Libor Čermák
A note on a discrete form of Friedrichs' inequality

Aplikace matematiky, Vol. 28 (1983), No. 6, 457–466

Persistent URL: http://dml.cz/dmlcz/104056

A NOTE ON A DISCRETE FORM OF FRIEDRICHS' INEQUALITY

LIBOR ČERMÁK

(Received July 7, 1982)

Discrete forms of Friedrichs' inequality in two-dimensional Euclidean space were introduced and analysed by Ženíšek [5], [6]. These inequalities enable us to extend the theory of curved finite elements introduced in [1] to the case of boundary value problems with various stable and unstable boundary conditions. The aim of this paper is to extend one of Ženíšek's results, namely to prove the discrete form of the "classical" Friedrichs' inequality in the n-dimensional Euclidean space for arbitrary $n \geqq 2$. The finite element space on which the discrete form of Friedrichs' inequality is proved originates from the 1-regular triangulation of simplicial isoparametric elements.

Following Ciarlet and Raviart, see e.g. [1], we introduce the 1-regular family $\{K\}_h$ of simplicial isoparametric finite elements K. We are given

(a) A set $\hat{\Sigma}_K = \bigcup_{i=1}^{\hat{N}_K} \{\hat{a}_{i,K}\}$ of $\hat{N}_K$ distinct points of R^n such that its closed convex hull $\hat{K}$ is the unit n-simplex.

(b) A finite dimensional space $\hat{P}_K$ of functions defined on $\hat{K}$ with $\dim \hat{P}_K = \hat{N}_K$ such that $\hat{\Sigma}_K$ is $\hat{P}_K$-unisolvent, i.e. the Lagrange interpolation problem: "Find $\hat{p}_K \in \hat{P}_K$ such that $\hat{p}_K(a_{i,K}) = \alpha_i$, $1 \leqq i \leqq \hat{N}_K$" has one and only one solution for any real numbers α_i.

(c) A set $\Sigma_K = \bigcup_{i=1}^{\hat{N}_K} \{a_{i,K}\}$ of $\hat{N}_K$ distinct points of R^n.

Then the finite element K is the image of the set $\hat{K}$ through the unique mapping $F_K : \hat{K} \to R^n$ which satisfies

$$F_K \in \hat{P}_K^n \quad \text{and} \quad F_K(\hat{a}_{i,K}) = a_{i,K}\,, \quad 1 \leqq i \leqq \hat{N}_K\,,$$

where the notation $F_K \in \hat{P}_K^n$ means that

$$F_K : \hat{x} \in \hat{K} \to F_K(\hat{x}) = (F_{K1}(\hat{x}), \ldots, F_{Kn}(\hat{x})) \in R^n$$

with

$$F_{Ki} \in \hat{P}_K\,, \quad 1 \leqq i \leqq n\,.$$

We suppose

(d) For all h the mapping F_K is a C^2-diffeomorphism and there exist constants c_0, c_1, c_2, independent of h, such that

$$\sup_{\hat{x}\in\hat{K}} \max_{|\alpha|=i} \left|D^\alpha F_K(\hat{x})\right| \leqq c_i h^i\,, \quad i = 1, 2\,, \tag{1}$$

$$0 < c_0 h^n \leqq \left|J_K(\hat{x})\right|\,, \tag{2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $J_K(\hat{x})$ is the Jacobian of the mapping F_K at the point $\hat{x} \in \hat{K}$.

Every element K is associated with the finite dimensional space P_K (with $\dim P_K = \hat{N}_K$) of functions

$$P_K = \{p_K \mid K \to R,\ p_K = \hat{p}_K(F_K^{-1}),\ \forall \hat{p}_K \in \hat{P}_K\}\,.$$

Let us denote by $\hat{S}$ one of the $n + 1$ surface $(n - 1)$-simplexes of the unit simplex $\hat{K}$ and by $\hat{H}$ one of the n surface $(n - 2)$-simplexes of the simplex $\hat{S}$. We denote by $\hat{\Sigma}_S$ and $\hat{P}_S$ the restrictions to $\hat{S}$ of $\hat{\Sigma}_K$ and $\hat{P}_K$, respectively, and similarly by $\hat{\Sigma}_H$ and $\hat{P}_H$ the restrictions to $\hat{H}$ of $\hat{\Sigma}_S$ and $\hat{P}_S$, respectively. Let $S = F_K(\hat{S})$, $H = F_K(\hat{H})$ and let Σ_S denote the restriction of Σ_K to S.

