Ivan Hlaváček

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OPTIMIZATION OF THE SHAPE OF AXISYMMETRIC SHELLS

IVAN HLAVÁČEK

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INTRODUCTION

Axisymmetric thin elastic shells of constant thickness are considered and the meridian curves of their middle surfaces taken for the design variable. Admissble functions are smooth curves of a given length, which are uniformly bounded together with their first and second derivatives, and such that the shell contains a given volume. The loading consists of the own weight, the hydrostatic pressure of a liquid and an external or internal pressure.

As a cost functional, the integral of the second invariant of the stress deviator on both surfaces of the shell is chosen.

In Section 1 we formulate an abstract optimal design problem and prove the existence of a solution. Section 2 contains the application of the abstract existence theorem to the design of axisymmetric shells. In Section 3 we introduce some approximate optimal design problems and in Section 4 we study the convergence of the approximate solutions. Some comments on the numerical solution of the approximate design problem are given in Section 5.

1. AN ABSTRACT OPTIMAL DESIGN PROBLEM

First we establish a general existence result for a class of optimal design problems.

Let U be a Banach space of controls and U_{ad} a set of admissible design variables. Assume that U_{ad} is compact in U.

Let a Hilbert space V be given with a norm $\|\cdot\|$. Consider a bilinear form $a(F; \cdot, \cdot)$ and a linear continuous functional $\langle f(F), \cdot \rangle$ on V, both depending on a parameter $F \in U$. Assume that there exist positive constants α_0, α_1 and a subset U^0. $U_{ad} \subset U^0 \subset$ $\subset U$, independent of F, u, v and such that

(1) $$a(F; u, v) \leqq \alpha_1 \|u\| \, \|v\| ,$$

(2) $$a(F; v, v) \geqq \alpha_0 \|v\|^2$$

hold for all $F \in U^0$ and $u, v \in V$.

Moreover, assume that:

(3) if $F, F_n \in U^0$, $F_n \to F$ in U and $u_n \rightharpoonup u$ (weakly) in V for $n \to \infty$.

then

$$a(F_n; u_n, v) \to a(F; u, v) \quad \forall v \in V;$$

(4) if $F, F_n \in U^0$, $F_n \to F$ in U, then $\langle f(F_n), v\rangle \to \langle f(F), v\rangle \quad \forall v \in V$;

(5) a positive constant γ exists, independent of F, v and such that

$$|\langle f(F), v\rangle| \leqq \gamma \|v\|$$

holds for all $F \in U^0$ and $v \in V$.

We consider the following *state problem*:

for $F \in U_{ad}$ find $u(F) \in V$ such that

(6) $$a(F; u(F), v) = \langle f(F), v\rangle \quad \forall v \in V.$$

Under the assumptions (1), (2), (5), the state problem (6) is uniquely solvable for any $F \in U^0$.

Let a functional

$$j : (U \times V) \to R,$$

be given, which satisfies the following condition

(7) $$\text{if } F_n, F \in U^0,\ F_n \to F \text{ in } U,\ u_n \rightharpoonup u \text{ in } V \text{ (weakly)} \Rightarrow \liminf j(F_n, u_n) \geqq j(F, u).$$

Defining the *cost functional* as

$$\mathcal{J}(F) = j(F, u(F)),$$

where $u(F)$ denotes the solution of (6), we may consider the *optimal design problem*: find $F^0 \in U_{ad}$ such that

(8) $$\mathcal{J}(F^0) \leqq \mathcal{J}(F) \quad \forall F \in U_{ad}.$$

We are able to prove the following existence result.

Theorem 1. *Under the assumptions* (1) *to* (5) *and* (7), *the optimal design problem* (8) *has at least one solution.*

Proof. Let $\{F_n\} \subset U_{ad}$ be a minimizing sequence for $\mathcal{J}$, i.e.

(9) $$\lim_{n \to \infty} \mathcal{J}(F_n) = \inf_{F \in U_{ad}} \mathcal{J}(F).$$

Let us denote the solution of (6) by $u_n = u(F_n)$. Using (2), (6), (5), we may write

$$\alpha_0 \|u_n\|^2 \leqq a(F_n; u_n, u_n) = \langle f(F_n), u_n\rangle \leqq \gamma \|u_n\|.$$

Consequently, the sequence $\{u_n\}$ is uniformly bounded in V. Then there exist a subsequence $\{u_m\}$ and an element $u \in V$ such that

$$u_m \rightharpoonup u \text{ (weakly) in } V .$$

Since U_{ad} is compact in U, there exist a subsequence $\{F_k\}$ of $\{F_m\}$ and $F \in U_{ad}$ such that

$$F_k \to F \quad \text{in} \quad U .$$

Recall that

$$a(F_k; u_k, v) = \langle f(F_k), v\rangle \quad \forall v \in V .$$

Passing to the limit with $k \to \infty$ and using (3), (4), we obtain

$$a(F; u, v) = \langle f(F), v\rangle \quad \forall v \in V .$$

Consequently, $u = u(F)$ follows from the uniqueness of the solution of (6).

By virtue of (7) and (9) we have

$$\inf_{F \in U_{ad}} \mathcal{J}(F) = \lim_{k \to \infty} \inf \mathcal{J}(F_k) = \lim \inf j(F_k, u_k) \geqq j(F, u(F)) = \mathcal{J}(F)$$

and therefore F is a solution of the problem (8).

2. SHAPE OPTIMIZATION OF AN AXISYMMETRIC SHELL

We shall apply the abstract theorem to the optimal design of a shape in the case of axisymmetric problems for thin elastic shells.

Let z and r denote the axial and radial coordinates, respectively. We describe the meridian curve by means of two functions F and G as follows:

$$r = F(s) , \quad z = G(s) , \quad 0 \leqq s \leqq l ,$$

where s denotes the arc parameter and the length l is given. Denoting the derivatives by primes, we set

$$G'(s) = [1 - (F'(s))^2]^{1/2} .$$

Let us choose $U = C^{(1)}(\bar{I})$, $I = (0, l)$,

$$U_{ad} = \{F \in C^{(1),1}(\bar{I}) : r_0 \leqq F(s) \leqq r_1, \ |F'(s)| \leqq C_1 < 1 ,$$
$$|F''(s)| \leqq C_2 , \int_0^l F^2(s)\, G'(s)\, ds = C_3 ,$$

where r_0, r_1, C_1, C_2, C_3 are given positive constants.

The integral condition means that the volume of the shell is prescribed. $C^{(1),1}(\bar{I})$ is the space of continuously differentiable functions in $\bar{I}$, the derivatives of which are Lipschitzian.

Moreover, we define an auxiliary set

$$U^0 = \{F \in C^{(1)}(\bar{I}),\ \tfrac{1}{2}r_0 \leqq F(s) \leqq 2r_1,\ |F'| \leqq \tfrac{1}{2}(1 + C_1) < 1\}\,.$$

We shall use the linear theory of shells (see e.g. Zienkiewicz [1]-Chapt. 12) and formulate the equilibrium in terms of the displacement vector $\boldsymbol{u} = (u, w)$, where u is the meridional and w the normal displacement (see Fig. 1). Let us define the following system of strains

$$(10) \qquad \begin{aligned} N_1(\boldsymbol{u}) &= u'\,, & N_2(\boldsymbol{u}) &= (F'u + G'w)/F\,,\\ N_3(\boldsymbol{u}) &= -w''\,, & N_4(\boldsymbol{u}) &= -F'w'/F\,, \end{aligned}$$

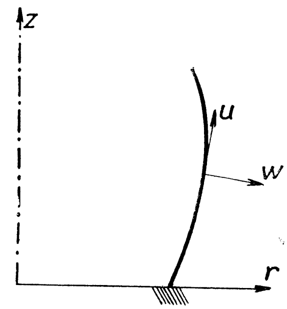

Fig. 1.

and the matrix

$$(11) \qquad K = \frac{Ee}{1 - \nu^2}\begin{bmatrix} 1 & \nu & 0 & 0\\ \nu & 1 & 0 & 0\\ 0 & 0 & e^2/12 & \nu e^2/12\\ 0 & 0 & \nu e^2/12 & e^2/12 \end{bmatrix},$$

where E is the Young modulus, e the (constant) thickness of the shell and ν Poisson's ratio $(0 \leqq \nu < 1/2)$.

We define

$$(12) \qquad a(F; \boldsymbol{u}, \boldsymbol{v}) = \int_I \sum_{i,j=1}^{4} K_{ij}\, N_i(\boldsymbol{u})\, N_j(\boldsymbol{v})\, F\, \mathrm{d}s\,,$$

$$(13) \qquad \langle f(F), \boldsymbol{u}\rangle = \int_I [k_0 w(G(l) - G(s)) + k_1(F'w - G'u) + k_3 w]\, F\, \mathrm{d}s\,,$$

where k_0 and k_1 are positive constants denoting the specific weight of a liquid and of the shell, respectively. The first part of the loading corresponds to the volume of the shell full of the liquid. The constant k_3 denotes an internal or external pressure.

Henceforth $H^k(I)$, $k = 1, 2$, denote the usual Sobolev spaces with square-integrable derivatives and $\|\cdot\|_k$ their norms. The norm in $L^2(I)$ will be denoted by $\|\cdot\|_0$ and the norm in $L_\infty(I)$ by $\|\cdot\|_\infty$. Let us consider the space

$$W = H^1(I) \times H^2(I)$$

and write for brevity $\|\boldsymbol{u}\| = \|\boldsymbol{u}\|_W = (\|u\|_1^2 + \|w\|_2^2)^{1/2}$.

We introduce the subspaces

$$V = \{\mathbf{u} = (u, w) \in W : u(0) = w(0) = w'(0) = 0\}, \tag{14}$$

$$\mathscr{P} = \{\mathbf{u} \in V : N_i(\mathbf{u}) = 0,\ i = 1, 2, 3, 4\}.$$

The boundary conditions in V correspond to the clamped edge $s = 0$. The subspace $\mathscr{P}$ represents the virtual displacements of a rigid shell.

It is easy to see that $\mathscr{P} = \{0\}$. In fact,

$$N_1(\mathbf{u}) = 0 \Rightarrow u = u_0 = \text{const.}, \tag{15}$$

$$N_3(\mathbf{u}) = 0 \Rightarrow w = w_0 + w_1 s, \quad w_0, w_1 = \text{const.}$$

Inserting the boundary conditions, we arrive at $u_0 = w_0 = w_1 = 0$.

If we define $a(F; \mathbf{u}, \mathbf{v})$ and $\langle f(F), \mathbf{v}\rangle$ by the formulas (10), (11), (12), (13) and V by (14), the state problem (6) corresponds to the equilibrium of a shell, the lower edge of which is clamped and the upper edge free, under the combined effect of its own weight, of the weight of a liquid and of a pressure.

Lemma 1. *The form a and the functional f satisfy the conditions* (1), (2), (3), (4), (5).

Proof. By virtue of the definition of U^0, the condition (1) is easy to see.

