Aplikace matematiky

Karel Segeth

On the choice of iteration parameters in the Stone incomplete factorization

Aplikace matematiky, Vol. 28 (1983), No. 4, 295–306

Persistent URL: http://dml.cz/dmlcz/104038

ON THE CHOICE OF ITERATION PARAMETERS IN THE STONE INCOMPLETE FACTORIZATION

KAREL SEGETH

(Received December 28, 1982)

1. INTRODUCTION

Recently, the development of science and technology has brought about problems whose treatment involves solving very large sparse linear algebraic systems. Various direct (finite) methods based on the Gaussian elimination procedure have been devised to solve special systems. These methods use the number of arithmetic operations proportional to the number of unknowns and are called fast methods. Their applicability, however, is limited to special sparse systems of a very regular structure.

For more general sparse systems, diverse iterative methods have been developed whose each step is fast. Such methods usually involve one or more parameters that have to be a priori chosen. The choice of these parameters may substantially influence the rate of convergence of the iterative method. On the other hand, it is usually very costly (and sometimes perhaps practically impossible) to find the optimal values of the iteration parameters.

A very frequently used class of methods is based on the incomplete factorization (triangular decomposition). Many authors (see e.g. Axelsson [1], Buleev [4], Dupont, Kendall and Rachford [5], Kershaw [7], Meijerink and van der Vorst [8], Segeth [10], Stone [13]) studied methods of this class in connection with various iterative processes. The Stone incomplete factorization [13] seems to be one of the most often used methods in computational practice. It has been examined and generalized by several authors (Bracha-Barak and Saylor [3], Gustafsson [6], Saylor [9], Segeth [11], Taranto [14]). In various fields, it is employed for solving large sparse systems arising from two- or three-dimensional boundary value problems (e.g. in geophysics), for solving systems obtained after the linearization of nonlinear problems (e.g. in electronics [12]), etc.

Up to now, the method has been given no rigorous theoretical foundation. We discuss numerical results obtained by solving a simple model problem to point out where there may be difficulties with the choice of iteration parameters. We have

chosen a particular model problem whose exact solution can be obtained in the first step of the used iterative procedure for a special choice of the parameters and the initial approximation. In general, we find out that these parameters can be chosen without a deeper a priori analysis of the system solved.

As an introduction we are concerned with a discretization of a one-dimensional boundary value problem in Sec. 2. The two-dimensional problem treated in Sec. 3 is a generalization of this factorization approach. We review the Stone method and its parameters in Sec. 3 and show some properties of the method in Sec. 4. In Sec. 5, we analyse a series of numerical experiments focused on the choice of the iteration parameters.

2. ONE-DIMENSIONAL PROBLEM

Consider a one-dimensional boundary value problem for the ordinary differential equation

$$-u''(x) = v(x) \quad \text{in} \quad R\,,$$

$$u(a) = g_a\,, \quad u(b) = g_b\,,$$

where $R = (a, b)$ is an interval. Choosing q interior nodes x_j, $j = 1, \ldots, q$, in R and employing the usual three-point finite-difference approximation (see e.g. Babuška, Práger, and Vitásek [2]), we come to the linear algebraic system

$$Au = v\,, \tag{2.1}$$

where $u = (u_j)$ is the column vector of the unknown values $u_j = u(x_j)$, $j = 1, \ldots, q$, of the approximate solution at the nodes x_j of the net, $v = (v_j)$ is the right-hand part vector, $v_j = v(x_j)$, and A is a $q \times q$ symmetric tridiagonal matrix.

The well-known factorization procedure for the solution of the system (2.1) (see e.g. [2]) now consists in three steps. The matrix A is first factored as

$$A = LU\,, \tag{2.2}$$

where L and U are lower and upper triangular matrices, respectively. In our case, both L and U have only two nonzero diagonals (i.e., the number of their nonzero diagonals is independent of q). The implementation of the factorization (2.2) then requires $O(q)$ arithmetic operations.

The two next steps in the solution of (2.1) now represent solving the systems

$$Lz = v\,, \quad Uu = z\,. \tag{2.3}$$

This process needs also $O(q)$ arithmetic operations and thus the whole factorization procedure requires altogether $O(q)$ operations, i.e., this procedure is fast as the amount of arithmetic operations is proportional to the number of unknowns. The number of operations (but not the order) is even lower if we use the symmetric factorization $LL^T = A$.