In the sequel, by Ω we mean a bounded domain in R^n $(n \geqq 2)$ with $\Omega \in \mathscr{C}^{1,1}$ (for the definition of such domains see e.g. Nečas [3], p. 55). Then we can suppose that there exist R coordinate systems $\{x^r\} = \{(x_1^r, \ldots, x_n^r)\}$, $r = 1, \ldots, R$, such that every point of the boundary Γ of the domain Ω can be described in one of such coordinate systems by an equation

$$x_n^r = \varphi^r(x'^r)\,, \quad x'^r \in \Delta^r\,,$$

see Fig. 1; here $x'^r = (x_1^r, \ldots, x_{n-1}^r)$, Δ^r is an $(n - 1)$-dimensional closed cube and φ^r is a function with Lipschitz continuous first derivatives on Δ^r, i.e. $\varphi^r \in C^{1,1}(\Delta^r)$.

Following Ciarlet and Raviart [1] we define the 1-regular triangulation of the domain Ω. Let Ω_h be the union of a finite number of simplicial elements K. The boundary of Ω_h is denoted by Γ_h. Every element $K = F_K(\hat{K})$ is determined by $\hat{N}_K$ points $a_{i,K}$. We suppose that all points $a_{i,K}$ belong to $\bar{\Omega}$. The family of elements constructed in this way is called a triangulation of Ω and denoted by τ_h. We say that a triangulation τ_h of Ω is 1-regular if:

(a) The family of all elements by which the triangulation is formed is 1-regular.

(b) The geometric shape of any "face" S of a given element $K \in \Omega_h$ must be completely determined by those points $a_{i,K}$ which belong to S, i.e. the set $\hat{\Sigma}_S$ contains a $\hat{P}_S$-unisolvent subset.

(c) The geometric shape of any "edge" H of a given element $S \in \Gamma_h$ must be completely determined by those points $a_{i,K}$ which belong to H, i.e. the set $\hat{\Sigma}_H$ contains a $\hat{P}_H$-unisolvent subset.

(d) There exist Lipschitz continuous functions ψ^r defined on Δ^r, $r = 1, \ldots, R$, such that every point of the boundary Γ_h may be described at least by one of the equations

$$x_n^r = \psi^r(x'^r), \quad x'^r \in \Delta^r ,$$

see Fig. 1. Moreover, we suppose

$$\max_{x'^r \in \Delta^r} \left|\varphi^r(x'^r) - \psi^r(x'^r)\right| \leqq ch^2 , \tag{3}$$

where c is a constant independent of h.

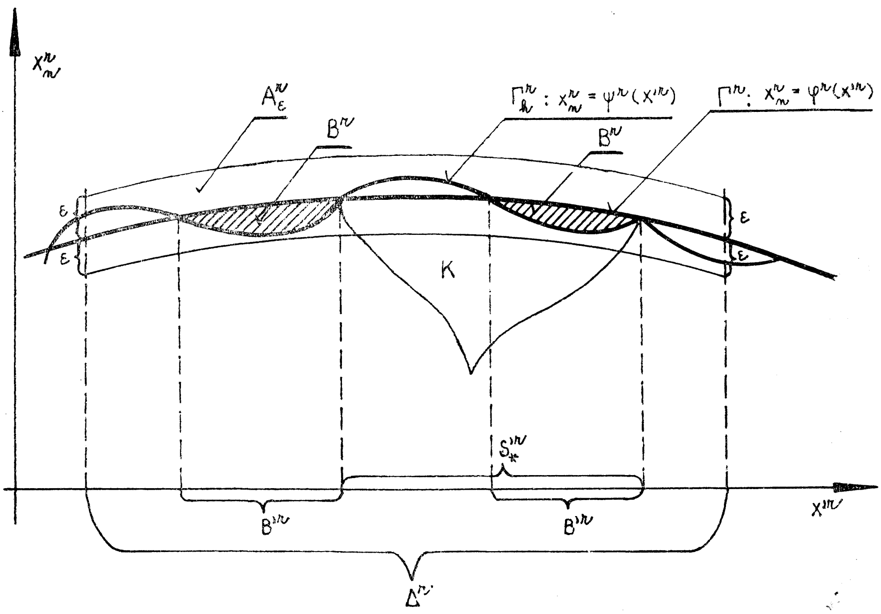

Fig. 1.

