^1(\Omega))^2 |\ \boldsymbol{q}|_G \in (L^\infty(G))^2\} \, . \tag{2.18}$$

Proof. Let $\boldsymbol{q} \in \tilde{Q}$ be arbitrary. By (1.7) there exists $v \in W$ such that $\boldsymbol{q} = \text{curl } v$. In our case, we have even $v \in \tilde{W}$. Setting again $R_h \boldsymbol{q} = \text{curl } r_h v$, we see that $R_h \boldsymbol{q} \in Q_h$ and from Lemma 2.2 and (2.11) we obtain

$$\|\boldsymbol{q} - R_h \boldsymbol{q}\|_{0,\Omega}^2 = |v - r_h v|_{1,\Omega}^2 \leqq C_1 h^2 (\|v\|_{2,\Omega} + |v|_{1,\infty,G})^2 \leqq$$
$$\leqq C_2 h^2 (|v|_{1,\Omega}^2 + |v|_{2,\Omega}^2 + |v|_{1,\infty,G}^2) = C_2 h^2 (\|\boldsymbol{q}\|_{1,\Omega}^2 + \|\boldsymbol{q}\|_{0,\infty,G}^2) \, . \ ∎$$

To generalize Theorems 2.1 and 2.2 even further, we extend the domain of the mapping r_h. The extended map will be denoted by r_h as well. Let us define the space $W'(\supset W)$ by

$$W' = \{v \in H^1(\Omega) |\ \text{if } n \geqq 2 \ \ \exists c_1, \ldots, c_{n-1} \in R^1 : v|_{\Gamma_g^i} = c_i \, , \quad i = 1, \ldots, n-1; \ v|_{\Gamma_g^0} = 0\} \, . \tag{2.19}$$

Here we note that the distances of the components Γ_g^i are positive due to the finiteness of the set M_1. Hence, for sufficiently small h, we can define the finite element subspace W_h' of the space W' in the following way:

$$W_h' = \{v \in W' |\ v|_{D^i_h} \in P_0(D_h^i)\ \forall i \in \{0, \ldots, n-1\},\ v|_K \in P_1(K)\ \forall K \in \mathscr{T}_h\},$$

where D_h^i is the union of all components D of the set D_h for which $\bar{D} \cap \Gamma_g^i \neq \emptyset$.

The operator $r_h : W' \cap H^2(\Omega) \to W_h'$ will be defined as follows. For $v \in W' \cap \cap H^2(\Omega)$ and $i \in \{0, \ldots, n-1\}$ we put

$$r_h v|_{D_h{}^i} = v|_{\Gamma_g{}^i} \tag{2.20}$$

and we demand as before that $(r_h v)(x) = v(x)$ for all nodal points x of the traingulation $\mathscr{T}_h$ such that $x \notin \Gamma_{gh}$. The function $r_h v \in W_h'$ is uniquely determined by these relations, and the following lemma is valid.

Lemma 2.3. *Let* $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$. *Then*

$$\|v - r_h v\|_{1,\Omega} \leqq Ch(\|v\|_{2,\Omega} + |v|_{1,\infty,G}) \quad \forall v \in \tilde{W}',$$

where

$$\tilde{W}' = \{v \in W' \cap H^2(\Omega)|\ v|_G \in W^{1,\infty}(G)\}.$$

Proof. The case $\Gamma_g = \emptyset$ was proved in Lemma 2.1. Consequently, let $\Gamma_g \neq \emptyset$. This case can be proved by an argument parallel to that of Lemma 2.2, since we immediately see that (2.12), (2.13), (2.14) and (2.15) hold for $v \in \tilde{W}'$, too. We show now that also (2.17) can be proved for $v \in \tilde{W}'$.

Since $r_h v = v$ on Γ_g (see (2.20)) and since $r_h v$ is constant on any D_h^i, $i = 0, \ldots \ldots, n-1$, we get from (2.16) that

$$|v(x) - (r_h v)(x)| \leqq Ch_K^2 |v|_{1,\infty,D_h{}^i},$$

where $x \in \Gamma_{gh}$ is a vertex of a triangle $K \subset E_h$ (see Fig. 3). However, $v(x) = v_K(x)$ and thus

$$|w_K(x)| = |v_K(x) - (r_h v)(x)| \leqq Ch_K^2 |v|_{1,\infty,G}.$$

But this estimate is true also for a vertex $x \notin \Gamma_{gh}$, because $w_K(x) = 0$. Since $w_K \in P_1(K)$, (2.17) holds for all $x \in K$. The rest of the proof is the same as in Lemma 2.2. According to (2.20), we get $v - r_h v \in W$ and therefore we can apply the Friedrichs inequality (2.11). ■

We shall use the preceding lemma in proving Theorem 2.3, which generalizes Theorems 2.1 and 2.2. Before that we introduce two important remarks.