3. STONE INCOMPLETE FACTORIZATION

We will now consider the Dirichlet boundary value problem for the Poisson partial differential equation

$$-\Delta u(x, y) = v(x, y) \quad \text{in} \quad R\,, \tag{3.1}$$
$$u(x, y) = g(x, y) \quad \text{on} \quad \Gamma\,,$$

where $R = (a, b) \times (c, d)$ is a rectangle with the boundary Γ.

The equation considered by Stone [13] is sligthly more general and the boundary condition is of the Neumann type

$$-\frac{\partial}{\partial x}\left(\lambda(x, y)\frac{\partial u(x, y)}{\partial x}\right) - \frac{\partial}{\partial y}\left(\mu(x, y)\frac{\partial u(x, y)}{\partial y}\right) = v(x, y) \quad \text{in} \quad R\,, \tag{3.2}$$
$$\frac{\partial u(x, y)}{\partial n} = \gamma(x, y) \quad \text{on} \quad \Gamma\,,$$

where $\partial u/\partial n$ is the normal derivative.

For our problem (3.1), we construct a rectangular net on the rectangle R with interior nodes (x_j, y_k) (denoted also by (j, k)); $j = 1, \ldots, m$; $k = 1, \ldots, n$. Suppose that $m \leqq n$ and put $q = mn$. The nodes with coordinates $x_0 = a$, $x_{m+1} = b$, $y_0 = c$, or $y_{n+1} = d$ are called boundary nodes. We further put

$$h_{xj} = x_{j+1} - x_j\,, \quad j = 0, \ldots, m\,,$$
$$h_{yk} = y_{k+1} - y_k\,, \quad k = 0, \ldots, n\,,$$
$$h_x = \max_{j=0,\ldots,m} h_{xj}\,, \quad h_y = \max_{k=0,\ldots,n} h_{yk}\,.$$

The simplest five-point finite-difference approximation (see e.g. Babuška, Práger, and Vitásek [2]) then leads to the linear algebraic system (here, too, we denote the exact solution as well as the discrete one by the same symbol)

$$Au = v\,, \tag{3.3}$$

where u is the vector of the unknown values $u(x_j, y_k)$ of the approximate solution at the nodes (x_j, y_k) of the net, $u = (u(x_1, y_1), u(x_2, y_1), \ldots, u(x_{m-1}, y_n), u(x_m, y_n))^{\mathsf{T}}$, v is the right-hand part vector, and $A = (a_{is})$ is a $q \times q$ symmetric sparse matrix. A has only five nonzero diagonals. If the equations are ordered in a proper way (in our case, if the numbering of the nodes follows the rows of the net), then in the ith row only the entries $a_{i,i-m}$, $a_{i,i-1}$, a_{ii}, $a_{i,i+1}$, $a_{i,i+m}$ may be nonzero. For the sake of simplicity, we speak about five nonzero entries in all the rows of A. In the first row, for example, there are indeed no entries $a_{1,1-m}$ and a_{10}. In this and analogous cases we assume that the entries which do not appear in the matrix are zero. (Note that for the problem (3.2), the finite-difference approximation is constructed also at boundary nodes. Moreover, the matrix obtained is singular.)

It is well-known that the factorization (2.2) now gives bandmatrices L and U of bandwidth $m + 1$, i.e., in the ith row the $m + 1$ entries $l_{i,i-m}, \ldots, l_{ii}$ and the $m + 1$ entries $u_{ii}, \ldots, u_{i,i+m}$ are nonzero. The implementation of the factorization (2.2) then requires $O(qm^2)$ arithmetic operations, the solution of the systems (2.3) requires $O(qm)$ operations. Thus the whole process described is not fast, which is an immediate consequence of the fact that the bandwidths of L and U depend on the number of nodes of the net.

Stone [13] proposed to construct, instead of (2.2), an incomplete factorization that, however, allows to solve a system "close" to (3.3) fast. Stone's idea (further developed e.g. by Bracha-Barak and Saylor [3], Meijerink and van der Vorst [8], and Saylor [9]) consists in computing only such entries of the factor matrices that are situated in the same positions as the nonzero entries of the original matrix A. All the other entries are set zero.