It can be proved that the assumption (d) in the above definition follows from the assumption

(d′) All points $a_{i,K}$ lying on Γ_h belong to Γ.

In the sequel we will suppose that for a 1-regular triangulation the assumption (d′) is fulfilled.

The parametr h has the usual geometrical meaning, i.e. h is the maximum of diameters of finite elements $\{K\}_h$. It is understood that h approaches zero in the limit.

A given 1-regular triangulation is associated with the finite dimensional space V_h of Lipschitz continuous test functions v defined by

$$V_h = \left\{v \mid v \in C^{0,1}(\bar{\Omega}_h),\ v_K \in P_K,\ \forall K \in \Omega_h\right\} ,$$

where v_K is the restriction of the function v to the set K.

Let us denote

$$\Gamma^r = \{(x'^r, \varphi^r(x'^r)) \mid x'^r \in \Delta^r\}, \quad \Gamma_h^r = \{(x'^r, \psi^r(x'^r)) \mid x'^r \in \Delta^r\}.$$

For a function v defined on Γ^r we denote

$$_{\varphi^r}v(x'^r) = v(x'^r, \varphi^r(x'^r)), \quad x'^r \in \Delta^r$$

and similarly for a function w defined on Γ_h^r we denote

$$_{\psi^r}w(x'^r) = w(x'^r, \psi^r(x'^r)), \quad x'^r \in \Delta^r.$$

In the sequel the constants independent of h are denoted by c. The notation is generic, i.e. c does not necessarily mean the same constant at any two places.

The norm in the Lebesgue space $L_2(A)$ is denoted by $\|\cdot\|_{0,A}$, the norm in the Sobolev space $H^m(A)$ by $\|v\|_{m,A} = (\sum_{i=0}^{m} |v|_{i,A}^2)^{1/2}$, where $|v|_{i,A} = (\sum_{|\alpha|=i} \|D^\alpha v\|_{0,A}^2)^{1/2}$, and the norm in the Sobolev space $W^{m,\infty}(A)$ by $\|v\|_{m,\infty,A} = \max_{0 \leqq i \leqq m} |v|_{i,\infty,A}$, where $|v|_{i,\infty,A} = \max_{|\alpha|=i} \operatorname{ess\,sup}_{x \in A} |D^\alpha v|$.

The norms in the Lebesgue spaces $L_2(\Gamma)$ and $L_2(\Gamma_h)$ are denoted by $\|v\|_{0,\Gamma} = (\int_\Gamma v^2 \, d\Gamma)^{1/2}$ and $\|v\|_{0,\Gamma_h} = (\int_{\Gamma_h} v^2 \, d\Gamma_h)^{1/2}$, respectively.

Let $\Phi(x)$ be a function defined on the element K. Then the function $\Phi(F_K(\hat{x}))$ is defined on $\hat{K}$ and we will denote it by $\hat{\Phi}(\hat{x})$.

Now we are able to formulate and to prove the discrete form of Friedrichs' inequality. The following theorem extends Ženíšek's results [6] to 3 and more dimensions.

Theorem. *Let τ_h be a 1-regular triangulation of the domain $\Omega \in \mathscr{C}^{1,1}$. Let Γ^* be part of the boundary Γ with meas $\Gamma^* > 0$. Then for any function $v \in V_h$ there exists a constant c independent of h and v such that the inequality*

$$\|v\|_{1,\Omega_h} \leqq c(\|v\|_{0,\Gamma_h^*} + |v|_{1,\Omega_h}) \tag{4}$$

holds true for all h sufficiently small. Here Γ_h^* denotes the set

$$\Gamma_h^* = \{x \mid x \in S,\ S \in \Gamma_h \text{ is such that } \Sigma_S \in \Gamma^*\}.$$

The proof of the theorem will be accomplished in five steps.

