Anukul De

Buckling of anisotropic shells. II

Aplikace matematiky, Vol. 28 (1983), No. 2, 129–137

Persistent URL: http://dml.cz/dmlcz/104014

BUCKLING OF ANISOTROPIC SHELLS II

ANUKUL DE

(Received February 9, 1982)

1. INTRODUCTION

The formulation of differential equations as well as solutions of the buckling problem for isotropic material are known from the literature on shells, e.g. Flügge [1]. Singer [3] solved the buckling problem of the orthotropic conical shell under external pressure in the isotropic case. Singer [2] solved the buckling of orthotropic and stiffened conical shells. De [4] obtained differential equations of the buckling problem for anisotropic cylindrical shells under the most general homogeneous stress action and also the solution of the differential equations of the buckling problem for anisotropic shells without shear load in case of two way compression, and deduced the corresponding differential equations as well as the corresponding solution for the isotropic case.

The object of this paper is to investigate the solution of the differential equation of the buckling problem for anistrotropic shells with shear load in the case of torsion of a long tube. The condition of neutral equilibrium and the critical value of the shear load T and also that of the total torque are obtained.

Solution for isotropic shells is deduced as a special case, which coincides with the known results, Flügge [1].

2. BASIC EQUATIONS

The differential equations for the buckling problem of an anisotropic shell, see De [4], are given by

$$u'' + A_1 u^{\bullet\bullet} + A_2 v'^{\bullet} + A_3 w' + K_1\{A_4(u^{\bullet\bullet} + w'^{\bullet\bullet}) - w'''\} - \\ - q_1(u^{\bullet\bullet} - w') - q_2 u'' - 2q_3 u'^{\bullet} = 0\,, \tag{1a}$$

$$A_5 u'^{\bullet} + v^{\bullet\bullet} + A_6 v'' + w^{\bullet} + K_1[3A_7 v'' - A_8 w''^{\bullet}] - \\ - A_9[q_1(v^{\bullet\bullet} + w^{\bullet}) + q_2 v'' + 2q(v' + w')] = 0\,, \tag{1b}$$

(1c)
$$A_{10}u' + v^{\bullet} + w + K_1[A_7u'^{\bullet\bullet} - A_9u''' - A_8v''^{\bullet} + A_9w'''' + + 2A_{11}w''^{\bullet\bullet} + A_{12}(w^{\bullet\bullet\bullet\bullet} + 2w^{\bullet\bullet} + w)] + + A_9[q_1(u' - v^{\bullet} + w^{\bullet\bullet}) + q_2w'' - 2q_3(v' - w'^{\bullet}) = 0\,,$$

where ( )′ and ( )˙ indicates $\partial/\partial x(\ )$ and $\partial/\partial\phi(\ )$, respectively,

(2)
$$A_1 = \frac{D_{x\varphi}}{D_x}\,,\quad A_2 = \frac{D_v + D_{x\varphi}}{D_x}\,,\quad A_3 = \frac{D_v}{D_x}\,,$$

$$A_4 = \frac{K_{x\varphi}}{K_x}\,,\quad A_5 = \frac{D_v + D_{x\varphi}}{D_\varphi}\,,\quad A_6 = \frac{D_{x\varphi}}{D_\varphi}\,,$$

$$A_7 = \frac{D_xK_{x\varphi}}{D_\varphi K_x}\,,\quad A_8 = \frac{D_x(3K_{x\varphi} + K_v)}{D_\varphi K_x}\,,\quad A_9 = \frac{D_x}{D_\varphi}\,,$$

$$A_{10} = \frac{D_v}{D_\varphi}\,,\quad A_{11} = \frac{D_x(2K_{x\varphi} + K_v)}{D_\varphi K_x}\,,\quad A_{12} = \frac{D_xK_\varphi}{D_\varphi K_x}$$