Remark 2.1. The functions $\boldsymbol{\alpha}^j$, $\boldsymbol{\beta}^j$ in Theorem 1.1 can be chosen for example in the following way (see [9], p. 59). If $n \geqq 2$, then we define

$$\boldsymbol{\beta}^j = \operatorname{curl} \bar{w}^j, \quad j = 1, \ldots, n-1, \tag{2.21}$$

where $\bar{w}^j \in H^1(\Omega)$ are arbitrary fixed functions satisfying

$$\bar{w}^j = \delta_{ij} \text{ on } \Gamma_g^i, \quad i = 0, \ldots, n-1, \quad j = 1, \ldots, n-1,$$

(δ_{ij} is Kronecker's symbol). Taking $\bar{w}^j \in C^\infty(\bar{\Omega})$, we get $\boldsymbol{\beta}^j \in (C^\infty(\bar{\Omega}))^2$.

Further, let $m \geqq 2$ and let $\partial\Omega_i \cap \Gamma_u \neq \emptyset$ for $i = 0, \ldots, m-1$ (otherwise we change the notation of the components of $\partial\Omega$). Then we can define the functions $\boldsymbol{\alpha}^j, j = 1, \ldots$ $\ldots, m-1$, by

$$\boldsymbol{\alpha}^j = \operatorname{curl} w^j \quad \text{on} \quad S_j, \tag{2.22}$$
$$\boldsymbol{\alpha}^j = 0 \quad \text{on} \quad \bar{\Omega} - S_j,$$

where $S_j \subset \Omega$ is an arbitrary simply connected domain with a Lipschitz boundary such that

$$\partial S_j \cap \partial\Omega_k = \emptyset \quad \forall k \in \{0, \ldots, H\} - \{j-1, j\} \tag{2.23}$$

and the sets

$$\partial S_j^1 = \partial S_j \cap \partial\Omega_{j-1}, \quad \partial S_j^3 = \partial S_j \cap \partial\Omega_j \tag{2.24}$$

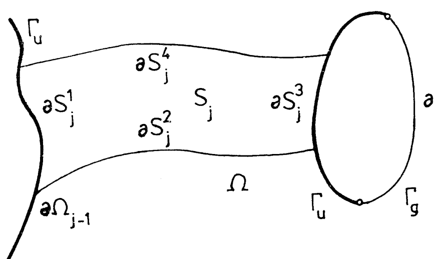

Fig. 4.

are nonempty, connected and included in Γ_u (see Fig. 4). Next, ∂S_j^2 and ∂S_j^4 are components of the set $\partial S_j - (\partial S_j^1 \cup \partial S_j^3)$, and $w^j \in H^1(S_j)$, $j = 1, \ldots, m-1$, are arbitrary fixed functions satisfying

$$w^j = 1 \quad \text{in a neighbourhood of the component} \quad \partial S_j^2, \tag{2.25}$$
$$w^j = 0 \quad \text{in a neighbourhood of the component} \quad \partial S_j^4.$$

One can see, using (2.22), that $\boldsymbol{\alpha}^j \in (C^\infty(\bar{\Omega}))^2$ follows for $w^j \in C^\infty(\bar{S}_j)$. Moreover, (2.23)–(2.25) imply that

$$\operatorname{dist}(\operatorname{supp} \boldsymbol{\alpha}^j, \Gamma_g) > 0, \quad j = 1, \ldots, m-1, \tag{2.26}$$

and ε (in the definition of the set G) can be chosen so small that

$$\boldsymbol{\alpha}^j|_G = 0, \quad j = 1, \ldots, m-1. \tag{2.27}$$

One can easily verify that the functions $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{m-1}$, $\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^{n-1}$ are linearly independent. Henceforth, we shall assume these functions to be fixed in the space $(C^\infty(\bar{\Omega}))^2$. ■