1. Given a function $v \in V_h$ we construct the extension $\tilde{v} \in H^1(\Omega_h \cup \Omega)$ such that

$$\|\tilde{v}\|_{1,\Omega-\Omega_h} \leqq ch^{1/2}\|v\|_{1,\Omega_h}. \tag{5}$$

2. We prove the inequality

$$\|v\|_{1,\Omega_h} \leqq c\|\tilde{v}\|_{1,\Omega}. \tag{6}$$

3. We can suppose that Γ^* may be completely described in one coordinate system $\{x^s\}$ for some $s \in \{1, \ldots, R\}$ (otherwise we take $\Gamma^* \cap \Gamma^s$ for a suitable s instead of Γ^*). Moreover, we can suppose that for all h sufficiently small the inclusion

$$(7) \quad \omega_s = \{x'^s \mid x'^s \in \Delta^s,\ (x'^s, \varphi^s(x'^s)) \in \Gamma^*\} \subset \{x'^s \mid x'^s \in \Delta^s,\ (x'^s, \psi^s(x'^s)) \in \Gamma_h^*\}$$

holds (otherwise we take a suitable subset of Γ^* with nonzero measure instead of Γ^*). From Friedrichs' inequality we deduce

$$(8) \qquad \|\tilde{v}\|_{1,\Omega} \leqq c(\|\tilde{v}\|_{0,\Gamma^*} + |\tilde{v}|_{1,\Omega}) .$$

4. We prove the inequality

$$(9) \qquad \|\tilde{v}\|_{0,\Gamma^*} \leqq c(\|v\|_{0,\Gamma_h^*} + |v|_{1,\Omega_h} + |\tilde{v}|_{1,\Omega-\Omega_h}) .$$

5. Combining (6), (8) and (9) we obtain

$$\|v\|_{1,\Omega_h} \leqq c(\|v\|_{0,\Gamma_h^*} + |v|_{1,\Omega_h} + |\tilde{v}|_{1,\Omega-\Omega_h}) .$$

Hence and from (5) we get the assertion of the theorem.

Proof of Step 1. We define the sets

$$A_\varepsilon^r = \{x \mid x = (x'^r, x_n^r),\ x'^r \in \Delta^r,\ \operatorname{dist}(x'^r, \partial\Delta^r) > 0,\ |x_n^r - \varphi^r(x'^r)| < \varepsilon\} ,$$

where ε is a sufficiently small positive number, see Fig. 1. Then for h sufficiently small we have

$$A_\varepsilon \equiv \bigcup_{r=1}^{R} A_\varepsilon^r \supset \Omega - \Omega_h .$$

The theorem on the partition of unity, see e.g. Yosida [4], p. 61, asserts that there exist functions $a^r \in C^\infty(R^n)$ such that $\operatorname{supp}(a^r) \subset A_\varepsilon^r$ and that $\sum_{r=1}^{R} a^r(x) = 1$ for any $x \in A_\varepsilon$.

For a function $v \in V_h$ we define functions

$$\tilde{v}^r(x) = \begin{cases} v(x) , & x \in A_\varepsilon^r \cap \Omega_h , \\ v(x'^r, \psi^r(x'^r)) , & x \in A_\varepsilon^r - \Omega_h , \quad r = 1, \ldots, R . \\ 0 , & x \in A_\varepsilon - A_\varepsilon^r , \end{cases}$$

Further, we define the function

$$\tilde{v}(x) = \sum_{r=1}^{R} a^r(x)\, \tilde{v}^r(x) , \quad x \in A_\varepsilon .$$

Evidently $\tilde{v}(x) = v(x)$ for $x \in A_\varepsilon \cap \Omega_h$. If we set

$$\tilde{v}(x) = v(x) \quad \text{for} \quad x \in \Omega_h - A_\varepsilon ,$$

then $\tilde{v}$ is the extension of the function v from Ω_h to $\Omega_h \cup A_\varepsilon$. Moreover, from the smoothness of the functions ψ^r we deduce that $\tilde{v} \in H^1(\Omega_h \cup A_\varepsilon)$ and hence also $\tilde{v} \in H^1(\Omega_h \cup \Omega)$.

