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Aplikace matematiky, Vol. 28 (1983), No. 3, 194–198

Persistent URL: `http://dml.cz/dmlcz/104026`

ISONEMALITY AND MONONEMALITY OF WOVEN FABRICS

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(Received August 24, 1982)

In the paper [2] combinational problems concerning woven fabrics are studied. The following conjecture is expressed: Every periodic mononemal fabric which is warp-isonemal is also weft-isonemal. We shall prove this conjecture for fabrics with $n \times n$ square fundamental blocks for n odd.

We shall consider diagrams of woven fabrics as they are used in [2] or in the Czech book [1]. Such a diagram is formed by a plane lattice in which some squares are white and the others are black. A white square denotes a place where a weft strand passes over a warp strand, and a black square denotes a place where a warp strand passes over a weft strand. A fabric is called periodic, if it can be obtained from a fundamental $n \times m$ block of squares by translations in horizontal and vertical directions through multiples of n and m units.

Consider a fundamental block of a given fabric $\mathscr{F}$. An example (the fabric No. 164 from [1]) is in Fig. 1. Let the warp strands be numbered by the numbers $1, \ldots, n$

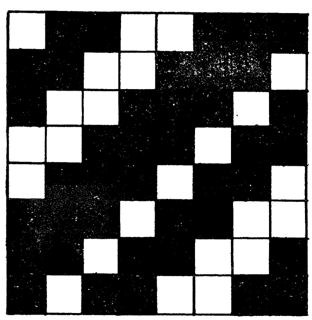

Fig. 1.

from the left end to the right end and let the weft strands be numbered by the numbers $1, \ldots, m$ from the upper end to the lower end. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$ put $a_{ij} = 1$ if the intersection of the i-th warp strand with the j-th weft strand is a black square, and $a_{ij} = 0$ if it is a white square. Now construct a bipartite graph $G(\mathscr{F})$ on the vertex sets $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_m\}$ in which the vertices u_i, v_j are adjacent if and only if $a_{ij} = 1$. This graph will be called the graph of the fabric $\mathscr{F}$. The graph of the fabric from Fig. 1 is in Fig. 2.

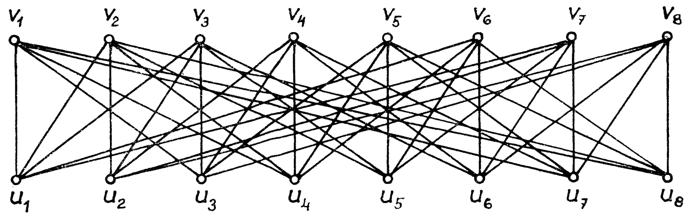

Fig. 2.

A fabric $\mathscr{F}$ is called warp-isonemal (or weft-isonemal), if for every two warp strands (or weft strands, respectively) there exists a mapping which maps one onto the other and is either a symmetry of the whole fabric (taken as infinite in all directions), or such a symmetry superposed with the interchange of the colours black and white.

A fabric $\mathscr{F}$ is called mononemal, if any two strands of $\mathscr{F}$ (each of them may be either a warp strand or a weft one) have the property that the two-way infinite sequences of white and black squares formed by these strands either are equal, or become equal after interchanging the colours black and white. Evidently if a fabric is mononemal, then it has a fundamental block which is a square. We shall always consider an $n \times n$ square and suppose that n is the least possible.

In the sequel we consider mononemal fabrics. The group of isometric mappings of the plane onto itself which map warp strands of a fabric $\mathscr{F}$ onto warp strands and weft strands onto weft strands will be denoted by $T_0(\mathscr{F})$. To each of these mappings, a certain mapping of the vertex set of $G(\mathscr{F})$ onto itself corresponds. The group $T_0(\mathscr{F})$ is generated by the elements φ_0, ψ_0, α_0, β_0 described in the sequel.

The mapping ψ_0 is a translation in the horizontal direction which maps every warp strand onto its neighbour from the right and leaves all weft strands fixed. To the mapping φ_0, a mapping φ of the vertex set of $G(\mathscr{F})$ onto itself corresponds; this mapping is defined by $\varphi(u_i) = u_{i+1}$, $\varphi(v_i) = v_i$ for $i = 1, \dots, n$.

The mapping ψ_0 is a translation in the vertical direction which maps every weft strand onto its neighbour from below and leaves all warp strands fixed. To the mapping ψ_0, a mapping ψ of the vertex set of $G(\mathscr{F})$ onto itself corresponds; this mapping is defined by $\psi(u_i) = u_1$, $\psi(v_i) = v_{i+1}$ for $i = 1, \dots, n$. (The subscripts are always taken modulo n.)