To prove the inequality (2), we first realize that K is positive definite, i.e. $\mathbf{x}^{\mathrm{T}}K\mathbf{x} \geqq \varkappa\mathbf{x}^{\mathrm{T}}\mathbf{x}\ \forall \mathbf{x} \in R^4$, $\varkappa > 0$, and we may write

$$a(F; \mathbf{u}, \mathbf{u}) \geqq \tfrac{1}{2}\varkappa r_0 \int_I [N_1^2(\mathbf{u}) + N_3^2(\mathbf{u})]\,\mathrm{d}s \quad \forall F \in U_{ad}, \quad \mathbf{u} \in V. \tag{16}$$

By virtue of (15) and the boundary conditions, we have

$$\int_I [(u')^2 + (w'')^2]\,\mathrm{d}s \geqq C\|\mathbf{u}\|^2 \quad \forall \mathbf{u} \in V \tag{17}$$

with $C > 0$ independent of $\mathbf{u}$ (see e.g. [2] – Chapt. 11, Lemma 3.2). Combining (16) and (17), we obtain (2).

Let us prove the condition (3). We may write

$$|a(F_n; \mathbf{u}_n, \mathbf{v}) - a(F; \mathbf{u}, \mathbf{v})| \leqq \tag{18}$$
$$= |a(F_n; \mathbf{u}_n, \mathbf{v}) - a(F; \mathbf{u}_n, \mathbf{v})| + |a(F; \mathbf{u}_n, \mathbf{v}) - a(F; \mathbf{u}, \mathbf{v})|,$$

$$|a(F_n; \mathbf{u}_n, \mathbf{v}) - a(F, \mathbf{u}_n, \mathbf{v})| = \tag{19}$$
$$= \int_0^l |N^{\mathrm{T}}(\mathbf{u}_n, F_n)\,KN(\mathbf{v}, F_n)\,F_n - N^{\mathrm{T}}(\mathbf{u}_n, F_n)\,KN(\mathbf{v}, F_n)\,F|\,\mathrm{d}s +$$
$$+ \int_0^l |N^{\mathrm{T}}(\mathbf{u}_n, F_n)\,KN(\mathbf{v}, F_n)\,F - N^{\mathrm{T}}(\mathbf{u}_n, F)\,KN(\mathbf{v}, F)\,F|\,\mathrm{d}s.$$

For the first integral we have

$$\int_0^l |N^{\mathrm{T}}(\boldsymbol{u}_n, F_n)\, K N(\mathbf{v}, F_n)| \,.\, |F_n - F| \,\mathrm{d}s \leqq$$

$$\leqq C\|F_n - F\|_{C(I)} \left[\int_0^l \sum_{j=1}^4 N_j^2(\boldsymbol{u}_n, F_n)\,\mathrm{d}s\right]^{1/2} \left[\int_0^l \sum_{j=1}^4 N_j^2(\mathbf{v}, F_n)\,\mathrm{d}s\right]^{1/2} \to 0\,,$$

since

$$(20) \qquad \sum_{j=1}^4 \|N_j(\boldsymbol{u}_n, F_n)\|_0^2 \leqq C\|\boldsymbol{u}_n\|^2 \leqq \tilde{C} \quad \forall n\,, \quad \forall F_n \in U^0$$

can be written on the basis of the weak convergence of $\boldsymbol{u}_n$.

For the second integral we have the following upper bound:

$$(21) \qquad \int_0^l \{|N^{\mathrm{T}}(\boldsymbol{u}_n, F_n)\, K N(\mathbf{v}, F_n) - N^{\mathrm{T}}(\boldsymbol{u}_n, F_n)\, K N(\mathbf{v}, F)| +$$

$$+ |N^{\mathrm{T}}(\boldsymbol{u}_n, F_n)\, K N(\mathbf{v}, F) - N^{\mathrm{T}}(\boldsymbol{u}_n, F)\, K N(\mathbf{v}, F)|\}\,\mathrm{d}s =$$

$$= \int_0^l \{|N^{\mathrm{T}}(\boldsymbol{u}, F_n)\, K(N(\mathbf{v}, F_n) - N(\mathbf{v}, F))| +$$

$$+ |(N^{\mathrm{T}}(\boldsymbol{u}_n, F_n) - N^{\mathrm{T}}(\boldsymbol{u}_n, F))\, K N(\mathbf{v}, F)|\}\,\mathrm{d}s \leqq$$

$$\leqq C\left[\int_0^l \sum_{j=1}^4 N_j^2(\boldsymbol{u}_n, F_n)\,\mathrm{d}s\right]^{1/2} \left[\int_0^l \sum_{j=1}^4 (N_j(\mathbf{v}, F_n) - N_j(\mathbf{v}, F))^2\right]^{1/2} +$$

$$+ C\left[\int_0^l \sum_{j=1}^4 (N_j(\boldsymbol{u}_n, F_n) - N_j(\boldsymbol{u}_n, F))^2\right]^{1/2} \left[\int_0^l \sum_{j=1}^4 N_j^2(\mathbf{v}, F)\right]^{1/2}.$$

From (10) we deduce that

$$(22) \qquad \int_0^l [N_2(\boldsymbol{u}_n, F_n) - N_2(\boldsymbol{u}_n, F)]^2 \,\mathrm{d}s \leqq$$

$$\leqq \int_0^l \left[|u_n| \,.\, \left|\frac{F_n'}{F_n} - \frac{F'}{F}\right| + |w_n| \,.\, \left|\frac{G_n'}{F_n} - \frac{G'}{F}\right|\right]^2 \mathrm{d}s \to 0\,,$$

since

$$\|u_n\|_0^2 + \|w_n\|_0^2 \leqq \|\boldsymbol{u}_n\|^2 < C$$

and

$$\lim_{n\to\infty} \left\|\frac{F_n'}{F_n} - \frac{F'}{F}\right\|_{C(I)} = 0\,, \quad \lim_{n\to\infty} \left\|\frac{G_n'}{F_n} - \frac{G'}{F}\right\|_{C(I)} = 0$$

holds if $F_n \to F$ in $C^{(1)}(\bar{I})$.

In a parallel way, we obtain

$$(23) \qquad \int_0^l [N_4(\boldsymbol{u}_n, F_n) - N_4(\boldsymbol{u}_n, F)]^2 \,\mathrm{d}s \leqq \int_0^l |w_n'|^2 \,.\, \left|\frac{F_n'}{F_n} - \frac{F'}{F}\right|^2 \mathrm{d}s \to 0\,.$$

Inserting (22), (23) and analogous relations with $\mathbf{u}_n$ replaced by $\mathbf{v}$ into (21) and using also (20), we are led to the assertion that the second integral in (19) tends to zero.

For any fixed $\mathbf{v} \in V$ and $F \in U^0$, the functional

$$\mathbf{u} \to a(F; \mathbf{u}, \mathbf{v})$$

is linear and continuous on V, as follows from (1). Consequently,

$$|a(F; \mathbf{u}_n, \mathbf{v}) - a(F; \mathbf{u}, \mathbf{v})| \to 0 . \tag{24}$$

Inserting (24) into (18) and using the above results for (19), we can verify the condition (3).

To prove the condition (4), we first realize that

$$\|G_n' - G'\|_{C(I)} \leqq C\|F_n' - F'\|_{C(I)} \to 0 \tag{25}$$

holds if $F_n \in U^0$, $F_n \to F$ in $C^{(1)}(\bar{I})$.

Then we have also

$$|G_n(l) - G_n(s) - (G(l) - G(s))| \leqq \tag{26}$$
$$\leqq \int_s^l |G_n' - G'| \, dt \leqq l\|G_n' - G'\|_{C(I)} \to 0 .$$

For any $\mathbf{v} = (u, w)$ we may write

$$|\langle f(F_n), \mathbf{v}\rangle - \langle f(F), \mathbf{v}\rangle| =$$
$$= \left|\int_0^l \{k_0 w[(G_n(l) - G_n(s)) F_n - (G(l) - G(s)) F] + \right.$$
$$\left. + k_1 w(F_n' F_n - F'F) - k_1 u(G_n' F_n - G'F) + k_3 w(F_n - F)\} \, ds \right| .$$

Using (25), (26) and the convergence of F_n in $C^{(1)}$, the condition (4) follows.

The condition (5) is an immediate consequence of the definition of U^0 and (13).

Lemma 2. *The set U_{ad} is compact in $C^{(1)}(\bar{I})$.*

Proof. Since the functions from U_{ad} are uniformly bounded and uniformly continuous, we apply Arzelà's theorem. In every sequence there is a subseuqnece $\{F_n\} \subset U_{ad}$ such that $F_n \to F$ uniformly on $[0, l]$. It is easy to see that F fulfils the condition $|F'| \leqq C_1$.

Since the derivatives F_n' are uniformly bounded and uniformly continuous, there exist a function H and a subsequence $\{F_m'\}$ such that $F_m' \to H$ uniformly on $[0, l]$. Using a classical theorem, we obtain $H = F'$, so that $F_m \to F$ in $C^{(1)}(\bar{I})$. Moreover, $|F''| \leqq C_2$ and

$$C_3 = \lim_{m \to \infty} \int_0^l F_m^2 G_m' \, ds = \int_0^l F^2 G' \, ds$$

follows. Q.E.D.

Next we define the cost functional. As in [3], let it be related to the second invariant of the stress tensor deviator (intensity of the shear stress or the von Mises equivalent stress)

$$I_2(\sigma) = \tfrac{4}{3}(\sigma_s^2 + \sigma_\vartheta^2 - \sigma_s\sigma_\vartheta)\,, \tag{27}$$

where σ_s and σ_ϑ denote the meridional and the circumferential normal stress, respectively. Thus we define

$$j(F, \boldsymbol{u}) = \int_0^l \sigma^{\mathrm{T}}(\boldsymbol{u})\, C\, \sigma(\boldsymbol{u})\, F\, \mathrm{d}s\,, \tag{28}$$

where

$$\sigma(\boldsymbol{u}) = \begin{bmatrix} \sigma_s^i \\ \sigma_s^e \\ \sigma_\vartheta^i \\ \sigma_\vartheta^e \end{bmatrix} = HKN(\mathrm{u}), \quad H = \begin{bmatrix} 1/e & 0 & -6/e^2 & 0 \\ 1/e & 0 & 6/e^2 & 0 \\ 0 & 1/e & 0 & -6/e^2 \\ 0 & 1/e & 0 & 6/e^2 \end{bmatrix},$$

the superscripts i and e denote that the stress is considered on the internal and external surface of the shell, respectively;

$$C = \begin{bmatrix} \beta_i(s) & 0 & -\frac{1}{2}\beta_i(s) & 0 \\ 0 & \beta_e(s) & 0 & -\frac{1}{2}\beta_e(s) \\ -\frac{1}{2}\beta_i(s) & 0 & \beta_i(s) & 0 \\ 0 & -\frac{1}{2}\beta_e(s) & 0 & \beta_e(s) \end{bmatrix},$$

where $\beta_i(s)$, $\beta_e(s)$ are (positive, bounded) weight functions.