Let us denote $B^r = A_\varepsilon^r \cap (\Omega - \Omega_h)$, $B'^r = \{x'^r \mid x'^r \in \Delta^r, (x'^r, x_n^r) \in B^r\}$. As $\Omega - \Omega_h \subset \bigcup_{r=1}^{R} B^r$ we have

$$\|\tilde{v}\|^2_{1,\Omega-\Omega_h} \leqq \sum_{r=1}^{R} \|\tilde{v}\|^2_{1,B^r} = \sum_{r=1}^{R} \Big\| \sum_{j=1}^{R} a^j \tilde{v}^j \Big\|^2_{1,B^r} \leqq \tag{10}$$

$$\leqq (2n+1)\, R \max_{j=1,\dots,R} \|a^j\|^2_{1,\infty,A_\varepsilon^j} \sum_{r=1}^{R} \sum_{j=1}^{R} \|\tilde{v}^j\|^2_{1,B^r \cap A_\varepsilon^j} \leqq c \sum_{r=1}^{R} \|\tilde{v}^r\|^2_{1,B^r}\,.$$

For $x^r \in B^r$ we have $\tilde{v}^r(x'^r, x_n^r) = v(x'^r, \psi^r(x'^r))$; then

$$\frac{\partial \tilde{v}^r}{\partial x_i^r}(x'^r, x_n^r) = \frac{\partial v(x'^r, \psi^r(x'^r))}{\partial x_i^r}\,, \quad i = 1, \dots, n-1 \quad \text{and} \quad \frac{\partial \tilde{v}^r(x'^r, \psi^r(x'^r))}{\partial x_n^r} = 0\,.$$

Hence

$$\|\tilde{v}^r\|^2_{1,B^r} = \int_{B'^r} \left| \int_{\varphi^r(x'^r)}^{\psi^r(x'^r)} \left\{ \sum_{i=1}^{n} \left[\frac{\partial \tilde{v}^r(x'^r, x_n^r)}{\partial x_i^r} \right]^2 + [\tilde{v}^r(x'^r, x_n^r)]^2 \right\} \mathrm{d}x_n^r \right| \mathrm{d}x'^r = \tag{11}$$

$$= \int_{B'^r} \left| \int_{\varphi^r(x'^r)}^{\psi^r(x'^r)} \left\{ \sum_{i=1}^{n-1} \left[\frac{\partial v(x'^r, \psi^r(x'^r))}{\partial x_i^r} \right]^2 + [v(x'^r, \psi^r(x'^r))]^2 \right\} \mathrm{d}x_n^r \right| \mathrm{d}x'^r \leqq$$

$$\leqq \|\varphi^r - \psi^r\|_{0,\infty,B'^r} \|_{\psi^r} v\|^2_{1,B'^r} \leqq \|\varphi^r - \psi^r\|_{0,\infty,\Delta^r} \|_{\psi^r} v\|^2_{1,\Delta^r}\,.$$

Let a "face" S have a non-empty intersection S_*^r with the set Γ_h^r. If we denote by $S_*'^r$ the projection of S_*^r into the hyperplane $x_n^r = 0$, then $\Delta^r = \bigcup S_*'^r$. We can and will suppose that the image $\hat{S}_* = F_K^{-1}(S_*^r)$ of the set S_*^r lies in the hyperplane $\hat{x}_n = 0$. Then

$$|_{\psi^r} v|^2_{0,S_*'^r} = \int_{S_*'^r} {}_{\psi^r} v^2 \,\mathrm{d}x'^r = \int_{\hat{S}_*} \hat{v}^2 J_K^{(n,n)} \,\mathrm{d}\hat{x}'\,,$$

where $J_K^{(n,n)}$ denotes the cofactor of the Jacobian J_K. Hence and from (1) we have

$$|_{\psi^r} v|^2_{0,S_*'^r} \leqq c h^{n-1} |\hat{v}|^2_{0,\hat{S}_*}\,.$$

From the trace theorem and from the equivalence of norms on the finite dimensional space $\hat{P}_K$ we immediately obtain