Remark 2.2. Let $\boldsymbol{q} \in Q \cap (H^1(\Omega))^2$. According to [5], p. 22, for this $\boldsymbol{q}$ a stream function exists if and only if

$$\int_{\partial\Omega_i} \boldsymbol{q}^{\mathrm{T}} \boldsymbol{v} \, \mathrm{d}s = 0 \quad \text{for all} \quad i = 0, \ldots, H .$$

Consequently, for $m \geqq 2$ the stream function does not exist, in general. There exists, however, precisely one linear combination $\boldsymbol{\alpha} = \sum_{j=1}^{m-1} c^j \boldsymbol{\alpha}^j$ such that a stream function $v \in W'$ exists for the difference $\boldsymbol{q} - \boldsymbol{\alpha}$, i.e.

$$\boldsymbol{q} - \boldsymbol{\alpha} = \operatorname{curl} v \tag{2.28}$$

holds. The coefficients $c^1, \ldots, c^{m-1}$ are the solution of the linear system of equations

$$\sum_{j=1}^{m-1} c^j \int_{\partial\Omega_i} (\boldsymbol{\alpha}^j)^{\mathrm{T}} \boldsymbol{v} \, \mathrm{d}s = \int_{\partial\Omega_i} \boldsymbol{q}^{\mathrm{T}} \boldsymbol{v} \, \mathrm{d}s , \quad i = 0, \ldots, m-1 , \tag{2.29}$$

which is uniquely solvable (see [9], p. 61). Therefore, we can consider the mapping $q \in Q \cap (H^1(\Omega))^2 \mapsto \boldsymbol{\alpha} \in \mathscr{L}(\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{m-1})$. It is readily seen that the mapping is linear. We shall prove that it is continuous as well.

Let $v^i \in C^\infty(\bar{\Omega})$, $i = 0, \ldots, m-1$, be fixed chosen functions such that

$$v^i|_{\partial\Omega_j} = \delta_{ij} , \quad i = 0, \ldots, m-1 , \quad j = 0, \ldots, H .$$

Using Green's Theorem, we obtain for $i = 0, \ldots, m-1$

$$\left| \int_{\partial\Omega_i} \boldsymbol{q}^{\mathrm{T}} \boldsymbol{v} \, \mathrm{d}s \right| = \left| \int_{\partial\Omega} v^i \boldsymbol{q}^{\mathrm{T}} \boldsymbol{v} \, \mathrm{d}s \right| = \left| (\operatorname{grad} v^i, \boldsymbol{q})_{0,\Omega} \right| \leqq \|\operatorname{grad} v^i\|_{0,\Omega} \|\boldsymbol{q}\|_{0,\Omega} \leqq C \|\boldsymbol{q}\|_{1,\Omega} ,$$

where C can be taken independent of i, since v^i are fixed. Consequently, making use also of (2.29) and of the equivalence of all norms in a finite-dimensional space, we are led to the continuity of the map $\boldsymbol{q} \mapsto \boldsymbol{\alpha}$, i.e.

$$\|\boldsymbol{\alpha}\|_{1,\Omega} \leqq C_1 \sum_{i=0}^{m-1} \left| \int_{\partial\Omega_i} \boldsymbol{q}^{\mathrm{T}} \boldsymbol{v} \, \mathrm{d}s \right| \leqq C_2 \|\boldsymbol{q}\|_{1,\Omega} . \tag{2.30}$$

Recall (see [5], p. 25) that the stream function is determined except for a constant. Thus the function v in (2.28) is uniquely determined in W' if $\Gamma_g \neq \emptyset$. In the case $\Gamma_g = \emptyset$, we choose (the unique) v such that

$$(v, 1)_{0,\Omega} = 0 . \quad ■ \tag{2.31}$$

Theorem 2.3. *Let $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$. Then there exists a linear operator $R_h : \tilde{Q} \to Q_h$ such that*

$$\|\boldsymbol{q} - R_h \boldsymbol{q}\|_{0,\Omega} \leqq Ch(\|\boldsymbol{q}\|_{1,\Omega} + \|\boldsymbol{q}\|_{0,\infty,G}) ,$$

where $\tilde{Q}$ and Q_h are defined by (2.18) *and* (2.3), *respectively.*

Proof. Let $\boldsymbol{q} \in \tilde{Q}$ be arbitrary. According to (2.27), we write

$$\boldsymbol{q} = \operatorname{curl} v + \sum_{j=1}^{m-1} c^j \boldsymbol{\alpha}^j ,$$

where $v \in W'$, $c', \ldots, c^{m-1} \in R^1$ and $\boldsymbol{\alpha}^j$ are chosen in $(C^\infty(\bar{\Omega}))^2$. Since $\boldsymbol{q} \in \tilde{Q}$, we even have $v \in \tilde{W}'$. First of all let us construct an approximation of the function $\boldsymbol{\alpha}^j$ by means of functions from Q_h. For the time being let the superscripts and subscripts j be omitted.