Note that

$$j(F, \boldsymbol{u}) = \tfrac{3}{4}\int_0^l (\beta_i I_2(\sigma^i(\boldsymbol{u})) + \beta_e I_2(\sigma^e(\boldsymbol{u}))\, r\, \mathrm{d}s\,. \tag{29}$$

Lemma 3. *The cost functional* (28) *satisfies the condition* (7).

Proof. We write

$$j(F_n, \boldsymbol{u}_n) - j(F, \boldsymbol{u}) = (j(F_n, \boldsymbol{u}_n) - j(F, \boldsymbol{u}_n)) + (j(F, \boldsymbol{u}_n) - j(F, \boldsymbol{u}))\,. \tag{30}$$

For any fixed $F \in U^0$ the functional $j(F, \cdot)$ is weakly lower semicontinuous in V. Indeed, it is differentiable and convex, since

$$D^2 j(F; \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 2\int_0^l \sigma^{\mathrm{T}}(\boldsymbol{v})\, C\, \sigma(\boldsymbol{v})\, F\, \mathrm{d}s = 2j(F, \boldsymbol{v})\,.$$

Combining (29) with positive definiteness of the form (27), we conclude that $j(F, \boldsymbol{v})$ is non-negative. Consequently,

$$\liminf_{n\to\infty} (j(F, \boldsymbol{u}_n) - j(F, \boldsymbol{u})) \geqq 0 \tag{31}$$

provided $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$.

Denoting

$$M = KH^{\mathrm{T}}CHK\,,$$

we may write

$$(32) \qquad |j(F_n, \mathbf{u}_n) - j(F, \mathbf{u}_n)| = \left|\int_0^l [N^{\mathrm{T}}(\mathbf{u}_n, F_n)\, MN(\mathbf{u}_n, F_n)\, F_n - N^{\mathrm{T}}(\mathbf{u}_n, F)\, MN(\mathbf{u}_n, F)\, F\, \mathrm{d}s\right| \leqq \int_0^l \{|N^{\mathrm{T}}(F_n)\, MN(F_n)\, F_n - N^{\mathrm{T}}(F_n)\, MN(F_n)\, F| + $$

$$+ |N^{\mathrm{T}}(F_n)\, MN(F_n)\, F - N^{\mathrm{T}}(F)\, MN(F)\, F|\}\, \mathrm{d}s \leqq$$

$$\leqq \int_0^l \{|N^{\mathrm{T}}(F_n)\, MN(F_n)|\, |F_n - F| + |N^{\mathrm{T}}(F_n)\, MN(F_n) - N^{\mathrm{T}}(F)\, MN(F)|\, |F|\}\, \mathrm{d}s\,.$$

Since the entries of M are bounded functions, we have

$$\int_0^l |N^{\mathrm{T}}(F_n)\, MN(F_n)|\, |F_n - F|\, \mathrm{d}s \leqq \|F_n - F\|_{C(I)}\, C \sum_{j=1}^4 \|N_j^2(\mathbf{u}_n, F_n)\|_0^2 \to 0\,,$$

where also (20) has been used.

The second part of the integral on the right-hand side of (32) has the following upeer bound:

$$(32') \quad 2r_1 \int_0^l \{|N^{\mathrm{T}}(F_n)\, M(N(F_n) - N(F))| + |(N^{\mathrm{T}}(F_n) - N^{\mathrm{T}}(F))\, MN(F)|\}\, \mathrm{d}s \leqq$$

$$\leqq r_1 C[\sum_1^4 \|N_j(F_n)\|_0^2]^{1/2}\, [\sum_1^4 \|N_j(F_n) - N_j(F)\|_0^2]^{1/2} +$$

$$+ r_1 C[\sum_1^4 \|N_j(F_n) - N_j(F)\|_0^2]^{1/2}\, [\sum_1^4 \|N_j(F)\|_0^2]^{1/2} \to 0\,,$$

by virtue of (20) and (22), (23).

Altogether, the right-hand side of (32) tends to zero. Combining this and (31) with (30), we obtain

$$\liminf\, (j(F_n, \mathbf{u}_n) - j(F, \mathbf{u})) \geqq \liminf\, (j(F, \mathbf{u}_n) - j(F, \mathbf{u})) \geqq 0\,. \qquad \text{Q.E.D.}$$

From Theorem 1 and Lemmas 1, 2, 3 one concludes the following assertion:

The optimal design problem (8), *where the data are defined as above, has at least one solution.*

3. APPROXIMATION BY FINITE ELEMENTS

The optimal design problem has to be solved approximately. To this end, we introduce the following approximate problem. Let N be a positive integer and $\mathscr{T}_h$ a partition of the interval $[0, l]$ into N subintervals $\Delta_k = [s_{k-1}, s_k]$ of the length $h = l/N$, $k = 1, 2, \ldots, N$; $s_0 = 0$, $s_N = l$. Let $P_m(\Delta_k)$ be the set of polynomials the order of which is at most m.

We define the following external approximations of the set U_{ad}:

$$U_{ad}^{h\varepsilon} = \{F_h \in C^{(1),1}(\bar{I}) : F_h|_{\Delta_k} \in P_3(\Delta_k)\,, \quad k = 1, 2, \ldots, N\,,$$

$$r_0 \leqq F_h(s_k) \leqq r_1\,, \quad |F_h'(s_k)| \leqq C_1\,, \quad |F_h''(s_k+)| \leqq C_2\,, \quad |F_h''(s_k-)| \leqq C_2\,,$$

$$k = 0, 1, \ldots, N\,,$$

$$|\sum_{k=1}^{N} h[F_h^2 G_h']_{s=\xi_k} - C_3| \leqq \varepsilon\}\,.$$

Here ε denotes a (small) positive constant; $F_h''(s_k \pm)$ denotes $\lim_{s\to s_k+} F_h''(s)$ and $\lim_{s\to s_k-} F_h''(s)$, respectively; $\xi_k = \frac{1}{2}(s_{k-1} + s_k)$.

Moreover, let us introduce

$$V_h = \{\boldsymbol{u} = (u, w) \in V : u|_{\Delta_k} \in P_1(\Delta_k),\ w|_{\Delta_k} \in P_3(\Delta_k)\ \forall k\}\,.$$

We shall employ some simple formulas of numerical integration and instead of $a(F_h; \boldsymbol{u}_h, \boldsymbol{v}_h)$ we introduce

(33) $$a_h(F_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{i,j=1}^{4} K_{ij} \sum_{k=1}^{N} A_{ij}^{hk}(F_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\,,$$

where

(34) $$A_{ij}^{hk} = h[N_i(F_h, \boldsymbol{u}_h)\, N_j(F_h, \boldsymbol{v}_h)\, F_h]_{s=\xi_k}$$

for $1 \leqq i, j \leqq 2$ and

$$A_{33}^{hk} = \int_{\Delta_k} w_h''(s)\, \delta\, w_h''(s)\, F_h(\xi_k)\, \mathrm{d}s\,,$$

$$A_{34}^{hk} = \int_{\Delta_k} w_h''(s)\, \delta\, w_h'(s)\, F_h'(\xi_k)\, \mathrm{d}s\,,$$

$$A_{43}^{hk} = \int_{\Delta_k} w_h'(s)\, \delta\, w_h''(s)\, F_h'(\xi_k)\, \mathrm{d}s\,,$$

$$A_{44}^{hk} = \int_{\Delta_k} w_h'(s)\, \delta\, w_h'(s)\, (F_h'(\xi_k))^2\, (F_h(\xi_k))^{-1}\, \mathrm{d}s$$

with $\boldsymbol{u}_h = (u_h, w_h)$, $\boldsymbol{v}_h = (\delta u_h, \delta w_h)$.

Instead of the functional $\langle f(F_h), \boldsymbol{u}_h\rangle$ – see (13) – we introduce

(36) $$\langle f_h(F_h), \boldsymbol{u}_h\rangle = \sum_{k=1}^{N} h[k_0 w_h \tilde{G}_h + k_1(F_h' w_h - G_h' u_h) + k_3 w_h]_{s=\xi_k}\, F_h(\xi_k)\,,$$

where

(37) $$\tilde{G}_h(\xi_k) = \sum_{m=k+1}^{N} h\, G_h'(\xi_m)\,.$$

We also introduce the approximate functional (assuming $\beta_i =$ const., $\beta_e =$ = const.)

(38) $$j_h(F_h, \boldsymbol{u}_h) = \sum_{i,j=1}^{4} M_{ij} \sum_{k=1}^{N} A_{ij}^{hk}(F_h; \boldsymbol{u}_h, \boldsymbol{u}_h)\,,$$

where $M = KH^{\mathrm{T}}CHK$ (cf. the proof of Lemma 3) and

$$A^{hk}_{13}(F_h;\, \boldsymbol{u}_h, \boldsymbol{u}_h) = \int_{\Delta_k} -u'_h w''_h\, F_h(\xi_k)\,\mathrm{d}s = A^{hk}_{31}\,,$$

$$A^{hk}_{14}(F_h;\, \boldsymbol{u}_h, \boldsymbol{u}_h) = \int_{\Delta_k} -u'_h w'_h\, F'_h(\xi_k)\,\mathrm{d}s = A^{hk}_{41}\,,$$

$$A^{hk}_{23}(F_h,\, \boldsymbol{u}_h, \boldsymbol{u}_h) = -h[(F'_h u_h + G'_h w_h)\, w''_h]_{s=\xi_k} = A^{hk}_{32}\,,$$

$$A^{hk}_{24}(F_h;\, \boldsymbol{u}_h, \boldsymbol{u}_h) = -\int_{\Delta_k} [(F'_h u_h + G'_h w_h)\, F'_h F_h^{-1}]_{s=\xi_k}\, w'_h\,\mathrm{d}s = A^{hk}_{42}\,.$$

The *approximate optimal design problem* will be defined as follows:

to find $F^0_h \in U^{h\varepsilon}_{ad}$ such that

$$(39) \qquad \mathscr{J}_h(F^0_h) \leqq \mathscr{J}_h(F_h) \quad \forall F_h \in U^{h\varepsilon}_{ad}\,,$$

where

$$\mathscr{J}_h(F_h) = j_h(F_h,\, \boldsymbol{u}_h(F_h))$$

and $\boldsymbol{u}_h(F_h) \in V_h$ solves the following *approximate state problem*:

$$(40) \qquad a_h(F_h;\, \boldsymbol{u}_h, \boldsymbol{v}_h) = \langle f_h(F_h);\, \boldsymbol{v}_h \rangle \quad \forall \boldsymbol{v}_h \in V_h\,.$$

Theorem 2. *The approximate optimal design problem has at least one solution for any sufficiently small h.*

Proof is based on several auxiliary lemmas.