We denote the factors constructed in this way by $\tilde{L}$ and $\tilde{U}$. In our case, $\tilde{L} = (\tilde{l}_{is})$ is a lower triangular matrix with three nonzero diagonals. In its ith row, only the three entries $\tilde{l}_{i,i-m}, \tilde{l}_{i,i-1}, \tilde{l}_{ii}$ are nonzero and we moreover put

$$\tilde{l}_{ii} = 1 .$$

Similarly, $\tilde{U} = (\tilde{u}_{is})$ is an upper triangular matrix with three nonzero diagonals and in its ith row only the three entries $\tilde{u}_{ii}, \tilde{u}_{i,i+1}, \tilde{u}_{i,i+m}$ are nonzero. In fact, for $i = 1, \ldots, q$ we successively calculate the entries $\tilde{l}_{i,i-m}, \tilde{l}_{i,i-1}, \tilde{u}_{ii}, \tilde{u}_{i,i+1}$, and $\tilde{u}_{i,i+m}$ from the equations

$$\begin{aligned} a_{i,i-m} &= \tilde{l}_{i,i-m}\tilde{u}_{i-m,i-m} , \\ a_{i,i-1} &= \tilde{l}_{i,i-1}\tilde{u}_{i-1,i-1} , \\ a_{ii} &= \tilde{l}_{i,i-m}\tilde{u}_{i-m,i} + \tilde{l}_{i,i-1}\tilde{u}_{i-1,i} + \tilde{u}_{ii} , \\ a_{i,i+1} &= \tilde{u}_{i,i+1} , \\ a_{i,i+m} &= \tilde{u}_{i,i+m} . \end{aligned} \tag{3.4}$$

Note that we need $O(q)$ arithmetic operations to construct the factorization (3.4) since we compute altogether only 5 nonzero entries in each row of $\tilde{L}$ and $\tilde{U}$ and we use only a fixed (independent of q) number of operations to determine each of these entries.

Putting now

$$\tilde{L}\tilde{U} = \tilde{A} , \tag{3.5}$$

we obtain instead of (2.2) that

$$A = \tilde{A} + E ,$$

where $E = (e_{is})$ is a nonzero matrix. In our case, the matrix $\tilde{A} = (\tilde{a}_{is})$ has seven nonzero diagonals. As compared with A, the entries $\tilde{a}_{i,i-m+1}$ and $\tilde{a}_{i,i+m-1}$ are, in ad-

dition, nonzero in the ith row of $\tilde{A}$. Indeed, by (3.4) we have $\tilde{a}_{is} = a_{is}$ except for

$$\tilde{a}_{i,i-m+1} = \tilde{l}_{i,i-m}\tilde{u}_{i-m,i-m+1},$$

$$\tilde{a}_{i,i+m-1} = \tilde{l}_{i,i-1}\tilde{u}_{i-1,i+m-1}.$$

Therefore the matrix E has two nonzero diagonals, i.e., in its ith row the entries $e_{i,i-m+1} = -\tilde{a}_{i,i-m+1}$ and $e_{i,i+m-1} = -\tilde{a}_{i,i+m-1}$ are nonzero. The matrix $\tilde{A}$ thus corresponds to a seven-point finite-difference scheme with two additional nodes, $(j+1, k-1)$ and $(j-1, k+1)$, which are associated with the two coefficients $\tilde{a}_{i,i-m+1}$ and $\tilde{a}_{i,i+m-1}$.

Conversely, note that modifying the seven-diagonal matrix $\tilde{A}$ in a proper way, we can construct its exact $\tilde{L}\tilde{U}$ factorization with the matrices $\tilde{L}$ and $\tilde{U}$ of the above described zero-nonzero structure. We will try to modify the matrix $\tilde{A}$ properly in order that it resemble the original matrix A in some sense as closely as possible, and then we will factorize this modified $\tilde{A}$ exactly.

Stone proposed to cancel the effect of the "parasitic" entries $\tilde{a}_{i,i-m+1}$ and $\tilde{a}_{i,i+m-1}$, expressing the value of the approximate solution at the additional two nodes $(j+1, k-1)$ and $(j-1, k+1)$ as a linear combination of the values at the adjacent original nodes and subtracting this linear combination from the seven-point finite-difference formula. The resulting (seven-point) difference formula should be close to the original five-point formula.