$$|\hat{v}|^2_{0,\hat{S}_*} \leqq c \|\hat{v}\|^2_{1,\hat{K}} \leqq c |\hat{v}|^2_{0,\hat{K}}$$

and consequently

$$|_{\psi^r} v|^2_{0,S_*'^r} \leqq c h^{n-1} |\hat{v}|^2_{0,\hat{K}}\,.$$

Using (2) we have

$$|_{\psi^r} v|^2_{0,S_*'^r} \leqq c h^{-1} |v|^2_{0,K}\,, \tag{12}$$

where K is an element such that $S_*^r \subset \partial K$. From the chain rule (on the differentiating of the composite function) we obtain

(13) $$\frac{\partial F_{Ki}^{-1}}{\partial x_j} = \frac{J_K^{(j,i)}}{J_K},$$

and consequently

$$|_{\psi^r} v|_{1,S_*'^r}^2 = \int_{S_*'^r} \sum_{j=1}^{n-1} \left[\frac{\partial_{\psi^r} v}{\partial x_j}\right]^2 \mathrm{d}x'^r =$$

$$= \int_{\hat{S}_*} \sum_{j=1}^{n-1} \left[\sum_{i=1}^{n-1} \frac{\partial \hat{v}}{\partial \hat{x}_i} \frac{\partial F_{Ki}^{-1}}{\partial x_j}\right]^2 J_K^{(n,n)} \,\mathrm{d}\hat{x}' = \sum_{j=1}^{n-1} \int_{\hat{S}_*} \left[\sum_{i=1}^{n-1} \frac{\partial \hat{v}}{\partial \hat{x}_i} \frac{J_K^{(j,i)}}{J_K}\right]^2 J_K^{(n,n)} \,\mathrm{d}\hat{x}'.$$

Here again $J_K^{(j,i)}$ is the cofactor of the Jacobian J_K. The last equality and (1), (2) yield

(14) $$|_{\psi^r} v|_{1,S_*'^r}^2 \leqq c h^{n-3} \sum_{i=1}^{n-1} \left|\frac{\partial \hat{v}}{\partial \hat{x}_i}\right|_{0,\hat{S}_*}^2.$$

Using the trace theorem and the equivalence of norms on a finite dimensional space we see that

$$\left|\frac{\partial \hat{v}}{\partial \hat{x}_i}\right|_{0,\hat{S}_*} \leqq c \left\|\frac{\partial \hat{v}}{\partial \hat{x}_i}\right\|_{1,\hat{K}} \leqq c \left\|\frac{\partial \hat{v}}{\partial \hat{x}_i}\right\|_{0,K}.$$

Since

$$\frac{\partial \hat{v}}{\partial \hat{x}_i} = \sum_{j=1}^{n} \frac{\partial v}{\partial x_j} \frac{\partial F_{Kj}}{\partial \hat{x}_i}$$

we obtain from the last two inequalities and from (1), (2) and (14)

(15) $$|_{\psi^r} v|_{1,S_*'^r}^2 \leqq c h^{-1} |v|_{1,K}^2.$$

Combining (12) and (15) and summing over all elements $S_*'^r$ we obtain

$$\|_{\psi^r} v\|_{1,\Delta^r}^2 \leqq c h^{-1} \|v\|_{1,\Omega_h}^2.$$

Together with (11) and (3) this yields

$$\|\tilde{v}^r\|_{1,B^r}^2 \leqq c h \|v\|_{1,\Omega_h}^2.$$

This together with (10) proves (5).

Proof of Step 2. For an element K having a non-empty intersection with the set $\Omega_h - \Omega$ we denote $K_* = K \cap (\Omega - \Omega_h)$ and $\hat{K}_* = F_K^{-1}(K_*)$. Using (1), (13) and (2) we have