Lemma 4. *Let $F_h \in U^{h\varepsilon}_{ad}$. Then*

$$(41) \qquad \|F''_h\|_\infty \leqq C_2\,,$$

$$(42) \qquad \|F'_h\|_\infty \leqq C_1 + \tfrac{1}{2}C_2 h\,,$$

$$(43) \qquad r_0 - \tfrac{1}{2}C_1 h - \tfrac{1}{4}C_2 h^2 \leqq F_h(s) \leqq r_1 + \tfrac{1}{2}C_1 h + \tfrac{1}{4}C_2 h^2 \quad \forall s \in \bar{I}$$

and there exist positive constants h_0 and C independent of h, F_h and such that

$$(44) \qquad \left|\int_0^l F_h^2 G'_h\,\mathrm{d}s - C_3\right| \leqq \varepsilon + Ch \quad \forall h \leqq h_0\,.$$

Proof. The estimate (41) follows from the linearity of F''_h in Δ_k. In any subinterval Δ_k we may write

$$|F'_h(s)| = \left|F'_h(s_j) + \int_{s_j}^{s} F''_h(t)\,\mathrm{d}t\right| \leqq C_1 + |s - s_j|\, C_2 \leqq C_1 + \tfrac{1}{2}hC_2\,,$$

taking for s_j the node closest to s.

In a parallel way, we have

$$F_h(s) = F_h(s_j) + \int_{s_j}^{s} F_h'(t)\,\mathrm{d}t \leqq r_1 + |s - s_j|\,(C_1 + \tfrac{1}{2}C_2 h) \leqq r_1 + \tfrac{1}{2}C_1 h + \tfrac{1}{4}C_2 h^2$$

and an analogous lower bound.

Using (41) and (42), the following estimates can be derived:

$$G_h' \geqq (1 - (C_1^0)^2)^{1/2} > 0\,, \quad |G_h''| = |F_h' F_h'' / G_h'| \leqq C$$

for sufficiently small $h \leqq h_0$. Consequently, for $h \leqq h_0$ we may write

$$\left| \int_0^l F_h^2 G_h'\,\mathrm{d}s - \sum_k h(F_h^k)^2\, G_h'^k \right| \leqq \sum_k \int_{\Delta_k} |F_h^2(s)\, G_h'(s) - (F_h^k)^2\, G_h'^k|\,\mathrm{d}s \leqq Ch$$

by virtue of the estimates

$$|G_h'(s) - G_h'^k| \leqq \tfrac{1}{2}h \|G_h''\|_\infty \leqq Ch\,,$$
$$|F_h(s) - F_h^k| \leqq \tfrac{1}{2}h \|F_h'\|_\infty \leqq Ch\,.$$

Here the superscript k denotes the value at the point $s = \xi_k$. Then we arrive at the estimate

$$\left| \int_0^l F_h^2 G_h'\,\mathrm{d}s - C_3 \right| \leqq \left| \int_0^l F_h^2 G_h'\,\mathrm{d}s - \sum_k h(F_h^k)^2\, G_k'^h \right| +$$
$$+ \left|\sum_k h(F_h^k)^2\, G_h'^k - C_3\right| \leqq Ch + \varepsilon \quad \forall h \leqq h_0\,.$$

Lemma 5. *Positive constants c, h_0 exist, independent of h, $\mathbf{u}_h$, $\mathbf{v}_h$, F_h, such that*

$$a_h(F_h;\, \mathbf{u}_h, \mathbf{u}_h) \geqq c\|\mathbf{u}_h\|^2$$

holds for all $F_h \in U_{ad}^{h\varepsilon}$, $\mathbf{u}_h \in V_h$, $h \leqq h_0$.

Proof. For sufficiently small h we may write $F_h \geqq \frac{1}{2}r_0$ by virtue of Lemma 4 and

(45) $$a_h(F_h;\, \mathbf{u}_h, \mathbf{u}_h) = \sum_{k=1}^{N} \left[\sum_{i,j=1}^{2} K_{ij} A_{ij}^{hk}(F_h;\, \mathbf{u}_h, \mathbf{u}_h) + \sum_{i,j=3}^{4} K_{ij} A_{ji}^{hk}(F_h;\, \mathbf{u}_h, \mathbf{u}_h) \right],$$

(46) $$\sum_{i,j=1}^{2} K_{ij} A_{ij}^{hk} = hF_h^k \sum_{i,j=1}^{2} K_{ij} N_i^k(\mathbf{u}_h)\, N_j^k(\mathbf{u}_h) \geqq$$
$$\geqq \tfrac{1}{2}hr_0\varkappa_1 \sum_{i=1}^{2} (N_i^k(\mathbf{u}_h))^2 \geqq \tfrac{1}{2}r_0\varkappa_1 \int_{\Delta_k} (u_h')^2\,\mathrm{d}s\,,$$

where $\varkappa_1$ is the minimal eigenvalue of the submatrix $[K_{ij}]_{i,j=1}^2$.

In a similar way, we obtain

(47) $$\sum_{i,j=3}^{4} K_{ij} A_{ij}^{hk} = \int_{\Delta_k} F_h^k \left[K_{33}(w_h'')^2 + 2K_{34} w_h'' \left(\frac{F_h'^k}{F_h^k}\, w_h' \right) + K_{44} \left(\frac{F_h'^k}{F_h^k}\, w_h' \right)^2 \right] \mathrm{d}s \geqq$$
$$\geqq \tfrac{1}{2}\varkappa_2 r_0 \int_{\Delta_k} (w_h'')^2\,\mathrm{d}s\,,$$

where $\varkappa_2$ denotes the minimal eigenvalue of the submatrix $[K_{ij}]_{i,j=3}^4$. Inserting (46), (47) in (45), we derive the estimate

$$a_h(F_h; \mathbf{u}_h, \mathbf{u}_h) \geqq c \int_0^l [(u_h')^2 + (w_h'')^2] \, ds .$$

Since $V_h \subset V$, we may employ (17) to complete the proof.

Lemma 6. *There exist positive constants C, h_0 independent of h, ε, F_h, $\mathbf{u}_h$, such that*

$$|\langle f_h(F_h); \mathbf{u}_h \rangle| \leqq C \|\mathbf{u}_h\|$$

holds for any $F_h \in U_{ad}^{h\varepsilon}$, $\mathbf{u}_h \in V_h$, $h \leqq h_0$.

Proof. For sufficiently small h we have $F_h \leqq 2r_1$ and

$$|\langle f_h(F_h), \mathbf{u}_h \rangle| \leqq$$

$$= 2r_1 \sum_{k=1}^N \int_{\Delta_k} [k_0 |w_h^k| \, |\tilde{G}_h^k| + k_1(|F_h'| \, |w_h^k| + |G_h'| \, |u_h^k|) + |k_3| \, |w_h^k|) \, ds \leqq$$

$$\leqq C \sum_{k=1}^N \int_{\Delta_k} (|w_h^k| + |u_h^k|) \, ds \leqq C(\|w_h\|_\infty + \|u_h\|_\infty) \int_{\Delta_k} ds \leqq$$

$$\leqq C(\|w_h\|_1 + \|u_h\|_1) \leqq C \|\mathbf{u}_h\| ,$$

where the Sobolev imbedding theorem and the following estimate has been used:

$$|\tilde{G}_h^k| = \sum_{m=k+1}^N |h G_h'(\xi_m)| \leqq \sum_{m=1}^N \int_{\Delta_m} |G_h'(\xi_m)| \, ds \leqq \sum_{m=1}^N \int_{\Delta_m} ds = l .$$

Lemma 7. *There exist positive constants C, h_0 independent of h, $\mathbf{u}_h$, $\mathbf{v}_h$ and such that*

$$|a_h(F_1; \mathbf{u}_h, \mathbf{v}_h) - a_h(F_2; \mathbf{u}_h, \mathbf{v}_h)| \leqq C \|F_1 - F_2\|_{C^1(I)} \|\mathbf{u}_h\| \, \|\mathbf{v}_h\| , \tag{48}$$

$$|\langle f_h(F_1, \mathbf{u}_h \rangle - \langle f_h(F_2), \mathbf{u}_h \rangle| \leqq C \|F_1 - F_2\|_{C^1(I)} \|\mathbf{u}_h\| \tag{49}$$

holds for any $\mathbf{u}_h, \mathbf{v}_h \in V_h$, $F_1, F_2 \in U^0$, $h \leqq h_0$.

Proof. We have

$$|a_h(F_1; \mathbf{u}_h, \mathbf{v}_h) - a_h(F_2; \mathbf{u}_h, \mathbf{v}_h)| \leqq$$

$$\leqq \sum_{i,j=1}^4 K_{ij} \sum_{k=1}^N |A_{ij}^{hk}(F_1; \mathbf{u}_h, \mathbf{v}_h) - A_{ij}^{hk}(F_2; \mathbf{u}_h, \mathbf{v}_h)| .$$

Let us show the estimate for $i = 1$, $j = 2$ in detail.

$$\sum_{k=1}^N |A_{12}^{hk}(F_1; \mathbf{u}_h, \mathbf{v}_h) - A_{12}^{hk}(F_2; \mathbf{u}_h, \mathbf{v}_h)| \leqq$$

$$\leqq \sum_{k=1}^N \int_{\Delta_k} |u_h'| \, (|\delta u_h^k| \, |F_1'^k - F_2'^k| + |\delta w_h^k| \, |G_1'^k - G_2'^k|) \, ds \leqq$$

$$\leqq C\|F_1 - F_2\|_{C^1} \sum_k \int_{\Delta_k} |u_h'| \, (|\delta u_h^k| + |\delta w_h^k|) \, ds \leqq$$

$$\leqq C\|F_1 - F_2\|_{C^1} (\|\delta u_h\|_1 + \|\delta w_h\|_1) \int_I |u_h'| \, ds \leqq$$

$$\leqq C\|F_1 - F_2\|_{C^1} \|\mathbf{u}_h\| \, \|\mathbf{v}_h\| \, .$$

Derivation of the estimates for the other terms is similar.

To verify (49), we write

$$|\langle f_h(F_1), \mathbf{u}_h \rangle - \langle f_h(F_2), \mathbf{u}_h \rangle| =$$

$$= \sum_{k=1}^{N} \int_{\Delta_k} [k_0 w_h(F_1 \tilde{G}_1 - F_2 \tilde{G}_2) + k_1 w_h(F_1' F_1 - F_2' F_2) +$$

$$+ k_1 u_h(G_2' F_2 - G_1' F_1) + k_3 w_h(F_1 - F_2)]_{s=\xi_k} \, ds \leqq$$

$$\leqq C\|F_1 - F_2\|_{C^1} \sum_{k=1}^{N} \int_{\Delta_k} (|w_h^k| + |u_h^k|) \, ds \leqq$$

$$\leqq C\|F_1 - F_2\|_{C^1} (\|w_h\|_1 + \|u_h\|_1) \leqq C\|F_1 - F_2\|_{C^1} \|\mathbf{u}_h\| \, ,$$

where also the following estimates have been used:

$$|G_1'^k - G_2'^k| \leqq C\|F_1 - F_2\|_{C^1} \, ,$$

$$|\tilde{G}_1^k - \tilde{G}_2^k| \leqq \sum_{m=2}^{N} \int_{\Delta_m} |G_1'^m - G_2'^m| \, ds \leqq Cl\|F_1 - F_2\|_{C^1} \, .$$

Proof of Theorem 2.