Assuming that the functions v and g are sufficiently smooth and using the Taylor expansion with $h_{x,j-1} \to 0$ and $h_{yk} \to 0$, we readily find that

$$u(x_{j-1}, y_{k+1}) = u(x_j, y_{k+1}) + u(x_{j-1}, y_k) - u(x_j, y_k) - \tag{3.6}$$

$$- h_{x,j-1}h_{yk}\frac{\partial^2 u(x_j, y_k)}{\partial x\,\partial y} - \tfrac{1}{3}h^3_{x,j-1}\frac{\partial^3 u(x', y')}{\partial x^3} + \tfrac{1}{2}h^2_{x,j-1}h_{yk}\frac{\partial^3 u(x'', y'')}{\partial x^2\,\partial y} -$$

$$- \tfrac{1}{2}h_{x,j-1}h^2_{yk}\frac{\partial^3 u(x'', y'')}{\partial x\,\partial y^2} + \tfrac{1}{3}h^3_{yk}\frac{\partial^3 u(x''', y''')}{\partial y^3},$$

where (x', y'), (x'', y'') and (x''', y''') are certain points of the rectangle with vertices (x_{j-1}, y_k), (x_j, y_k), (x_j, y_{k+1}) and (x_{j-1}, y_{k+1}). A similar expression holds at (x_{j+1}, y_{k-1}). The cancellation suggested by (3.6), however, is performed only partially, i.e. with some weight α. This number α, $0 \leqq \alpha \leqq 1$, is a parameter of the Stone method. We thus replace the approximate solution u at $(j-1, k+1)$ by

$$u(x_{j-1}, y_{k+1}) - \alpha(u(x_j, y_{k+1}) + u(x_{j-1}, y_k) - u(x_j, y_k)) \tag{3.7}$$

and the solution at $(j+1, k-1)$ by

$$u(x_{j+1}, y_{k-1}) - \alpha(u(x_j, y_{k-1}) + u(x_{j+1}, y_k) - u(x_j, y_k)). \tag{3.8}$$

The coefficients in the ith row of the system change, but no new nonzero coefficient arises (cf. Fig. 1). The matrix with this partial cancellation performed is denoted

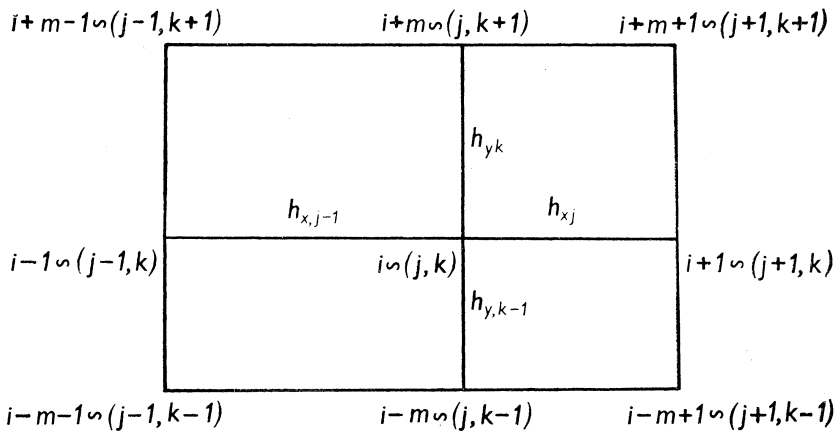

Fig. 1.

by $\hat{A}(\alpha) = (\hat{a}_{is}(\alpha))$. It is unsymmetric in general; clearly, we have

$$
\begin{aligned}
\hat{a}_{i,i-m}(\alpha) &= \tilde{a}_{i,i-m} - \alpha\tilde{a}_{i,i-m+1}\,, \\
\hat{a}_{i,i-m+1}(\alpha) &= \tilde{a}_{i,i-m+1}\,, \\
\hat{a}_{i,i-1}(\alpha) &= \tilde{a}_{i,i-1} - \alpha\tilde{a}_{i,i+m-1}\,, \\
\hat{a}_{ii}(\alpha) &= \tilde{a}_{ii} + \alpha\tilde{a}_{i,i-m+1} + \alpha\tilde{a}_{i,i+m-1}\,, \\
\hat{a}_{i,i+1}(\alpha) &= \tilde{a}_{i,i+1} - \alpha\tilde{a}_{i,i-m+1}\,, \\
\hat{a}_{i,i+m-1}(\alpha) &= \tilde{a}_{i,i+m-1}\,, \\
\hat{a}_{i,i+m}(\alpha) &= \tilde{a}_{i,i+m} - \alpha\tilde{a}_{i,i+m-1}\,.
\end{aligned} \tag{3.9}
$$

We can obtain the exact factorization of $\hat{A}(\alpha)$ in the form

$$\hat{A}(\alpha) = \hat{L}(\alpha)\,\hat{U}(\alpha)\,, \tag{3.10}$$

where $\hat{L}(\alpha)$ and $\hat{U}(\alpha)$ are lower and upper triangular matrices, respectively, and each of them has the same three nonzero diagonals as described above for $\tilde{L}$ and $\tilde{U}$.