(16) $$\|v\|_{1,K_*}^2 = |v|_{0,K_*}^2 + |v|_{1,K_*}^2 = \int_{\hat{K}_*} \hat{v}^2 J_K \,\mathrm{d}\hat{x} + \int_{\hat{K}_*} \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \frac{\partial \hat{v}}{\partial \hat{x}_i} \frac{\partial F_{Ki}^{-1}}{\partial x_j}\right]^2 J_K \,\mathrm{d}\hat{x} \leqq$$

$$\leqq c h^n (|\hat{v}|_{0,\hat{K}_*}^2 + h^{-2} |\hat{v}|_{1,\hat{K}_*}^2) \leqq c h^n \operatorname{meas} \hat{K}_* (|\hat{v}|_{0,\infty,\hat{K}}^2 + h^{-2} |\hat{v}|_{1,\infty,\hat{K}}^2).$$

From the equivalence of norms on finite dimensional spaces we deduce

$$|\hat{v}|_{i,\infty,\hat{K}} \leqq c|\hat{v}|_{i,\hat{K}}, \quad i = 0, 1 .$$

Hence, from (16), (1) and (2) we get

(17) $$\|v\|_{1,K_*}^2 \leqq ch^n \operatorname{meas} \hat{K}_*(|\hat{v}|_{0,\hat{K}}^2 + h^{-2}|\hat{v}|_{1,\hat{K}}^2) \leqq c \operatorname{meas} \hat{K}_*\|v\|_{1,K}^2 .$$

Let us suppose that

(18) $$\operatorname{meas} \hat{K}_* \leqq ch .$$

Then summing over all elements K_* we obtain

(19) $$\|v\|_{1,\Omega_h-\Omega}^2 \leqq ch\|v\|_{1,\Omega_h}^2 .$$

Since $\|v\|_{1,\Omega_h}^2 \leqq \|\tilde{v}\|_{1,\Omega}^2 + \|v\|_{1,\Omega_h-\Omega}^2$, (6) follows for all h sufficiently small from the last two inequalities.

To prove (18) it suffices to prove that for any point $\hat{x} \in \hat{K}_*$ the inequality

(20) $$\operatorname{dist}(\hat{x}, \partial\hat{K}) \leqq ch$$

holds, see Fig. 2. Let $\hat{x} \in \hat{K}_*$ and let $x = (x'^r, x_n^r) = F_K(\hat{x})$, $z = (x'^r, \psi^r(x'^r))$ and $\hat{z} = F_K^{-1}(z)$. Then

$$\operatorname{dist}(\hat{x}, \partial\hat{K}) = |\hat{x} - \hat{x}_P| \leqq |\hat{x} - \hat{z}| = |F_K^{-1}(x) - F_K^{-1}(z)| \leqq$$
$$\leqq \max_{|\alpha|=1} \sup_{y\in K} |D^\alpha F_K^{-1}(y)| \max_{y'^r\in\Delta^r} |\varphi^r(y'^r) - \psi^r(y'^r)|$$

and (20) follows from (13), (1), (2) and (3).

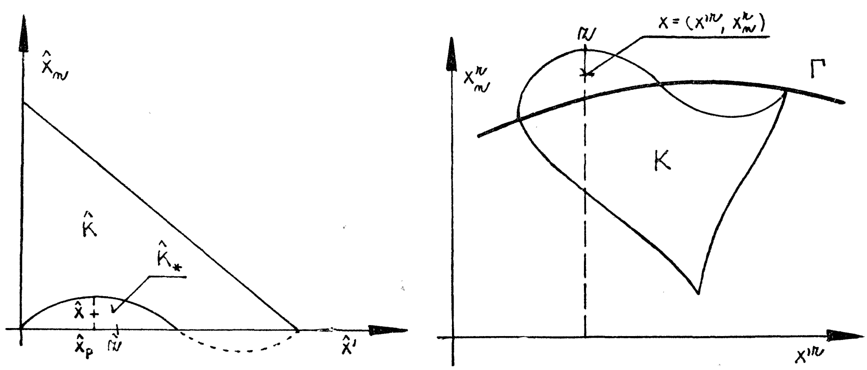

Fig. 2.