1° The problem (40) has a unique solution for $h \leqq h_0$. In fact, the inequality (1) holds also for the bilinear form $a_h(F_h; \cdot, \cdot)$ as follows from Lemma 4.

By virtue of Lemmas 5 and 6, we may apply the Lax-Milgram Theorem in the space V_h.

2° The set $U_{ad}^{h\varepsilon}$ is compact in $C^1(\bar{I})$. Infact, $U_{ad}^{h\varepsilon}$ is a finite-dimensional, bounded set. Its closedness follows from the definition.

3° We show that the mapping $F_h \to \mathbf{u}_h(F_h)$ is continuous from $U_{ad}^{h\varepsilon}$ into V_h. Let h, $\mathscr{T}_h$, V_h be fixed. Consider a sequence of $F_h^n \in U_{ad}^{h\varepsilon}$, $n \to \infty$, such that

$$F_h^n \to F_h \quad \text{in} \quad C^1(\bar{I}) \, .$$

Consequently, $F_h \in U_{ad}^{h\varepsilon}$. Denote for brevity $F_h^n = F^n$, $F_h = F$, $\mathbf{u}_h(F^n) = \mathbf{u}^n$, $\mathbf{u}_h(F) = \mathbf{u}$. From Lemmas 5 and 6 we obtain

$$c\|\mathbf{u}^n\|^2 \leqq a_h(F^n; \mathbf{u}^n, \mathbf{u}^n) = \langle f_h(F^n), \mathbf{u}^n \rangle \leqq C\|\mathbf{u}^n\| \, ,$$

so that the sequence $\{\mathbf{u}^n\}$ is uniformly bounded.

By definition, we have

$$a_h(F^n; \mathbf{u}^n, \mathbf{v}) = \langle f_h(F^n), \mathbf{v} \rangle \, , \quad a_h(F; \mathbf{u}, \mathbf{v}) = \langle f_h(F), \mathbf{v} \rangle$$

for any $\mathbf{v} \in V_h$. For sufficiently small h, $U_{ad}^{h\varepsilon} \subset U^0$ holds. Inserting $\mathbf{v} = \mathbf{u} - \mathbf{u}^n$ and using Lemma 7, we may write

$$c\|\mathbf{u}^n - \mathbf{u}\|^2 \leqq a_h(F; \mathbf{u} - \mathbf{u}^n, \mathbf{u} - \mathbf{u}^n) = a_h(F; \mathbf{u}, \mathbf{u} - \mathbf{u}^n) -$$
$$- a_h(F^n, \mathbf{u}^n, \mathbf{u} - \mathbf{u}^n) + \{a_h(F^n; \mathbf{u}^n, \mathbf{u} - \mathbf{u}^n) - a_h(F; \mathbf{u}^n, \mathbf{u} - \mathbf{u}^n)\} =$$
$$= \langle f_h(F), \mathbf{u} - \mathbf{u}^n \rangle - \langle f_h(F^n), \mathbf{u} - \mathbf{u}^n \rangle + a_h(F^n; \mathbf{u}^n, \mathbf{u} - \mathbf{u}^n) -$$
$$- a_h(F; \mathbf{u}^n, \mathbf{u} - \mathbf{u}^n) \leqq C\|F - F^n\|_{C^1(\bar{I})} \|\mathbf{u} - \mathbf{u}^n\| (1 + \|\mathbf{u}^n\|) .$$

Since $\|\mathbf{u}^n\|$ are bounded and $F^n \to F$ in $C^1(\bar{I})$, $\mathbf{u}^n \to \mathbf{u}$ in V follows.

4° Let us show that $\mathcal{J}_h(F_h)$ is continuous in $U_{ad}^{h\varepsilon} \subset C^1(\bar{I})$. To this end, we use the abbreviations of the point 3° and write

$$|\mathcal{J}_h(F^n) - \mathcal{J}_h(F)| = |j_h(F^n, \mathbf{u}^n) - j_h(F, \mathbf{u})| =$$
$$= \Big| \sum_{i,j=1}^{4} M_{ij} \sum_{k=1}^{N} (A_{ij}^{hk}(F^n; \mathbf{u}^n, \mathbf{u}^n) - A_{ij}^{hk}(F; \mathbf{u}, \mathbf{u}))\Big| \leqq$$
$$\leqq \sum_{i,j=1}^{4} |M_{ij}| \sum_{k=1}^{N} (|A_{ij}^{hk}(F^n; \mathbf{u}^n, \mathbf{u}^n) - A_{ij}^{hk}(F^n; \mathbf{u}, \mathbf{u})| +$$
$$+ |A_{ij}^{hk}(F^n; \mathbf{u}, \mathbf{u}) - A_{ij}^{hk}(F; \mathbf{u}, \mathbf{u})|) \leqq$$
$$\leqq C((\|\mathbf{u}^n\| + \|\mathbf{u}\|) \|\mathbf{u}^n - \mathbf{u}\| + \|F^n - F\|_{C^1(\bar{I})} \|\mathbf{u}\|^2) ,$$

where an argument similar to that of Lemma 7 (48) has been used. Using also the results of the point 3°, we arrive at the continuity of $\mathcal{J}_h$.

5° The existence of a minimum follows from the continuity of $\mathcal{J}_h$ and the compactness of $U_{ad}^{h\varepsilon}$ in $C^1(\bar{I})$. Q.E.D.

4. CONVERGENCE OF THE APPROXIMATE SOLUTIONS

We can show that some subsequence of the approximate solutions converges to a function for which the cost functional is lower than for any $F \in U_{ad}$. To this end we introduce a new definition and establish several auxiliary lemmas.

Let us define

$$U_{ad}^{\varepsilon\delta} = \Big\{F \in C^{(1),1}(\bar{I}) : -\delta + r_0 \leqq F(s) \leqq r_1 + \delta , \quad |F^{(j)}(s)| \leqq C_j + \delta ,$$
$$j = 1, 2 \quad \forall s \in \bar{I} , \left| \int_0^l F^2 G' \, ds - C_3 \right| \leqq \varepsilon + \delta \Big\} ,$$

where ε and δ are (small) positive constants.

Lemma 8. *There exists a positive constant C independent of h, F, $\mathbf{u}_h$, $\mathbf{v}_h$ and such that*

(50) $$|a_h(F; \mathbf{u}_h, \mathbf{v}_h) - a(F; \mathbf{u}_h, \mathbf{v}_h)| \leqq Ch\|\mathbf{u}_h\| \|\mathbf{v}_h\| ,$$

(51) $$\left|\langle f_h(F), \mathbf{u}_h\rangle - \langle f(F); \mathbf{u}_h\rangle\right| \leqq Ch\|\mathbf{u}_h\|$$

holds for any $F \in U_{ad}^{\varepsilon\delta}$, $\mathbf{u}_h$, $\mathbf{v}_h \in V_h$ *provided* δ *is sufficiently small.*

Proof. Let us consider $\delta < \min(r_0, 1 - C_1)$. We have

$$\left|a_h(F; \mathbf{u}_h, \mathbf{v}_h) - a(F; \mathbf{u}_h, \mathbf{v}_h)\right| \leqq \sum_{i,j=1}^{4} K_{ij} \sum_{k=1}^{N(h)} \left|A_{ij}^{hk} - \int_{\Delta_k} N_i(\mathbf{u}_h)\, N_j(\mathbf{v}_h)\, F\, \mathrm{d}s\right|$$

and it suffices to estimate the particular terms individually. Inserting $\mathbf{u}_h = (u_h, v_h)$, $\mathbf{v}_h = (\delta u_h, \delta w_h)$ and realizing that the first components are piecewise linear, we may write for $i = 1$ and $j = 1, 2$:

(52) $$\sum_k \left|A_{1j}^{hk} - \int_{\Delta_k} u_h'\, N_j(\mathbf{v}_h)\, F\, \mathrm{d}s\right| = \sum_k \left|\int_{\Delta_k} u_h'(N_j^k(\mathbf{v}_h)\, F^k - N_j(\mathbf{v}_h)\, F)\, \mathrm{d}s\right| \leqq$$

$$\leqq \sum_k \int_{\Delta_k} |u_h'|\, |g(\xi_k) - g(s)|\, \mathrm{d}s \leqq \sum_k \|u_h'\|_{0,\Delta_k} \left(\int_{\Delta_k} |g(\xi_k) - g(s)|^2\, \mathrm{d}s\right)^{1/2},$$

where $g(s) = N_j(\mathbf{v}_h)\, F$. Using the estimate

(53) $$|g(s) - g(\xi_k)| \leqq (s - \xi_k)^{1/2} \left(\int_{\xi_k}^{s} |g'(t)|^2\, \mathrm{d}t\right)^{1/2} \leqq h^{1/2} \|g'\|_{0,\Delta_k}$$

we obtain the upper bound

(54) $$\sum_k \|u_h'\|_{0,\Delta_k}\, h \|g'\|_{0,\Delta_k} \leqq h \|u_h'\|_0 \cdot \|g'\|_0 \,.$$

Since $g' = F'\delta u_h'$ for $j = 1$,

$$g' = F'\delta u_h' + F''\delta u_h + G'\delta w_h' + G''\delta w_h \quad \text{for} \quad j = 2$$

and

$$1 \geqq G' = [1 - (F')^2]^{1/2} \geqq c > 0$$

holds for sufficiently small δ, we have

$$|G''| \leqq |F'| \cdot |F''|/G' \leqq (C_1 + \delta)(C_2 + \delta)\, c^{-1}$$

and consequently

(55) $$|g'| \leqq C(|\delta u_h| + |\delta u_h'| + |\delta w_h| + |\delta w_h'|)\,,$$

$$\int_I |g'|^2\, \mathrm{d}s \leqq C(\|\delta u_h\|_1^2 + \|\delta w_h\|_1^2) \leqq C\|\mathbf{v}_h\|^2\,.$$

Inserting (55) into (54), we obtain the upper bound $Ch\|\mathbf{u}_h\|\, \|\mathbf{v}_h\|$.
Next we may write

$$\sum_{k=1}^{N} \left|A_{22}^{hk} - \int_{\Delta_k} N_2(\mathbf{u}_h)\, N_2(\mathbf{v}_h)\, F\, \mathrm{d}s\right| = \sum_k \left|\int_{\Delta_k} \{u_h(s)\,[g_1(\xi_k) - g_1(s)] + \right.$$

$$\left. + \delta\, u_h(s)\,[g_2(\xi_k) - g_2(s)] + g_3(\xi_k) - g_3(s)\}\, \mathrm{d}s\right|,$$

where $$g_1 = (F'\delta u_h + G'\delta w_h) F^{-1}F' , \quad g_2 = F'G'F^{-1}w_h ,$$
$$g_3 = (G')^2 F^{-1} w_h \delta w_h .$$

Here we have again utilized the piecewise linearity of u_h, δu_h. It is not difficult to derive the estimates

$$\|g_1'\|_0 \leqq C\|\mathbf{v}_h\| , \quad \|g_2'\|_0 \leqq C\|\mathbf{u}_h\| , \quad \|g_3'\|_0 \leqq C\|\mathbf{u}_h\| \, \|\mathbf{v}_h\| .$$

Combining these results in a way similar to (52), (54), we obtain the desired bound.