Note that the value $\alpha = 0$ corresponds to no cancellation, i.e. to the factorization described e.g. in [7] and [8]. Clearly, $\hat{L}(0) = \tilde{L}$ and $\hat{U}(0) = \tilde{U}$ in our notation (3.5). On the other hand, the value $\alpha = 1$ leads to the best possible result as far as the above mentioned five-point and seven-point finite-difference schemes are compared.

Let us now complete the description of the Stone algorithm. We thus solve the linear algebraic system (3.3) after having constructed the factorization (3.10), where the matrix $\hat{A}(\alpha)$ is in some sense close to A.

Stone used a one-point iterative method

$$\hat{A}(\alpha)\,u_r = \hat{A}(\alpha)\,u_{r-1} - \beta(Au_{r-1} - v)\,, \quad r = 1, 2, \ldots\,, \tag{3.11}$$

where u_0 is an arbitrary initial approximation and β is another iteration parameter. Putting

$$t_r = u_r - u_{r-1}\,, \tag{3.12}$$

we may rewrite (3.11) in the more convenient form

$$\hat{A}(\alpha)\, t_r = -\beta(Au_{r-1} - v)\,, \quad r = 1, 2, \ldots\,. \tag{3.13}$$

To compute u_r, we have to solve the system (3.13) with a known right-hand part and then use (3.12). The evaluation of this right-hand part requires $O(q)$ arithmetic operations as A has five nonzero entries in every row. The solution of the system (3.13) requires also $O(q)$ operations since we have constructed the factorization (3.10) with the triangular matrices $\hat{L}(\alpha)$ and $\hat{U}(\alpha)$ having only three nonzero entries in every row. This $\hat{L}\hat{U}$ factorization also requires only $O(q)$ arithmetic operations as we mentioned above. Finally, we need $O(q)$ operations to compute u_r from (3.12). Hence, the whole iteration step (3.11) (or (3.13)) needs altogether $O(q)$ arithmetic operations and it is therefore fast.

To reach a certain symmetry of the computation, Stone numbered the nodes rowwise, but ordered the rows alternately from below upwards (displayed in Fig. 1) in one iteration step and from above downwards in the next step. We thus work with two different matrices A (and also $\hat{A}(\alpha)$) in each two consecutive steps and the additional two nodes in the difference scheme are alternately $(j-1, k+1), (j+1, k-1)$ or $(j-1, k-1), (j+1, k+1)$. These two consecutive iteration steps are called a double-step in what follows. Every double-step is performed with a fixed value of α.

The Fourier analysis of the rate of convergence of the iteration (3.11), which is similar for both the problems (3.1) and (3.2), has led Stone [13] to the following recommendations: We choose a positive integer P. If the functions $\lambda(x, y)$ and $\mu(x, y)$ and the quantities h_{xj} and h_{yk} are constant, we put $\lambda(x, y) = \Lambda$ and $\mu(x, y) = M$, and determine the quantity $\alpha_{\max}$ from the relation

$$1 - \alpha_{\max} = \min\left(\frac{2h_x^2}{1 + Mh_x^2/(\Lambda h_y^2)}, \frac{2h_y^2}{1 + \Lambda h_y^2/(Mh_x^2)}\right). \tag{3.14}$$

Note that Kershaw [7] expressed his objections to this ad hoc formula. A set of P iteration parameters α_p is then defined as

$$1 - \alpha_p = (1 - \alpha_{\max})^{p/(P-1)}\,, \quad p = 0, 1, \ldots, P-1\,, \tag{3.15}$$

if $P > 1$; we put

$$\alpha_0 = \alpha_{\max}$$

for $P = 1$. These P parameters are used cyclically, each of them for one double-step. For the $(P+1)$st double-step, the same parameter α is used as for the first double-step etc. Note that the formula (3.15) implies $\alpha_0 = 0$ for $P > 1$ and $\alpha_{P-1} = \alpha_{\max}$.