Consider $w \in C^\infty(\bar{S})$, which satisfies (2.25), and define (for sufficiently small h) a function $\pi_h w \in C^0(\bar{S})$ by

$$\begin{aligned} &\pi_h w|_{K \cap S} \in P_1(K \cap S)\,, \quad K \in \mathscr{T}_h\,, \quad K \cap S \neq \emptyset\,, \\ &\pi_h w = w \quad \text{on} \quad \partial S^2 \cup \partial S^4\,, \\ &(\pi_h w)(x) = w(x) \end{aligned} \tag{2.32}$$

for all nodal points x of the triangulation $\mathscr{T}_h$ such that $x \in S$. Let us put

$$\begin{aligned} \Pi_h \boldsymbol{\alpha} &= \operatorname{curl} \pi_h w \quad &&\text{on} \quad S\,, \\ \Pi_h \boldsymbol{\alpha} &= 0 &&\text{on} \quad \Omega - S\,. \end{aligned} \tag{2.33}$$

We can show that $\Pi_h \boldsymbol{\alpha} \in Q_h$. In fact, from (2.32) it follows that $\pi_h w$ fulfils (2.25) for sufficiently small h. Consequently, using Theorem 1.1 and Remark 2.1, we obtain $\Pi_h \boldsymbol{\alpha} \in Q$, since $\pi_h w \in H^1(S)$. Further, with regard to (2.26) and (2.33) we get $\Pi_h \boldsymbol{\alpha}|_{D_h} = 0$, and $\Pi_h \boldsymbol{\alpha}|_K \in (P_0(K))^2$ follows from (2.32) and (2.33) for all $K \in \mathscr{T}_h$. Consequently, $\Pi_h \boldsymbol{\alpha} \in Q_h$ (cf. (2.3)).

Using (2.22) and (2.33), we derive that

$$\|\boldsymbol{\alpha} - \Pi_h \boldsymbol{\alpha}\|_{0,\Omega} = \|\operatorname{curl}(w - \pi_h w)\|_{0,S} \leqq \|w - \pi_h w\|_{1,S} \leqq Ch\|w\|_{2,S}\,. \tag{2.34}$$

The last inequality is standard and can be proved in a way parallel to that of (2.6) (taking into account the fact that $w = \pi_h w$ in a certain neighbourhood of the components ∂S^2 and ∂S^4).

The Friedrichs inequality together with the fact that $w|_{\partial S_j^4} = 0$ and with (2.34) and (2.22) yields

$$\|\boldsymbol{\alpha}^j - \Pi_h^j \boldsymbol{\alpha}^j\|_{0,\Omega}^2 \leqq Ch^2(|w^j|_{1,S_j}^2 + |w^j|_{2,S_j}^2) = Ch^2\|\boldsymbol{\alpha}^j\|_{1,S_j}^2 = Ch^2\|\boldsymbol{\alpha}^j\|_{1,\Omega}^2\,.$$

Putting

$$\Pi_h \boldsymbol{\alpha} = \sum_{j=1}^{m-1} c^j \Pi_h^j \boldsymbol{\alpha}^j \quad \text{for} \quad \boldsymbol{\alpha} = \sum_{j=1}^{m-1} c^j \boldsymbol{\alpha}^j\,, \quad c^j \in R^1\,,$$

we can easily find that

$$\|\boldsymbol{\alpha} - \Pi_h \boldsymbol{\alpha}\|_{0,\Omega} \leqq Ch\|\boldsymbol{\alpha}\|_{1,\Omega} \tag{2.35}$$

holds for all $\boldsymbol{\alpha}$ from the finite-dimensional space $\mathscr{L}(\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^{m-1})$.