The mapping α_0 is an axial symmetry with respect to the vertical axis going through the centre of a fundamental block. The corresponding mapping α of the vertex set of $G(\mathscr{F})$ onto itself is defined by $\alpha(u_i) = u_{n+1-i}$, $\alpha(v_i) = v_i$ for $i = 1, \dots, n$.

The mapping β_0 is an axial symmetry with respect to the horizontal axis going through the centre of a fundamental block. The corresponding mapping β of the vertex set of $G(\mathscr{F})$ onto itself is defined by $\beta(u_i) = u_i$, $\beta(v_i) = v_{n+1-i}$ for $i = 1, \ldots, n$.

By $T(\mathscr{F})$ denote the group generated by the elements φ, ψ, α, β.

Now let φ_1, α_1 be the restrictions of φ, α, respectively, onto U and let φ_2, β_2 be the restrictions of ψ, β, respectively, onto V. Let $T_1(\mathscr{F})$ (or $T_2(\mathscr{F})$) be the group formed by the restrictions of elements of $T(\mathscr{F})$ onto U (or V, respectively). As every mapping $\eta \in T(\mathscr{F})$ maps U onto U and V onto V, there exist mappings $\eta_1 \in T_1(\mathscr{F})$ and $\eta_2 \in T_2(\mathscr{F})$ such that $\eta(x) = \eta_1(x)$ for $x \in U$ and $\eta(x) = \eta_2(x)$ for $x \in V$; we may write $\eta = [\eta_1, \eta_2]$.

Evidently, $T_1(\mathscr{F})$ is generated by φ_1, α_1 and $T_2(\mathscr{F})$ is generated by β_2, ψ_2. Let $A(\mathscr{F})$ be the automorphism group of $G(\mathscr{F})$ and let $B(\mathscr{F})$ be the group consisting of all automorphisms of $G(\mathscr{F})$ and all isomorphisms of $G(\mathscr{F})$ onto its bipartite complement. (The bipartite complement of $G(\mathscr{F})$ is the bipartite graph on the vertex sets U, V such that a vertex of U is adjacent to a vertex of V in it if and only if these vertices are not adjacent in $G(\mathscr{F})$.) Let $A_0(\mathscr{F}) = A(\mathscr{F}) \cap T(\mathscr{F})$, $B_0(\mathscr{F}) = B(\mathscr{F}) \cap \cap T(\mathscr{F})$. The mappings from $B_0(\mathscr{F})$ are exactly those mappings of the vertex set of $G(\mathscr{F})$ onto itself which correspond to the symmetries of $\mathscr{F}$ and to those symmetries superposed with the interchange of the colours black and white. If $G(\mathscr{F})$ is not isomorphic to its bipartite complement, then evidently $B(\mathscr{F}) = A(\mathscr{F})$ and $B_0(\mathscr{F}) = A_0(\mathscr{F})$.

Now let $B_1(\mathscr{F})$ (or $B_2(\mathscr{F})$) be the set of all mappings $\eta_1 \in T_1(\mathscr{F})$ (or $\eta_2 \in T_2(\mathscr{F})$) to which there exists a mapping $\eta_2 \in T_2(\mathscr{F})$ (or $\eta_1 \in T_1(\mathscr{F})$, respectively) such that $\eta = [\eta_1, \eta_2] \in B_0(\mathscr{F})$. Analogously $A_1(\mathscr{F})$, $A_2(\mathscr{F})$ may be defined.

We shall prove some theorems and a lemma. Here $\mathscr{F}$ is always a fabric with an $n \times n$ square fundamental block and n is supposed to be the least possible.

Theorem 1. *Let $\mathscr{F}$ be a warp-isonemal fabric with an $n \times n$ square fundamental block for n odd. Then $\varphi_1 \in B_1(\mathscr{F})$.*

Proof. There are two mappings from $T_1(\mathscr{F})$ which map u_1 onto u_2; they are φ_1 and $\varphi_1^2\alpha_1$. As $\mathscr{F}$ is warp-isonemal, at least one of them must be in $B_1(\mathscr{F})$. If $\varphi_1 \in B_1(\mathscr{F})$, the assertion is true; thus suppose that $\varphi_1^2\alpha_1 \in B_1(\mathscr{F})$. Similarly there are two mappings from $T_1(\mathscr{F})$ which map u_1 onto u_3; they are φ_1^2 and $\varphi_1^3\alpha_1$. If $\varphi_1^2 \in B_1(\mathscr{F})$, then $\varphi_1 = (\varphi_1^2)^{(n+1)/2} \in B_1(\mathscr{F})$. If $\varphi_1^3\alpha_1 \in B_1(\mathscr{F})$, then $\varphi_1 = (\varphi_1^3\alpha_1) \cdot \cdot (\varphi_1^2\alpha_1)^{-1} \in B_1(\mathscr{F})$.