Within a cycle, the order of the application of the parameters is not critical (Stone [13]), but e.g. for $P = 9$ Stone recommended the order $\alpha_8, \alpha_5, \alpha_2, \alpha_7, \alpha_4, \alpha_1, \alpha_6, \alpha_3, \alpha_0$. Stone also claimed that the value $\beta = 1$ is satisfactory in general.

In conclusion, we briefly estimate the storage requirements of the method. We need $5q$ words to store the matrix A and q words to store the right-hand part v. Further, we use q words for the approximate solution u_r. Finally, $5q$ words are required to compute and store the matrices $\hat{L}(\alpha)$ and $\hat{U}(\alpha)$ of (3.10). These factors are evaluated in each step anew. The total storage requirements thus are $12q$ words. Note that if we considered A a bandmatrix, the storage requirements for the exact factors L and U would be $O(qm)$.

In view of the values of q and P, it may be possible to accomplish the factorization (3.10) in advance for all α_p, $p = 0, \ldots, P-1$, and to store all the $2P$ results (two factorizations correspond to each double-step). The storage requirements are then much greater, but the amount of the words needed remains $O(q)$. Significant time savings result from this modification of the process.

4. THE SECOND ORDER PROPERTY OF THE STONE METHOD

The formula (3.6) shows that if $\partial^2 u/(\partial x\, \partial y) = 0$ in R and also all the third partial derivatives of the exact solution u vanish in R, then the cancellations (3.7) and (3.8) are complete for $\alpha = 1$. Thus the original five-point and the modified seven-point difference schemes are equivalent, i.e., the systems with the matrices A and $\hat{A}(1)$ have the same solution if u satisfies the above assumptions. A factorization possessing this property is called the second order method by Saylor [9]. This result can be formulated in the following way.

Theorem. *Consider the problem* (3.1) *and its discretization* (3.3). *Let* $\hat{A}(1)$ *be given by* (3.9). *If* $\partial^2 u/(\partial x\, \partial y) = 0$ *in* R *and all the third partial derivatives of the solution* u *vanish in* R, *then*

$$(\hat{A}(1) - A)\, u = 0\,. \tag{4.1}$$

Proof. The statement follows from the fact that the second and higher order terms in (3.6) vanish for the function u satisfying the hypothesis of the theorem and thus (3.7) and (3.8) vanish for $\alpha = 1$. Hence $\hat{A}(1)\, u = Au$.

Corollary. *Under the assumptions of the theorem, the Stone method described in Section* 3 *is a finite process for* $P = 1$, $\alpha_0 = 1$, $\beta = 1$ *and the initial approximation* $u_0 = 0$.

Proof. The first iteration step in (3.13) gives $t_1 = u_1$ so that

$$\hat{A}(1)\, t_1 = \hat{A}(1)\, u_1 = v\,.$$

On the other hand, (3.3) and (4.1) imply

$$\hat{A}(1)\, u - v = 0\,,$$

i.e. $u_1 = u$ since $\hat{A}\,(1)$ may be shown to be nonsingular [3].

Hence, for the problem whose exact solution u possesses the above property, the optimal choice of the parameters for the Stone method is $P = 1$, $\alpha_{\max} = \alpha_0 = 1$, and $\beta = 1$. This property of the exact solution cannot be verified a priori in practice. In practical computation, we are thus forced to employ some guesses for P, $\alpha_{\max}$, and β which may be based e.g. on (3.14).

5. NUMERICAL EXAMPLES

The author's experience with solving numerous two-dimensional elliptic problems (including problems with complex coefficients) has confirmed that the Stone method is not overly sensitive to the choice of the parameters involved (i.e., P, $\alpha_{\max}$ and β, whereas α_p is chosen according to (3.15)).

We will now present a numerical example of the application of the Stone method to a simple model problem.