Now, for $\boldsymbol{q} \in \tilde{Q}$ we define

$$R_h \boldsymbol{q} = \operatorname{curl} r_h v + \Pi_h \boldsymbol{\alpha} \tag{2.36}$$

on the basis of the relation (2.28). Then obviously $R_h \boldsymbol{q} \in Q_h$ and we may write, making use of (2.28), (2.36), Lemma 2.3, (2.35), (2.31), the Friedrichs inequality if $\Gamma_g \neq \emptyset$ or the Poincaré inequality (2.10) if $\Gamma_g = \emptyset$, (2.27) and (2.30),

$$\begin{aligned} \|\boldsymbol{q} - R_h \boldsymbol{q}\|_{0,\Omega} &\leqq \|\operatorname{curl}(v - r_h v)\|_{0,\Omega} + \|\boldsymbol{\alpha} - \Pi_h \boldsymbol{\alpha}\|_{0,\Omega} \leqq C_1 h(\|v\|_{2,\Omega} + |v|_{1,\infty,G} + \\ &+ \|\boldsymbol{\alpha}\|_{1,\Omega}) \leqq C_2 h(|v|_{1,\Omega} + |v|_{2,\Omega} + \|\operatorname{curl} v\|_{0,\infty,G} + \|\boldsymbol{\alpha}\|_{1,\Omega}) \leqq \\ &\leqq C_3 h(\|\boldsymbol{q} - \boldsymbol{\alpha}\|_{1,\Omega} + \|\boldsymbol{q}\|_{0,\infty,G} + \|\boldsymbol{\alpha}\|_{1,\Omega}) \leqq C_4 h(\|\boldsymbol{q}\|_{1,\Omega} + \|\boldsymbol{q}\|_{0,\infty,G}) . \end{aligned}$$ ∎

3. ERROR ESTIMATES AND CONVERGENCE

In this chapter we shall estimate the difference $\boldsymbol{p} - \boldsymbol{p}_h$, where $\boldsymbol{p}$ and $\boldsymbol{p}_h$ are the solution of the dual problem and its internal approximation (from the space Q_h – (2.3)), respectively (see Chap. 1).

Theorem 3.1. *Let* $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$ *and let* $\boldsymbol{p} \in \tilde{Q}$ (cf. (2.18)). *Then*

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,\Omega} \leqq Ch(\|\boldsymbol{p}\|_{1,\Omega} + \|\boldsymbol{p}\|_{0,\infty,G}) .$$

Proof. The well-known Céa's Lemma ([1], p. 104) yields

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,\Omega} \leqq C_1 \inf_{q_h \in Q_h} \|\boldsymbol{p} - \boldsymbol{q}_h\|_{0,\Omega} \leqq C_1 \|\boldsymbol{p} - R_h \boldsymbol{p}\|_{0,\Omega} .$$

The assertion follows then from Theorem 2.3. ∎

When no regularity of the solution $\boldsymbol{p} \in Q$ is assumed, we obtain a convergence of $\boldsymbol{p}_h$ to $\boldsymbol{p}$ by virtue of the following density theorem.

Theorem 3.2. *Let* $\Omega \subset R^2$ *be a bounded domain with a Lipschitz boundary and let* Γ_u *and* Γ_g *satisfy* (1.2). *Then the set* $Q \cap (C^\infty(\bar{\Omega}))^2$ *is dense in* Q (*with respect to the* $\|\cdot\|_{0,\Omega}$*-norm*).

Proof. Let $\boldsymbol{q} \in Q$ be arbitrary. Then by Theorem 1.1 we have

$$\boldsymbol{q} = \operatorname{curl} w + \boldsymbol{\alpha} + \boldsymbol{\beta} ,$$

where $w \in W$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (C^\infty(\bar{\Omega}))^2$ – see Remark 2.1. According to [3], p. 618, there exists a sequence $\{w_k\} \subset W \cap C^\infty(\bar{\Omega})$ such that

$$\|w - w_k\|_{1,\Omega} \to 0 \quad \text{if} \quad k \to \infty .$$

Hence, putting $q_k = \operatorname{curl} w_k + \boldsymbol{\alpha} + \boldsymbol{\beta} \in (C^\infty(\bar{\Omega}))^2$, we obtain $\boldsymbol{q}_k \in Q$,

$$\|\boldsymbol{q} - \boldsymbol{q}_k\|_{0,\Omega} = \|\operatorname{curl}(w - w_k)\|_{0,\Omega} \leqq \|w - w_k\|_{1,\Omega} \to 0 .$$ ∎

Remark 3.1. Similar results obtained under a little stronger assumptions can be found in [8, 10]. ■

Theorem 3.3. *If* $(\Omega, \Gamma_g) \in \mathscr{C}^{(2)}$, *then* $\|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,\Omega} \to 0$ *for* $h \to 0$.