We also have

$$\sum_{k=1}^{N} \left| A_{44}^{hk} - \int_{\Delta_k} N_4(\mathbf{u}_h) N_4(\mathbf{v}_h) F \,\mathrm{d}s \right| =$$
$$= \sum_k \left| \int_{\Delta k} w_h' \delta w_h' [(F'^k)^2 (F^k)^{-1} - (F')^2 F^{-1}] \,\mathrm{d}s \right| \leqq$$
$$\leqq \sum_k \int_{\Delta_k} |w_h'| \, |\delta w_h'| \, |g(\xi_k) - g(s)| \,\mathrm{d}s \leqq$$
$$\leqq \sum_k C\|w_h'\|_1 \, \|\delta w_h'\|_1 \, h^{3/2} \, \|g'\|_{0,\Delta_k} \leqq$$
$$\leqq C\|w_h\|_2 \, \|\delta w_h\|_2 \, h \|g'\|_0 ,$$

and $|g'| \leqq C$ holds for sufficiently small δ. Consequently, the upper bound can be again $Ch\|\mathbf{u}_h\| \, \|\mathbf{v}_h\|$. Similar arguments apply to the remaining terms and therefore (50) is true.

Furthermore,

$$|\langle f(F), \mathbf{u}_h \rangle - \langle f_h(F), \mathbf{u}_h \rangle| =$$
$$= \Big| \sum_{k=1}^{N} \int_{\Delta_k} \{[k_0 w_h(G(l) - G(s)) + k_1(F'w_h - G'u_h) + k_3 w_h] F -$$
$$- [k_0 w_h^k(G(l) - G^k) + k_1(F'^k w_h^k - G'^k u_h^k) + k_3 w_h^k] F^k\} \,\mathrm{d}s +$$
$$+ \sum_{k=1}^{N} \int_{\Delta_k} k_0 w_h^k [G(l) - G^k - \tilde{G}^k] F^k \,\mathrm{d}s \Big| .$$

Denoting

$$g = [k_0 w_h(G(l) - G) + k_1(F'w_h - G'u_h) + k_3 w_h] F ,$$

we can estimate the first sum from above as follows (cf. (53), (54)):

$$\sum_{k=1}^{N} \int_{\Delta_k} |g(s) - g(\xi_k)| \,\mathrm{d}s \leqq h l^{1/2} \|g'\|_0 .$$

Using the boundedness of G'' and the estimates

$$|g'| \leqq C(|u_h| + |u_h'| + |w_h| + |w_h'|) , \quad \|g'\|_0 \leqq C\|\mathbf{u}_h\| ,$$

we arrive at the desired upper bound for the first sum.

Using (53), we may write

$$|G(l) - G^k - \tilde{G}^k| = \left|\int_{\xi_k}^{l} G'(t)\,\mathrm{d}t - \sum_{m=k+1}^{N} hG'^m\right| =$$

$$= \left|\int_{\xi_k}^{s_k} G'\,\mathrm{d}t + \sum_{m=k+1}^{N} \int_{\Delta_m} (G'(t) - G'^m)\,\mathrm{d}t\right| \leqq$$

$$\leqq \int_{\xi_k}^{s_k} |G'|\,\mathrm{d}t + \sum_{m=2}^{N} \int_{\Delta_m} h|G''|\,\mathrm{d}s \leqq \tfrac{1}{2}h + Chl \leqq Ch\,.$$

Therefore the second sum can be estimated as follows:

$$\sum_{k=1}^{N} \int_{\Delta_k} |k_0 w_h^k [G(l) - G^k - \tilde{G}^k]\, F^k|\,\mathrm{d}s \leqq$$

$$\leqq Ch \sum_k \int_{\Delta_k} \|w_h\|_\infty\,\mathrm{d}s \leqq Ch\|w_h\|_2\, l \leqq Clh\|\mathbf{u}_h\|\,. \qquad \text{Q.E.D.}$$

Lemma 9. *Let $F \in U_{ad}^{\varepsilon 0}$ and a sequence $\{F_h\}$, $F_h \in U_{ad}^{h\varepsilon}$ be given such that* $\lim_{h\to 0} F_h = F$ *in $C^1(\bar{I})$. Let $\mathbf{u}_h(F_h)$ be the corresponding solutions of the problem* (40) *and $\mathbf{u}(F)$ the solution of* (6).

Then

$$\|\mathbf{u}_h(F_h) - \mathbf{u}(F)\| \to 0 \quad \textit{for} \quad h \to 0\,.$$

Proof. Denote for brevity $\mathbf{u}_h(F_h) = \mathbf{u}_h$, $\mathbf{u}(F) = \mathbf{u}$. By virtue of Lemmas 5, 6 we have for $h \leqq h_0$

$$c\|\mathbf{u}_h\|^2 \leqq a_h(F_h; \mathbf{u}_h, \mathbf{u}_h) = \langle f_h(F_h), \mathbf{u}_h\rangle \leqq C\|\mathbf{u}_h\|\,.$$

Consequently,

(56) $$\|\mathbf{u}_h\| \leqq C/c \quad \forall h \leqq h_0$$

and there exists a subsequence, denoted again by $\{\mathbf{u}_h\}$, such that

(57) $$\mathbf{u}_h \rightharpoonup \mathbf{u}^* \ \text{(weakly) in } V.$$

We shall show that $\mathbf{u}^*$ satisfies the condition (6). Let $\mathbf{v} \in V$ be arbitrary, $\mathbf{v} = (y, z)$, $y \in H^1(I)$, $z \in H^2(I)$. There exists a sequence of $\mathbf{v}_\varkappa = (y_\varkappa, z_\varkappa)$ such that $\mathbf{v}_\varkappa \in V \cap \cap [C^\infty(\bar{I})]^2$ and

(58) $$\|\mathbf{v}_\varkappa - \mathbf{v}\| \to 0 \quad \text{for} \quad \varkappa \to 0\,.$$

Let us construct the function

$$\varphi_h = R_h \mathbf{v}_\varkappa = (R_h^1 y_\varkappa, R_h^3 z_\varkappa)\,,$$

where $R_h^1 y_\varkappa$ denotes the linear Lagrange and $R_h^3 z_\varkappa$ the cubic Hermite interpolate on the mesh $\mathscr{T}_h$, respectively. Then $\varphi_h \in V_h$ and

(59) $$\|\varphi_h - \mathbf{v}_\varkappa\| = (\|R_h^1 y_\varkappa - y_\varkappa\|_1^2 + \|R_h^3 z_\varkappa - z_\varkappa\|_2^2)^{1/2} \leqq$$
$$\leqq Ch(\|y_\varkappa\|_2^2 + h^2\|z_\varkappa\|_4^2)^{1/2}\,.$$

Consequently, combining (58) and (59), we obtain

(60) $$\|\varphi_h - \mathbf{v}\| \to 0 \quad \text{for} \quad h \to 0 .$$

Inserting $\mathbf{v}_h = \varphi_h$ into (40),

$$a_h(F_h; \mathbf{u}_h, \varphi_h) = \langle f_h(F_h), \varphi_h \rangle$$

follows. This equation can be rewritten in the form

(61) $$a(F_h, \mathbf{u}_h, \varphi_h) + \{a_h(F_h; \mathbf{u}_h, \varphi_h) - a(F_h; \mathbf{u}_h, \varphi_h)\} =$$
$$= \langle f(F_h), \varphi_h \rangle + \{\langle f_h(F_h), \varphi_h \rangle - \langle f(F_h), \varphi_h \rangle\} .$$

It is easy to deduce that

(62) $$\lim_{h \to 0} a(F_h; \varphi_h) = a(F; \mathbf{u}^*, \mathbf{v}) .$$

In fact, since $U_{ad}^{h\varepsilon} \subset U^0$ for sufficiently small h,

$$|a(F_h; \mathbf{u}_h, \varphi_h) - a(F_h; \mathbf{u}_h, \mathbf{v})| \leqq \alpha_1 \|\mathbf{u}_h\| \, \|\varphi_h - \mathbf{v}\| \to 0$$

according to (1), (56) and (60);

$$|a(F_h; \mathbf{u}_h, \mathbf{v}) - a(F; \mathbf{u}^*, \mathbf{v})| \to 0$$

by virtue of (3) and (57). Combining these two results, we arrive at (62).

Moreover,

(63) $$\lim_{h \to 0} \langle f(F_h), \varphi_h \rangle = \langle f(F), \mathbf{v} \rangle$$

holds. It is a consequence of (5), (60) and (4), since

$$|\langle f(F_h), \varphi_h - \mathbf{v} \rangle| \leqq \gamma \|\varphi_h - \mathbf{v}\| \to 0 ,$$
$$|\langle f(F_h), \mathbf{v} \rangle - \langle f(F), \mathbf{v} \rangle| \to 0 .$$

By virtue of Lemma 8 and (60), (56), we have

(64) $$|a_h(F_h; \mathbf{u}_h, \varphi_h) - a(F_h; \mathbf{u}_h, \varphi_h)| \leqq Ch \|\mathbf{u}_h\| \, \|\varphi_h\| \to 0 ,$$

(65) $$|\langle f_h(F_h), \varphi_h \rangle - \langle f(F_h), \varphi_h \rangle| \leqq Ch \|\varphi_h\| \to 0 .$$

Passing to the limit with $h \to 0$ and using (62), (63), (64), (65), we arrive at

(66) $$a(F; \mathbf{u}^*, \mathbf{v}) = \langle f(F), \mathbf{v} \rangle .$$

Since $U_{ad}^{\varepsilon 0} \subset U^0$, (66) is uniquely solvable, $\mathbf{u}^* = \mathbf{u}(F)$ and the whole sequence converges: $\mathbf{u}_h \rightharpoonup \mathbf{u}$.

To prove the strong convergence $\mathbf{u}_h \to \mathbf{u}$ in V, it sufficies to show that $\|\mathbf{u}_h\| \to \|\mathbf{u}\|$. First we realize that

(67) $$a_h(F_h; \mathbf{u}_h, \mathbf{u}_h) = \langle f_h(F_h), \mathbf{u}_h \rangle$$

and

(68) $$\lim_{h \to 0} \langle f_h(F_h), \mathbf{u}_h \rangle = \langle f(F), \mathbf{u} \rangle = a(F; \mathbf{u}, \mathbf{u}) .$$

In fact,

$$(69) \qquad |\langle f(F_h), \mathbf{u}_h\rangle - \langle f(F), \mathbf{u}_h\rangle| \leqq C\|F_h - F\|_{C^1}\,\|\mathbf{u}_h\|$$

can be proved by an argument similar to that of (4) in Lemma 1.