Proof of Step 4. For any point $x'^s \in \omega_s$, for the definition of ω_s see (7), we have

$$\tilde{v}(x'^s, \varphi^s(x'^s)) = v(x'^s, \psi^s(x'^s)) + \int_{\psi^s(x'^s)}^{\varphi^s(x'^s)} \frac{\partial}{\partial x_n^s} \tilde{v}(x'^s, \tau) \, d\tau .$$

Squaring, using Cauchy's inequality, integrating over the set ω_s and using (7) we get

$$\int_{\omega_s} \tilde{v}^2(x'^s, \varphi^s(x'^s))\, \mathrm{d}x'^s \leqq c\left[\int_{\omega_s} v^2(x'^s, \psi^s(x'^s))\, \mathrm{d}x'^s + \right. \tag{21}$$

$$\left. + \int_{\omega_s} \left|\int_{\psi^s(x'^s)}^{\varphi^s(x'^s)} \left[\frac{\partial}{\partial x_n^s} \tilde{v}(x'^s, \tau)\right]^2 \mathrm{d}\tau\right| \mathrm{d}x'^s\right] \leqq c(\|v\|^2_{0,\Gamma_h^*} + |\tilde{v}|^2_{1,\Omega-\Omega_h\cup\Omega_h-\Omega}).$$

Due to the smoothness of the boundary Γ we have

$$\|\tilde{v}\|^2_{0,\Gamma^*} = \int_{\omega_s} \tilde{v}^2(x'^s, \varphi^s(x'^s)) \sqrt{\left(1 + \sum_{i=1}^{n-1}\left[\frac{\partial \varphi^s(x'^s)}{\partial x_i^s}\right]^2\right)}\, \mathrm{d}x'^s \leqq c\int_{\omega_s} \tilde{v}^2(x'^s, \varphi^s(x'^s))\, \mathrm{d}x'^s.$$

From the last two inequalities we obtain (9). □

Using the same technique we can prove the discrete form of Friedrichs' inequality also for other isoparametric triangulations, e.g. for the triangulation based on using quadrilateral isoparametric finite elements.

The discrete form of Friedrichs' inequality is applied in the analysis of the finite element solution of elliptic partial differential equations of the second order with unstable boundary conditions, see [5], [6], [2]. To illustrate it let us consider the bilinear form

$$b_h(v, w) = \int_{\Omega_h} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}\, \mathrm{d}\Omega_h + \int_{\Gamma_h^*} a(x)\, vw\, \mathrm{d}\Gamma_h^*$$

such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geqq c_1 \sum_{i=1}^{n} \xi_i^2 \quad \forall x \in \Omega_h\,, \quad (\xi_1, \ldots, \xi_n) \in R^n\,, \quad c_1 > 0\,,$$

and

$$a(x) \geqq c_2 > 0 \quad \forall x \in \Gamma_h^*\,.$$

Then the theorem on the discrete form of Friedrichs' inequality yields

$$b_h(v, v) \geqq c\|v\|^2_{1,\Omega_h} \quad \forall v \in V_h\,, \quad c > 0\,,$$

i.e. the bilinear form $b_h(v, w)$ is uniformly (with respect to h) V_h-elliptic.

References

[1] *P. G. Ciarlet, P. A. Raviart:* The combined effect of curved boundaries and numerical integration in isoparametric finite element methods. In: The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A. K. Aziz Editor). Academic Press. New York and London. 1972.

[2] *L. Čermák:* The finite element solution of second order elliptic problems with the Newton boundary condition. Apl. Mat., 28 (1983), 430—456.

[3] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia. Prague. 1967.
[4] *K. Yosida:* Functional Analysis. Springer-Verlag. Berlin—Heidelberg—New York. 1966.
[5] *A. Ženíšek:* Nonhomogeneous boundary conditions and curved triangular finite elements. Apl. Mat., 26 (1981), 121—141.
[6] *A. Ženíšek:* Discrete forms of Friedrichs' inequalities in the finite element method. R.A.I.R.O Numer. Anal., 15 (1981), 265—286.

Souhrn

POZNÁMKA O DISKRÉTNÍM TVARU FRIEDRICHSOVY NEROVNOSTI

LIBOR ČERMÁK

Práce je věnována důkazu Friedrichsovy nerovnosti na třídě konečně dimensionálních prostorů používaných v metodě konečných prvků. Konkrétně jsou uvažovány aproximační prostory generované simpliciálními izoparametrickými elementy.

Author's address: Dr. *Libor Čermák,* CSc., OVC VUT, Tř. Obránců míru 21, 602 00 Brno.