(3)
$$k_1 = \frac{K_x}{a^2D_x}\,,\quad q_1 = \frac{p\,.\,a}{D_x}\,,\quad q_2 = \frac{P}{D_x}\,,\quad q_3 = \frac{T}{D_x}\,,$$

where the rigidities D and K are given by

(i) external rigidities

(4a)
$$D_x = E_1t_1 + 2E_2t_2\,,\quad D_\varphi = E_2t_1 + 2E_1t_2\,,\quad D_v = E_vt\,;$$

(ii) shear rigidity:

(4b)
$$D_{x\varphi} = Gt\,;$$

(iii) bending rigidities:

(4c)
$$K_x = \tfrac{1}{12}[E_2(t^3 - t_1^3) + E_1t^3]\,,\quad K_\varphi = \tfrac{1}{12}[E_1(t^3 - t_1^3) + E_2t_1^3]\,,\quad K_v = \tfrac{1}{12}E_vt^3\,;$$

(iv) twisting rigidity:

(4d)
$$K_{x\varphi} = \tfrac{1}{12}Gt^3\,,$$

in which E_1, E_2, E_v and G are four moduli of elasticity and $t = t_1 + 2t_2$ is the thickness of the shell (Fig. 2) and the shell is simultaneously subject to three simple loads (Fig. 1):

(i) a uniform normal pressure on its wall, $P_r = -P$,

(ii) an axial compression applied at the edges, the force per unit circumference being P,

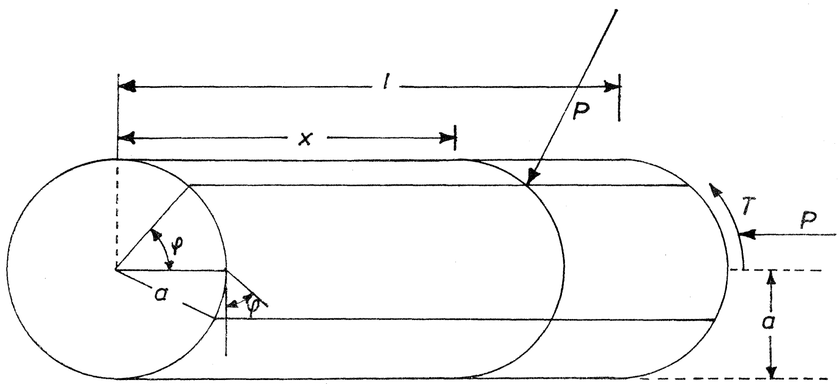

Fig. 1.

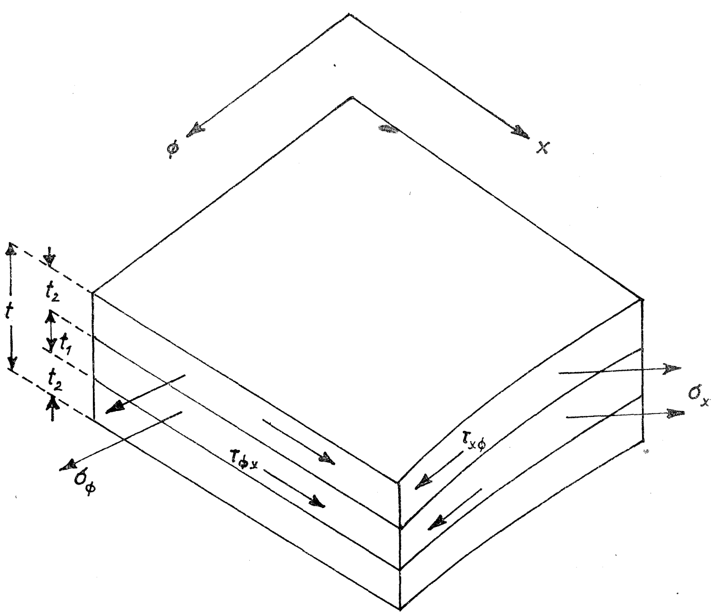

Fig. 2.

(iii) a shear load applied at the edges so as to produce a torque in the cylinder, the shearing force (shear flow) being T.

3. SOLUTION FOR SHELLS WITH SHEAR LOAD

The equations (1) describe the buckling of a cylindrical shell under the most general homogeneous membrane stress action in the anisotropic case.