Moreover,

$$(70) \qquad |\langle f(F), \mathbf{u}_h\rangle - \langle f(F), \mathbf{u}\rangle| \to 0$$

follows from the weak convergence of $\{\mathbf{u}_h\}$. Combining (70) and (69) with (56), we arrive at (68).

Since $U_{ad}^{\varepsilon 0} \subset U_{ad}^{\varepsilon\delta}$ for any $\delta > 0$, we may use (50) for F. Since $U_{ad}^{h\varepsilon} \subset U_{ad}^{\varepsilon\delta} \subset U^0$ follows from Lemma 4 for sufficiently small h and δ, (48) can also be emploeyd with $\mathbf{u}_h = \mathbf{v}_h$. Thus we obtain, using also (56), (67), (68), the following result:

$$(71) \qquad |a(F; \mathbf{u}_h, \mathbf{u}_h) - a(F; \mathbf{u}, \mathbf{u})| \leqq |a(F; \mathbf{u}_h, \mathbf{u}_h) - a_h(F; \mathbf{u}_h, \mathbf{u}_h)| +$$
$$+ |a_h(F; \mathbf{u}_h, \mathbf{u}_h) - a_h(F_h; \mathbf{u}_h, \mathbf{u}_h)| + |a_h(F_h; \mathbf{u}_h, \mathbf{u}_h) - a(F; \mathbf{u}, \mathbf{u})| \to 0\,.$$

By virtue of (1), (2), the bilinear form $a(F; \cdot, \cdot)$ can be introduced for the scalar product in V. Then (71) implies that the associated norms $\|\mathbf{u}_h\|_A$ tend to $\|\mathbf{u}\|_A$. Since the norms $\|\cdot\|$ and $\|\cdot\|_A$ are equivalent (see (1), (2)), combining the convergence of norms with the weak convergence, we deduce the strong convergence $\mathbf{u}_h \to \mathbf{u}$ in V.

Q.E.D.

Lemma 10. *Let the assumptions of Lemma 9 be satisfied. Then*

$$\lim_{h\to 0} \mathscr{J}_h(F_h) = \mathscr{J}(F)\,.$$

Proof. We may write

$$|\mathscr{J}_h(F_h) - \mathscr{J}(F)| = |j_h(F_h, \mathbf{u}_h(F_h)) - j(F, \mathbf{u}(F))| =$$
$$= |\sum_{i,j=1}^{4} M_{ij} \sum_{k=1}^{N} A_{ij}^{hk}(F_h; \mathbf{u}_h, \mathbf{u}_h) - \int_0^l N^{\mathsf{T}}(\mathbf{u}, F)\, MN(\mathbf{u}, F)\, F\, \mathrm{d}s| \leqq$$
$$\leqq |\sum_{i,j} M_{ij} \sum_k A_{ij}^{hk}(F_h; \mathbf{u}_h, \mathbf{u}_h) - \sum_k \int_{\Delta_k} N^{\mathsf{T}}(\mathbf{u}_h, F_h)\, MN(\mathbf{u}_h, F_h)\, F_h\, \mathrm{d}s| +$$
$$+ \int_I |N^{\mathsf{T}}(\mathbf{u}_h, F_h)\, MN(\mathbf{u}_h, F_h)\, F_h - N^{\mathsf{T}}(\mathbf{u}_h, F)\, MN(\mathbf{u}_h, F)\, F\, \mathrm{d}s +$$
$$+ \int_I |N^{\mathsf{T}}(\mathbf{u}_h, F)\, MN(\mathbf{u}_h, F)\, F - N^{\mathsf{T}}(\mathbf{u}, F)\, MN(\mathbf{u}, F)\, F|\, \mathrm{d}s\,.$$

Since $U_{ad}^{\varepsilon\delta} \supset U_{ad}^{h\varepsilon}$ for sufficiently small h, the argument of Lemma 8 (50) can be applied to the first term on the right-hand side, to obtain the upper bound $Ch\|\mathbf{u}_h\|^2 \leqq \tilde{C}h$ (by virtue of (56)).

Since $U_{ad}^{h\varepsilon} \subset U^0$ for h small enough, we can use the estimates parallel to (20), (22), (23), (32), (32′) to show that the second term tends to zero.

Finally,

$$\|N_j(\mathbf{u}_h, F) - N_j(\mathbf{u}, F)\|_0 \to 0$$

easily follows from Lemma 9. Hence the third term tends to zero as well and the proof is completed.

Lemma 11. *For any $F \in U_{ad}$ there exist a sequence $\{F_h\}$ and a positive constant $h_0(\varepsilon, F)$ such that $F_h \in U_{ad}^{h\varepsilon}$ $\forall h \leqq h_0(\varepsilon, F)$ and $F_h \to F$ in $C^1(\bar{I})$ for $h \to 0$.*

Proof.

1° Introduce a new coordinate $x = s - l/2$ and denote

$$F(x + l/2) = \tilde{F}(x),$$

$$F_\lambda(x) = \tilde{F}((1 - \lambda)\, x), \quad \lambda \in (0, 1).$$

Then F_λ is defined on the interval

$$I_\lambda = [(1 - \lambda)^{-1}\, l/2,\ (1 - \lambda)^{-1}\, l/2],$$

$$F_\lambda \in C^{(1),1}(I_\lambda) \quad \text{and} \quad r_0 \leqq F_\lambda(x) \leqq r_1, \quad |F_\lambda^{(j)}| \leqq (1 - \lambda)^j\, C_j,$$

$$j = 1, 2, \quad \forall x \in I_\lambda,$$

$$\|F_\lambda^{(j)} - \tilde{F}^{(j)}\|_{\infty, I} \leqq C\lambda, \quad (I = [-l/2, l/2]).$$

2° Applying the regularization

$$R_H f = \frac{1}{\varkappa H} \int_{-\infty}^{\infty} \omega_1(x - y, H)\, f(y)\, dy, \quad \text{where} \quad H = \text{const.} > 0,$$

$$\omega_1(z, H) = \exp\left(\frac{|z|^2}{|z|^2 - H^2}\right), \quad \text{if} \quad |z| < H, \quad \omega_1 = 0 \quad \text{if} \quad |z| \geqq H,$$

$$\varkappa H = \int_{|z|<H} \omega_1(z, H)\, dz,$$

we obtain

$$R_H F_\lambda \in C^\infty(\bar{I}),$$

$$r_0 \leqq R_H\, F_\lambda(x) \leqq r_1 \quad \forall x \in [-l/2, l/2],$$

$$|(R_H F_\lambda)^{(j)}(x)| = |R_H(F_\lambda^{(j)}(x)| \leqq \frac{1}{\varkappa H} \int_{-\infty}^{\infty} \omega_1(x - y, H) |F_\lambda^{(j)}|\, dy \leqq$$

$$\leqq (1 - \lambda)^j\, C_j, \quad \forall H \leqq \frac{l}{2} \frac{\lambda}{1 - \lambda}, \quad j = 1, 2.$$

Moreover, since $F_\lambda \in C^{(1),1}(\bar{I}) \subset W^{2,p}(I)$ $\forall p > 1$ and

$$\|f\|_{C(I)} \leqq C\|f\|_{W^{1,p}(I)},$$

we obtain for $j = 0, 1$

$$\|R_H F_\lambda^{(j)} - F_\lambda^{(j)}\|_{C(I)} \leqq C \|R_H F_\lambda - F_\lambda\|_{W^{2,p}(I)} \, .$$

The right-hand side tends to zero with $H \to 0$ and therefore

$$R_H F_\lambda \to F_\lambda \quad \text{in} \quad C^1(\bar{I}) \quad \text{for} \quad H \to 0 \, .$$

3° Let us define an auxiliary mapping $\mathscr{L}_\mu$ as follows:

$$\mathscr{L}_\mu f(x) = \frac{r_0 + r_1}{2} + (1 - \mu)\left(f(x) - \frac{r_0 + r_1}{2}\right),$$

where $\mu = \text{const.} > 0$. Then it is easy to see that

$$(\mathscr{L}_\mu f)^{(j)} = (1 - \mu) f^{(j)} \, , \quad j = 1, 2,$$

$$r_0 + \mu \frac{r_1 - r_0}{2} \leqq \mathscr{L}_\mu f \leqq r_1 - \mu \frac{r_1 - r_0}{2} \quad \text{for} \quad r_0 \leqq f \leqq r_1 \, ,$$

$$\|\mathscr{L}_\mu f - f\|_{\infty,I} = \mu \left\| \frac{r_0 + r_1}{2} - f \right\|_{\infty,I}$$

$$\|(\mathscr{L}_\mu f)' - f'\|_{\infty,I} = \mu \|f'\|_{\infty,I} \, .$$

4° Let us introduce the cubic spline interpolation Spf of f on the mesh $\mathscr{T}_h$ (see [4], [5]) with $(\text{Sp}f)'' = f''$ at the endpoints.

We define

$$F_h = \text{Sp}(\mathscr{L}_\mu R_H F_\lambda) \, ,$$

where

$$h \leqq \frac{C_1}{6C_2} \lambda \, , \quad H \leqq \tfrac{1}{2} l \lambda / (1 - \lambda) \, ,$$

$$\mu = 5 \omega(h, (R_H F_\lambda)'') / C_2 \, ,$$

$\omega(h, f)$ denoting the modulus of continuity of f on $\bar{I}$.

We shall utilize the error estimates

$$\|(\text{Sp}f)'' - f''\|_{\infty,I} \leqq 5 \omega(h, f'') \, ,$$

$$\|(\text{Sp}f)^{(j)} - f^{(j)}\|_{\infty,I} \leqq 12 \cdot 2^{j-2} h^{2-j} \|f''\|_{\infty,I} \, , \quad j = 0, 1$$

(see [5], Theorems 9, 10 in Chapter II).

Then we may write

$$\|F_h'' - (\mathscr{L}_\mu R_H F_\lambda)''\|_{\infty,I} \leqq 5 \omega(h, (R_H F_\lambda)'') = \mu C_2 \, ,$$

since

$$\omega(h, (\mathscr{L}_\mu R_H F_\lambda)'') = (1 - \mu) \, \omega(h, (R_H F_\lambda)'') \, ;$$

$$|F_h''| \leqq (1 - \mu) \, |(R_H F_\lambda)''| + \mu C_2 \leqq (1 - \mu)(1 - \lambda)^2 \, C_2 + \mu C_2 \leqq C_2 \, ,$$

$$\|F_h' - (\mathscr{Z}_\mu R_H F_\lambda)'\|_{\infty,I} \leqq 6C_2 h\,,$$

$$|F_h'| \leqq (1-\mu)\,|(R_H F_\lambda)'| + 6C_2 h \leqq (1-\mu)(1-\lambda)\,C_1 + C_1\lambda \leqq C_1\,.$$

At the nodal points we have

$$r_0 \leqq r_0 + \mu\,\frac{r_1 - r_0}{2} \leqq F_h = \mathscr{Z}_\mu R_H F_\lambda \leqq r_1 - \mu\,\frac{r_1 - r_0}{2} \leqq r_1\,.$$

Let us estimate the error

$$|F_h - \tilde{F}| \leqq |\mathrm{Sp}(\mathscr{Z}_\mu R_H F_\lambda) - \mathscr{Z}_\mu R_H F_\lambda| + |\mathscr{Z}_\mu R_H F_\lambda - R_H F_\lambda| +$$
$$+ |R_H F_\lambda - F_\lambda| + |F_\lambda - \tilde{F}| \leqq 3h^2(1-\mu)(1-\lambda)^2\,C_2 +$$
$$+ \mu\|\tfrac{1}{2}(r_0 + r_1) - R_H F_\lambda\|_{\infty,I} + \|R_H F_\lambda - F_\lambda\|_{\infty,I} + C\lambda\,.$$

Passing to the limit with $\lambda \to 0$, $H \to 0$, $h \to 0$ and $\mu \to 0$, we conclude that $F_h \to \tilde{F}$ in $C(\bar{I})$ for $h \to 0$. A parallel estimate is valid for $|F_h' - \tilde{F}'|$.