Theorem 2. *Let $\mathscr{F}$ be a fabric with an $n \times n$ square fundamental block, where n is odd. Then no mapping which is a superposition of an isometric mapping of the plane onto itself and the interchange of colour black and white maps $\mathscr{F}$ onto itself.*

Proof. All fundamental blocks of $\mathscr{F}$ are obtained from one of them by cyclic permutations of warp strands and cyclic permutations of weft strands; therefore all of them have the same number of black squares and the same number of white squares. As n is odd, the number of squares of any fundamental block is odd and such a block cannot contain the same number of black and white squares. Hence the interchange of colours black and white transforms the fabric $\mathscr{F}$ into a fabric non-isomorphic to $\mathscr{F}$.

Lemma. *Let $\mathscr{F}$ be a warp-isonemal and mononemal fabric with an $n \times n$ square fundamental block for n odd. Let η_2 be the mapping from $B_2(\mathscr{F})$ such that $\eta = [\varphi_1, \eta_2] \in A_0(\mathscr{F}) = B_0(\mathscr{F})$. Then the degree of η_2 in $T_2(\mathscr{F})$ is equal to n.*

Remark. The equality $B_0(\mathscr{F}) = A_0(\mathscr{F})$ follows from Theorem 2.

Proof. Evidently, the degree of η_2 is either 2 or a divisor of n. Let it be $k \neq n$. If $\eta = [\varphi_1, \eta_2] \in B_2(\mathscr{F})$, then $\eta^k = [\varphi_1^k, \eta_2^k] = [\varphi_1^k, \varepsilon_2] \in B_0(\mathscr{F})$, where ε_2 is the identity mapping of V. The mapping η^k is an automosphism of $G(\mathscr{F})$, therefore the neighbourhoods of u_i and u_{i+m} are equal for each i, where m is the greatest common divisor of n and k. (No mapping from $B_0(\mathscr{F})$ maps $G(\mathscr{F})$ onto its bipartite complement, therefore each of them maps it onto itself; this follows from Theorem 2.) Hence n is not the least possible period of the two-way infinite sequence of black and white squares on a strand; hence m is such a period and there exists an $m \times m$ square fundamental block of $\mathscr{F}$, which is a contradiction with the assumption that the fundamental block of $\mathscr{F}$ is an $n \times n$ square.

Theorem 3. *Let $\mathscr{F}$ be a fabric with an $n \times n$ square fundamental block, where n is odd. Let $\mathscr{F}$ be mononemal and warp-isonemal. Then $\mathscr{F}$ is weft-isonemal.*

Proof. According to Theorem 1 we have $\varphi_1 \in B_1(\mathscr{F})$. According to Lemma there exists $\eta_2 \in B_2(\mathscr{F})$ such that $\eta = [\varphi_1, \eta_2] \in B_0(\mathscr{F}) = A_0(\mathscr{F})$, and the degree of η_2 is n. As the degree of $\psi_2^k \beta$ is 2 for each k, we have $\eta_2 = \psi_2^l$, where l is relatively prime to n. Among the powers of ψ_2^l there are all powers of ψ_2, hence each v_i can be mapped onto each v_j by a mapping from $B_2(\mathscr{F})$ and $\mathscr{F}$ is weft-isonemal.

References

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[2] *B. Grünbaum, G. C. Shephard*: Satins and Twills: An Introduction to the Geometry of Fabrics. Math. Magazine 53 (1980), 139—161.

Souhrn

ISONEMALITA A MONONEMALITA TKANIN

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V článku se zkoumají diagramy tkanin složené z bílých a černých čtverečků jakožto geometrické útvary a popisují se jejich symetrie. Užívá se pojmů isonemality a mononemality, které zavedli B. Grünbaum a G. C. Shephard. Dokazuje se, že periodická mononemální útkově isonemální tkanina, jejíž střída je čtverec o straně liché délky, je rovněž osnovně isonemální.

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