Let $R = (0, 1) \times (0, 1)$. We solve the problem

$$\begin{aligned} -\Delta u(x, y) &= 0 \quad \text{in} \quad R\,, \\ u(x, y) &= x \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{5.1}$$

with the exact solution

$$u(x, y) = x \quad \text{in} \quad R \cup \Gamma\,.$$

This problem represents just the case considered in Section 4 when the Stone method is a finite process for $P = 1$, $\alpha_0 = 1$, $\beta = 1$ and $u_0 = 0$ and when it requires only one iteration step to obtain the solution of the linear algebraic system. The exact solution $u(x, y)$ is independent of y and all its second and higher order derivatives vanish. Further, the five-point finite-difference scheme is exact for this problem and the approximate solution is equal to the exact one except for round-off.

We present a comparison of numerical experiments performed for various choices of P, $\alpha_{\max}$ and β. We have used the equidistant square net with $m = n = 19$, that is, $q = 361$. We have employed the iterative process (3.13) with the initial vector $u_0 = 0$. Since $t_r = (t_r(x_1, y_1), t_r(x_2, y_1), \ldots, t_r(x_m, y_n))^{\mathrm{T}}$ in (3.12), we may write the stopping criterion: The process is finished after the rth step if

$$|t_r(x_j, y_k)| \leqq \varepsilon |u_r(x_j, y_k)|\,,$$

where $\varepsilon > 0$ is a parameter, holds for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$. This value of r is denoted by r_{stop}. We put $\varepsilon = 10^{-5}$ in single precision arithmetic.

Several tables illustrate the properties of the Stone factorization. Table 1 corresponds to the most straightforward use of the method, i.e. with $P = 1$, $\alpha_0 = 0$, and various values of β. For the problem (5.1), the optimal value of β lies near 1·6.

Tab. 1. $P = 1$, $\alpha_0 = 0$

β	0·9	1	1·5	1·59	1·6	1·61	1·62	1·65	1·7
r_{stop}	134	121	83	79	78	78	79	106	>300

Tab. 2. $\alpha_{\max} = 0{\cdot}9975$ (cf. (3.14) and (3.15)), $\beta = 1$

P	1	2	3	4	5	6	7
r_{stop}	74	23	17	15	17	15	17

In Table 2 we used the formulae (3.14) and (3.15), which are due to Stone [13], to determine $\alpha_{\max} = 0{\cdot}9975$ and α_p, $p = 0, \ldots, P - 1$; we have $\beta = 1$ and P varies from 1 to 7. The dependence of the rate of convergence on P is rather weak. The values of P between 3 and 7 give about the same results. With the very small number of iterations (15 to 17) it is of no use to test larger values of P. Hence, the exact choice of P is not of primary importance.

If $P = 1$, notice that for $\alpha_0 = 1$ one iteration step gives the solution of the linear algebraic system, whereas for $\alpha_0 = 0{\cdot}9975$ we needed 74 iteration steps to reach the accuracy prescribed.

Tab. 3. $\alpha_{\max} = 0{\cdot}9975$ (cf. (3.14) and (3.15))

	β	0·6	0·7	0·8	0·9	1·0	1·1	1·2	1·3	1·4	1·5	1·6
r_{stop}	$P = 4$	23	21	19	15	15	15	15	14	15	20	27
	$P = 5$	26	19	19	16	17	17	17	17	17	19	27

In Table 3 we use again $\alpha_{\max} = 0{\cdot}9975$ and compare the rate of convergence for various values of β, $P = 4$ and $P = 5$. The dependence of the rate of convergence on β is rather weak in the range from 0·9 to 1·4, which confirms that the choice of $\beta = 1{\cdot}0$ is satisfactory.

We have tried to test also the dependence of the rate of convergence on the order in which the values α_p are employed within a cycle. The order usually used for the

Tab. 4. $\alpha_{max} = 0{\cdot}9975$ (cf. (3.14) and (3.15)), $\beta = 1{\cdot}3$, $P = 4$

sequence of α's	3, 2, 1, 0	2, 3, 1, 0	3, 1, 2, 0	0, 2, 1, 3	0, 3, 1, 2	0, 1, 2, 3
r_{stop}	14	14	16	17	20	22

computation of the results presented was $\alpha_{P-1}, \ldots, \alpha_0$. For $P = 4$ and $\beta = 1{\cdot}3$ the number of iterations for various sequences of α's ranges between 14 and 22; some particular values are shown in Tab. 4. Notice that the order suggested by Stone [13] is $\alpha_3, \alpha_1, \alpha_2, \alpha_0$. For $P = 4$ and $\beta = 1{\cdot}6$ the number of iterations is 27 or 28 in all the cases, the dependence on the order of α_p thus being very weak.