5° Using the convergence of F_h in C^1, we may write

(72)
$$\left|\int_I F_h^2 G_h'\,\mathrm{d}x - C_3\right| = \left|\int_I (F_h^2 G_h' - \tilde{F}^2 \tilde{G}')\,\mathrm{d}x\right| \leqq$$
$$\leqq \int_I (|F_h^2 - \tilde{F}^2|\,|G_h'| + \tilde{F}^2|G_h' - \tilde{G}'|)\,\mathrm{d}x \leqq$$
$$\leqq \int_I (2r_1|F_h - \tilde{F}| + r_1^2 C_1(1 - C_1^2)^{-1/2}\,|F_h' - \tilde{F}'|)\,\mathrm{d}x \leqq$$
$$\leqq C\|F_h - \tilde{F}\|_{C^1(I)} \to 0 \quad \text{if} \quad h \to 0\,.$$

Furthermore, we have

(73)
$$\left|\sum_{k=1}^N h(F_h^k)^2\,G_h'^k - \int_I F_h^2 G_h'\,\mathrm{d}x\right| \leqq$$
$$= \sum_k \int_{\Delta_k} |F_h^2(x)\,G_h'(x) - (F_h^k)^2\,G_h'^k|\,\mathrm{d}x \leqq Ch$$

by virtue of the estimates

$$G_h' \geqq (1 - C_1^2)^{1/2} > 0\,, \quad |G_h''| = |F_h' F_h''/G_h| \leqq C\,,$$
$$|G_h'(x) - G_h'^k| \leqq \tfrac{1}{2}h\|G_h''\|_{\infty,I} \leqq Ch\,,$$
$$|F_h(x) - F_h^k| \leqq \tfrac{1}{2}h\|F_h'\|_{\infty,I} \leqq Ch\,.$$

Combining (72) and (73) we obtain

$$\left|\sum_{k=1}^N h(F_h^k)^2\,G_h'^k - C_3\right| \leqq C(h + \|F_h - \tilde{F}\|_{C^1}) \leqq \varepsilon$$

for sufficiently small $h \leqq h_0(\varepsilon, F)$. Q.E.D.

Theorem 3. *Let $\{F_h\}$, $h \to 0$, be a sequence of solutions of the approximate optimal design problems* (39).

Then there exists subsequence $\{F_{\tilde{h}}\}$ such that

$$F_{\tilde{h}} \to F \quad in \quad C^1(\bar{I}), \tag{74}$$

$$\mathbf{u}_{\tilde{h}}(F_{\tilde{h}}) \to \mathbf{u}(F) \quad in \quad V \tag{75}$$

holds for $\tilde{h} \to 0$ and $F \in U_{ad}^{\varepsilon 0}$,

$$\mathscr{J}(F) \leqq \mathscr{J}(\Phi) \quad \forall \Phi \in U_{ad}. \tag{76}$$

Proof. Since $U_{ad}^{h\varepsilon} \subset U_{ad}^{\varepsilon\delta}$ $\forall h \leqq h_0(\delta)$ by virtue of Lemma 4, and $U_{ad}^{\varepsilon\delta}$ is compact in $C^1(\bar{I})$ (see the proof of Lemma 2), there exists a subsequence $\{F_{\tilde{h}}\}$, $F_{\tilde{h}} \in U_{ad}^{\tilde{h}\varepsilon}$ such that (74) holds, where $F \in U_{ad}^{\varepsilon\delta}$. On the other hand, using Lemma 4 and passing to the limit with $\tilde{h} \to 0$, we deduce $F \in U_{ad}^{\varepsilon 0}$.

Let us apply Lemma 11 to an arbitrary function $\Phi \in U_{ad}$. Consequently, there exists a sequence $\Phi_h \in U_{ad}^{h\varepsilon}$ such that

$$\Phi_h \to \Phi \quad \text{in} \quad C^1(\bar{I}) \quad \text{for} \quad h \to 0\,.$$

By definition, we have

$$\mathscr{J}_{\tilde{h}}(F_{\tilde{h}}) \leqq \mathscr{J}_{\tilde{h}}(\Phi_{\tilde{h}}) \quad \forall \tilde{h}\,. \tag{77}$$

Since $U_{ad} \subset U_{ad}^{\varepsilon 0}$, Lemma 9 and 10 hold for both the sequences $\{F_{\tilde{h}}\}$ and $\{\Phi_{\tilde{h}}\}$. Passing to the limit in (77), we obtain (76). Q.E.D.

Remark. A question arises whether $\varepsilon = \varepsilon(h)$ in Theorem 3 can be chosen so that $\varepsilon(h) \to 0$ for $h \to 0$, $F_{\tilde{h}} \to F \in U_{ad}$ and (76) hold. Unfortunately, I was not able to solve this problem.

5. SOME REMARKS ON THE NUMERICAL SOLUTION

Let us consider the approximate optimal design problem (39) and discuss some possible algorithms of solving this problem. The functional $\mathscr{J}_h$ is differentiable, nonconvex and we can choose some of the methods of nonlinear programming for constrained minimization of a differentiable functional, e.g. the Frank-Wolfe algorithm [6]. In any case, one will need an efficient method for evaluating the gradient $\nabla \mathscr{J}_h(F_h)$. To this end, we employ an adjoint state problem, which is classical in Optimal Control.

Lemma 12. *The state equation* (40) *is equivalent to the linear system*

$$\mathscr{A}_h(\varphi_h)\, x_h = \mathscr{F}_h(\varphi_h)\,, \tag{78}$$

where $\mathscr{A}_h(\varphi_h)$ *is a symmetric, positive definite matrix* $n \times n$, $n = 3N$ *and* $\mathscr{F}_h(\varphi_h)$ *is an* $n \times 1$ *matrix. Denote the solution of* (78) *by* $x_h(\varphi_h)$.

Let us introduce another linear system (the adjoint problem)

$$\mathscr{A}_h(\varphi_h)\, p_h = \frac{\partial j_h}{\partial x_h}\,(F_h;\, x_h(\varphi_h)) \tag{79}$$

and denote its solution by $p_h(\varphi_h)$.

Then the gradient of the cost functional is given by the formula

$$\nabla \mathscr{J}_h(\varphi_h) = \frac{\partial j_h}{\partial \varphi_h}\,(\varphi_h,\, x_h(\varphi_h)) + \left[\frac{\mathrm{d}\mathscr{F}_h}{\mathrm{d}\varphi_h}\,(\varphi_h)\right]^{\mathrm{T}} p_h(\varphi_h) - \tag{80}$$
$$- \left[\frac{\mathrm{d}\mathscr{A}_h}{\mathrm{d}\varphi_h}\,(\varphi_h)\, x_h(\varphi_h)\right]^{\mathrm{T}} p_h(\varphi_h)\,.$$

Proof. The equivalence of (40) and (78) follows from the expansion of F_h and $\boldsymbol{u}_h$ in terms of Hermite basic functions. The vectors of nodal values of F_h, $\boldsymbol{u}_h$ and of their derivatives are denoted by φ_h and x_h, respectively. The positive definiteness of $\mathscr{A}_h$ is a consequence of Lemma 4.

We may write (omitting the subscripts h everywhere)

$$\mathrm{d}\mathscr{J}(\varphi) = \left(\frac{\partial j}{\partial \varphi}\,(\varphi,\, x(\varphi)),\, \delta\varphi\right)_{R^m} + \left(\frac{\partial j}{\partial x}\,(\varphi,\, (x(\varphi)),\, \delta x\right)_{R^n}, \tag{81}$$

where $m = 2N + 2$ (note that the nodal values of $F_h \in U_{ad}^{h\varepsilon}$ belong to a subset of R_m).

Differentiating the equation (78), we obtain

$$\mathscr{A}(\varphi)\, \delta\, x(\varphi) + \frac{\mathrm{d}\mathscr{A}(\varphi)}{\mathrm{d}\varphi}\, x(\varphi)\, \delta\varphi = \frac{\mathrm{d}\mathscr{F}(\varphi)}{\mathrm{d}\varphi}\, \delta\varphi\,. \tag{82}$$

Using (79) and (82), we may write

$$\left(\frac{\partial j}{\partial x}\,(\varphi,\, x(\varphi)),\, \delta\, x(\varphi)\right)_{R^n} = (\mathscr{A}(\varphi)\, p,\, \delta\, x(\varphi))_{R^n} = \tag{83}$$
$$= (\mathscr{A}(\varphi)\, \delta\, x(\varphi),\, p)_{R^n} = \left(\frac{\mathrm{d}\mathscr{F}(\varphi)}{\mathrm{d}\varphi}\, \delta\varphi - \frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\varphi}\,(\varphi)\, x(\varphi)\, \delta\varphi,\, p\right)_{R^n} =$$
$$= \left(\left[\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}\varphi}\,(\varphi)\right]^{\mathrm{T}} p - \left[\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\varphi}\,(\varphi)\, x(\varphi)\right]^{\mathrm{T}} p,\, \delta\varphi\right)_{R^m}.$$

Substituting from (83) into (81), we arrive at (80).

Remark. The systems (78) and (79) differ only in the right-hand sides, which simplifies the algorithm.

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Souhrn

OPTIMALIZACE TVARU ROTAČNĚ SYMETRICKÝCH SKOŘEPIN

IVAN HLAVÁČEK

Uvažují se pružné rotačně symetrické skořepiny konstantní tloušťky a jejich meridiánová křivka se bere za návrhovou proměnnou. Je předepsána její délka a objem, který jí odpovídá, derivace do 2. řádu jsou v daných mezích. Zatížení se skládá z hydrostatického tlaku, vlastní váhy a přetlaku. Cenový funkcionál je integrál druhého invariantu napětí při obou površích skořepiny.

Dokazuje se existence řešení optimalizačního problému, a to nejprve na abstraktní úrovni. Jsou navrženy aproximační úlohy a dokázána konvergence jejich řešení k funkci, která je v jistém smyslu blízká řešení spojitého problému.

Author's address: Ing. *Ivan Hlaváček,* CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.