Proof. Let $\varepsilon > 0$ be given. By Theorem 3.2 there exists $\boldsymbol{q} \in Q \cap (C^\infty(\bar{\Omega}))^2 \subset \tilde{Q}$ such that $\|\boldsymbol{p} - \boldsymbol{q}\|_{0,\Omega} < \varepsilon/2$ and by Theorem 2.3, $\|\boldsymbol{q} - R_h\boldsymbol{q}\|_{0,\Omega} < \varepsilon/2$ for sufficiently small h. Thus on the basis of Céa's Lemma, we get

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,\Omega} \leqq C \inf_{q_h \in Q_h} \|\boldsymbol{p} - \boldsymbol{q}_h\|_{0,\Omega} \leqq C(\|\boldsymbol{p} - \boldsymbol{q}\|_{0,\Omega} + \|\boldsymbol{q} - R_h\boldsymbol{q}\|_{0,\Omega}) \leqq C\varepsilon\,. \quad ■$$

4. EQUILIBRIUM FINITE ELEMENT SPACES GENERATED BY POLYNOMIALS OF HIGHER ORDERS

Let us consider again problems of the class $\mathscr{C}^{(2)}$. Assume now that each of the smooth arcs belonging to $C^{(2)}$, from which the boundary $\partial\Omega$ is composed, has a parametric representation

$$x = \varphi(s)\,, \quad y = \psi(s)\,, \quad \varphi, \psi \in C^{(2)}\,,$$

and the functions φ, ψ are available.

Let $\mathfrak{M}$ be a regular family of triangulations of the domain Ω, including curved elements — triangles with one curved side along the curved part of $\partial\Omega$.

To define subspace $W'_h \subset W'$ (see (2.19)) generated by polynomials of higher orders, we can use the approach of Zlámal ([18], [17] p. 28). Let us introduce the mapping

$$\begin{aligned} (4.1) \qquad x = x(\xi, \eta) &= x_1 + \bar{x}_2\xi + \bar{x}_3\eta + \\ &+ (1 - \xi - \eta)(1 - \eta)^{-1}(\varphi(s_1 + \bar{s}_3\eta) - x_1 - \bar{x}_3\eta)\,, \\ y = y(\xi, \eta) &= y_1 + \bar{y}_2\xi + \bar{y}_3\eta + \\ &+ (1 - \xi - \eta)(1 - \eta)^{-1}(\psi(s_1 + \bar{s}_3\eta) - y_1 - \bar{y}_3\eta)\,, \end{aligned}$$

where

$$\bar{x}_j = x_j - x_1\,, \quad \bar{y}_j = y_j - y_1\,, \quad j = 1, 2\,,$$
$$\bar{s}_3 = s_3 - s_1\,,$$

which maps the closed triangle $\hat{K}$ with the vertices $R_1 = (0, 0)$, $R_2 = (1, 0)$, $R_3 = (0, 1)$ in the ξ, η-plane onto a closed triangle $K \in \mathscr{T}_h$ with vertices $P_j = (x_j, y_j)$, $j = 1, 2, 3$, in the x, y-plane. Then the side $\overline{R_1R_3}$ is mapped onto the arc $\widehat{P_1P_3}$, the sides $\overline{R_1R_2}$ and $\overline{R_2R_3}$ are linearly mapped onto the sides $\overline{P_1P_2}$ and $\overline{P_2P_3}$, respectively (see Fig. 5). Finally, s_1, s_3 are the values of the arc parameter corresponding to the vertices P_1 and P_3, respectively.

Zlámal has proved the following assertion (see [18]): Let the boundary $\partial\Omega$ belong piecewise to $C^{(2)}$. Then for sufficiently small h and any triangle $K \in \mathscr{T}_h$ the mapping

(4.1) maps one-to-one the closed triangle $\hat{K}$ onto the closed triangle K and the Jacobian $J(\xi, \eta)$ of (4.1) is different from zero on $\hat{K}$.

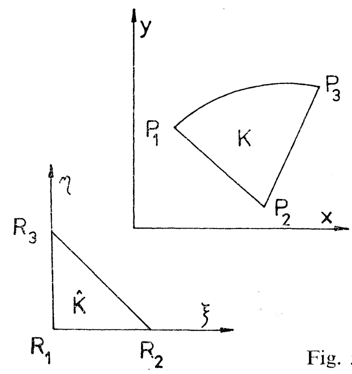

Fig. 5.