Summarizing the experience with the choice of iteration parameters, we can say that the Stone incomplete factorization may be well used without a deeper a priori analysis of the problem solved. The computation of the electromagnetic field in some problems of geophysics carried out by the author confirmed this conclusion.

References

[1] *O. Axelsson:* Solution of linear systems of equations: iterative methods. Sparse Matrix Techniques. (Advanced Course, Copenhagen 1976.) Lecture Notes in Mathematics, Vol. 572. Springer-Verlag, Berlin 1977, 1—51.

[2] *I. Babuška, M. Práger, E. Vitásek:* Numerical Processes in Differential Equations. SNTL, Praha 1966.

[3] *A. Bracha-Barak, P. Saylor:* A symmetric factorization procedure for the solution of elliptic boundary value problems. SIAM J. Numer. Anal. 10 (1973), 190—206.

[4] *N. I. Buleev:* A numerical method for solving two- and three-dimensional diffusion equations. (Russian.) Mat. Sb. 51 (1960), 227—238.

[5] *T. Dupont, R. P. Kendall, H. H. Rachford:* An approximate factorization procedure for solving self-adjoint elliptic difference equations. SIAM J. Numer. Anal. 5 (1968), 559—573.

[6] *I. Gustafsson:* On first and second order symmetric factorization methods for the solution of elliptic difference equations. Res. Rep. 78.01 R, Dept. of Computer Sciences, Chalmers University of Technology and the University of Göteborg, Göteborg 1978.

[7] *D. S. Kershaw:* The incomplete Choleski-conjugate gradient method for iterative solution of systems of linear equations. J. Comput. Phys. 26 (1978), 43—65.

[8] *J. A. Meijerink, H. A. van der Vorst:* An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix. Math. Comp. 31 (1977), 148—162.

[9] *P. Saylor:* Second order strongly implicit symmetric factorization methods for the solution of elliptic difference equations. SIAM J. Numer. Anal. 11 (1974), 894—908.

[10] *K. Segeth:* The iterative use of fast algorithms for the solution of elliptic partial differential equations. (Lecture at the summer school Software a algoritmy numerické matematiky 4, Karlovy Vary 1981.) Matematický ústav ČSAV, Praha 1983.

[11] *K. Segeth:* Numerical experiments with the Stone incomplete triangular decomposition. Mathematical Models in Physics and Chemistry and Their Numerical Realization 3. (School-Seminar, Visegrád 1982.) To appear.

[12] *S. Selberherr, A. Schütz, W. Petzl:* MINIMOS — a two-dimensional MOS transistor singular analyser. IEEE J. Solid-State Circuits SC-15 (1980), 605—623.
[13] *H. L. Stone:* Iterative solution of implicit approximations of multidimensional partial differential equations. SIAM J. Numer. Anal. 5 (1968), 530—558.
[14] *R. J. Taranto:* Numerical studies of Stone's factorization and the iteration parameters, α and τ. Rep. 423, Dept. of Computer Science, University of Illinois, Urbana, Ill., 1971.

Souhrn

O VÝBĚRU ITERAČNÍCH PARAMETRŮ VE STONEOVĚ NEÚPLNÉ FAKTORIZACI

KAREL SEGETH

Článek se zabývá iteračním řešením řídkých soustav lineárních algebraických rovnic metodou Stoneovy neúplné faktorizace. Jako úvod se v odst. 2 uvažuje diskretizace jednorozměrné okrajové úlohy a přímé řešení vzniklé soustavy lineárních algebraických rovnic faktorizací. Iterační řešení dvourozměné okrajové úlohy, zkoumané v odst. 3, je zobecněním faktorizačního přístupu k řešení soustavy. Algoritmus Stoneovy neúplné faktorizace je pro přehlednost celý popsán v odst. 3 a některé vlastnosti této metody jsou odvozeny v odst. 4.

Závěrečný odstavec je věnován numerickým experimentům, jež jsou zaměřeny na volbu iteračních parametrů ve Stoneově metodě. Jak ukazuje zkoumaná modelová úloha, lze všeobecně říci, že vhodné hodnoty parametrů je možno úspěšně zvolit bez hlubší apriorní analýzy řešené soustavy lineárních algebraických rovnic.

Author's address: RNDr. *Karel Segeth*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.