Choosing a polynomial $r(\xi, \eta)$ in $\hat{K}$ we define

$$p(x, y) = r(\xi(x, y), \eta(x, y)) \quad \text{on} \quad K,$$

where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ is the inverse mapping to (4.1). The polynomials $r(\xi, \eta)$ are such that their values on each side of $\hat{K}$ are uniquely determined by some (nodal) parmeters associated with some points (nodes) lying on this side. The trial functions are now defined on the whole domain Ω by the values of the nodal parameters at the nodes.

If K runs through the partition $\mathcal{T}_h$ of $\bar{\Omega}$, we get all nodes of $\bar{\Omega}$. Evidently, the trial functions form a finite-dimensional subspace of $H^1(\Omega)$. As the boundary $\partial\Omega$ is mapped piecewise onto $\overline{R_1R_3}$, the conditions $v|_{\Gamma_g{}^i} = c_i$, $i = 0, 1, \ldots, n - 1$, where $c_0 = 0$ (see (2.19)), are easy to satisfy by choosing the boundary nodal parameters in such a way that $r(0, \eta) \equiv c_i$. Thus we obtain a subspace W'_h of W'.

Remark 4.1. If the boundary $\partial\Omega$ is piecewise polynomial, one can use the so-called isoparametric finite elements to construct the subspaces W'_h of W'. For details we refer to the book [1], Chap. 4. ∎

We may therefore consider an arbitrary finite element space W'_h such that

$$W'_h \subset W'. \tag{4.2}$$

First of all let us suppose that the part Γ_u of $\partial\Omega$ is contained in at most one component of the boundary $\partial\Omega$. Then by (2.19), (2.21) and by Theorem 1.1 for $m < 2$ we see that

$$Q = \operatorname{curl} W'. \tag{4.3}$$

We can define the space of equilibrium finite elements as follows:

$$Q_h = \operatorname{curl} W'_h . \tag{4.4}$$

The desired inclusion $Q_h \subset Q$ results from (4.2) and (4.3).

Let $r_h : W' \cap H^{k+1}(\Omega) \to W'_h$, $k \geqq 1$, be an operator with the following approximation property:

$$\|v - r_h v\|_{1,\Omega} \leqq Ch^k \|v\|_{k+1,\Omega} .$$

(See [17, 18], where such estimates have been proved for subspaces of curved finite elements.) Then we can define an operator $R_h : Q \cap (H^k(\Omega))^2 \to Q_h$ by

$$R_h \boldsymbol{q} = \operatorname{curl}(r_h v) ,$$

where $\boldsymbol{q} = \operatorname{curl} v$, and R_h has this approximation property:

$$\|\boldsymbol{q} - R_h \boldsymbol{q}\|_{0,\Omega} = \|\operatorname{curl}(v - r_h v)\|_{0,\Omega} \leqq C_1 h^k \|v\|_{k+1,\Omega} \leqq C_2 h^k \|\boldsymbol{q}\|_{k,\Omega} . \tag{4.5}$$

The last inequality has been obtained by means of the Friedrichs or Poincaré inequality, respectively. Therefore, if the solution $\boldsymbol{p}$ of the dual problem belongs to $Q \cap (H^k(\Omega))^2$, we get by Céa's Lemma that

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,\Omega} \leqq Ch^k \|\boldsymbol{p}\|_{k,\Omega} .$$

If the set $\bigcup_h W'_h$ is dense in the space W' (with the $\|\cdot\|_{1,\Omega}$-norm), then the set $\bigcup_h Q_h$ is dense in Q (with the $\|\cdot\|_{0,\Omega}$-norm) and the convergence $\|\boldsymbol{p} - \boldsymbol{p}_h\|_{0,\Omega} \to 0$ can be derived in a way analogous to that of Theorem 3.3.

When Γ_u is contained in at least two components of $\partial\Omega$ (i.e. $m \geqq 2$), then the space of the equilibrium finite elements can be defined as follows:

$$Q_h = \mathscr{L}(\operatorname{curl} W'_h \cup \{\boldsymbol{\alpha}_h^1, \ldots, \boldsymbol{\alpha}_h^{m-1}\}) , \tag{4.6}$$

where $\boldsymbol{\alpha}_h^j$ are determined by (2.22), where $w_h^j \in \{w|_{S_j} \mid w \in W'_h\}$ are functions satisfying (2.25). The definition of Q_h is independent of the particular choice of $\boldsymbol{\alpha}_h^j$, since any other $\bar{\boldsymbol{\alpha}}_h^j$ can be expressed as $\bar{\boldsymbol{\alpha}}_h^j = \boldsymbol{\alpha}_h^j + \operatorname{curl} w_h$ for convenient $w_h \in W'_h$. Now an approach parallel to that in the proof of Theorem 2.3 can be used to obtain the approximation property (4.5) of Q_h.

5. APPENDIX

The internal approximation $\boldsymbol{p}_h$ of the dual problem can be easily found via the following theorem.

Theorem 5.1. *Let $\Gamma_g \neq \emptyset$, let Γ_u be contained in at most one component of $\partial\Omega$ and let $\{w_i\}_{i=1}^d$ be a basis of the space $W'_h \subset W'$. Then we have*

$$\boldsymbol{p}_h = \sum_{i=1}^d x_i \operatorname{curl} w_i ,$$

where $x_1, \ldots, x_d$ is the solution of the system of linear algebraic equations with a positive definite matrix

$$\sum_{j=1}^{d} b(\operatorname{curl} w_i, \operatorname{curl} w_j)\, x_j = l(\operatorname{curl} w_i)\,, \quad i = 1, \ldots, d\,, \tag{5.1}$$

where $b(\cdot, \cdot)$ and $l(\cdot)$ are defined by (1.5) and (1.6), respectively.

Proof. Since $\Gamma_g^0 \neq \emptyset$, the kernel of the mapping curl : $W_h' \to Q_h$ reduces to the zero element. Consequently, the relation dim $W_h' =$ dim Q_h follows. Thus $\{\operatorname{curl} w_i\}_{i=1}^d$ generate a basis of the space Q_h and the ellipticity of the bilinear form $b(\cdot, \cdot)$ implies that the matrix $(b(\operatorname{curl} w_i, \operatorname{curl} w_j))_{i,j=1}^d$ of the system (5.1) is positive definite. The rest of the assertion is obvious. ∎

Remark 5.1. Since supp curl $w_i =$ supp grad w_i, by a suitable labelling of the basis functions we can reach that the matrix of the system has a structure similar to that of the corresponding system of the primal finite element method. Moreover, if the material is isotropic and homogeneous (i.e. if A in (1.1) is an identity matrix), the relation

$$(\operatorname{curl} w_i, \operatorname{curl} w_j)_{0,\Omega} = (\operatorname{grad} w_i, \operatorname{grad} w_j)_{0,\Omega}$$

holds, i.e. the inner products in the matrix of the system (5.1) can be calculated in the same way as in the primal finite element method. ∎

Remark 5.2. In the case $\Gamma_g = \emptyset$ we have dim $W_h' = 1 +$ dim Q_h and for the choice of the basis functions in Q_h see the paper [9], p. 46. If Γ_u is contained in at least two components of the boundary $\partial\Omega$ and if $\{q^j\}_{j=1}^r$ is a basis of the space curl W_h', then $\{q^1, \ldots, q^r, \alpha_h^1, \ldots, \alpha_h^{m-1}\}$, where α_h^j are the same as those in (4.6), will be a basis of the space Q_h. For the details we refer to the paper [9]. ∎

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Souhrn

VNITŘNÍ APROXIMACE KONEČNÝMI PRVKY V DUÁLNÍ VARIAČNÍ METODĚ PRO ELIPTICKÉ PROBLÉMY DRUHÉHO ŘÁDU SE ZAKŘIVENÝMI HRANICEMI

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Na oblastech s po částech hladkou hranicí jsou zkonstruovány pomocí proudové funkce podprostory konečných prvků v prostorech vektorových funkcí, jejichž divergence je rovna nule a jejichž normálová komponenta je na části hranice rovněž nulová. Pomocí těchto podprostorů je definována vnitřní aproximace duální úlohy pro eliptické rovnice 2. řádu. Je dokázána konvergence této metody (bez předpokladu na regularitu řešení) a pro dostatečně hladké řešení je dokázána i optimální rychlost konvergence. Vnitřní aproximaci lze získat řešením soustavy lineárních algebraických rovnic s pozitivně definitní maticí.

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