When there is no shear load on the shell ($T = 0$, hence $q_3 = 0$), the solution in that case considerably simplifies but when there is a shear load $q_3 \neq 0$, the solution which is applicable in this case is given by

$$\begin{aligned} u &= A \sin(\lambda x/a + m), \\ v &= B \sin(\lambda x/a + m), \\ w &= C \cos(\lambda x/a + m), \end{aligned} \tag{5}$$

where

$$\lambda = n\pi a/l, \quad l = \text{length of the shell and } n \text{ is an integer}. \tag{6}$$

The solution may be used for any combination of loads p, P and T. The zeros of u, v, w and of their derivatives are found on lines $m\varphi + \lambda x/a = \text{const}$, winding around the cylinder (Fig. 3). It is therefore not possible to satisfy reasonable boundary conditions on lines $x = \text{const}$, and the solution (5) cannot be used to deal with cylinders of finite length, We shall use it here to study the buckling of an infinitely long cylinder subjected to a shear load T only. Thus if $q_1 = q_2 = 0$, $q_3 \neq 0$, substitute the values of u, v, w given by linear equations for A, B, C:

$$\begin{aligned} &A[\lambda^2 + m^2(A_1 + k_1A_4) - 2q_3\lambda m] + B[A_2\lambda m] + \\ &\quad + C[-A_3\lambda + k_1(\lambda^3 - A_4\lambda m^2)] = 0 \end{aligned} \tag{6a}$$

$$\begin{aligned} &A[A_5\lambda m] + B[m^2 + \lambda^2(A_6 + 3k_1A_7) - 2q_3\lambda m] + \\ &\quad + C[m + k_1A_8\lambda^2 m - 2q_3\lambda] = 0 \end{aligned} \tag{6b}$$

$$\begin{aligned} &A[A_{10}\lambda + k_1(A_9\lambda^3 - A_7\lambda m^2)] + B[m + A_8k_1\lambda^2 m - 2q_3\lambda] + \\ &+ C[1 + k_1\{A_9\lambda^4 + 2A_{11}\lambda^2m^2 + A_{12}(m^2 - 1)^2\} - 2q_3\lambda m] = 0 \end{aligned} \tag{6c}$$

The equations (6) are three linear equations with buckling amplitudes A, B, C as unknowns, and with the brackets as coefficients. Since the equations are homogeneous, they admit only the solution $A = B = C = 0$, which shows that the shell is not in neutral equilibrium. Non-vanishing solutions A, B, C, are possible if and only if the determinant of the nine coefficients of the equations (6) is equal to zero. Thus the vanishing of this determinant is the buckling condition of the shell. When the buckling condition is fulfilled, any two of the three equations (6) determine the ratios A/C and B/C and thus the buckling mode according to (5). As in all cases of neutral equilibrium, the magnitude of the possible deformation remains arbitrary.

The buckling condition contains three unknowns: the dimensionless parameters q_3, and the modal parameters m and λ. Also we know that m must be an integer (0, 1, 2, 3, ...) and λ must be an integral multiple of $\pi a/1$ ($n = 1, 2, 3, 4, \ldots$); q_3

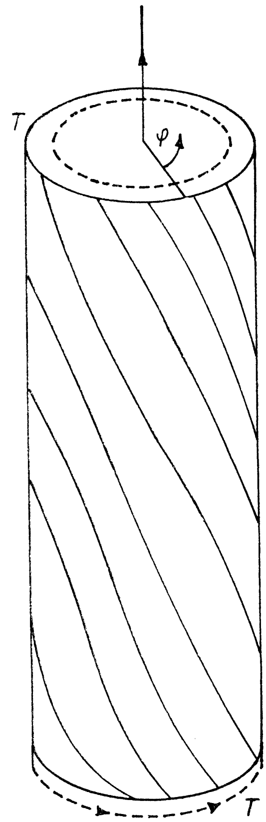

Fig. 3.

is a small quantity. Also k_1 is a small quantity since we are interested in thin shells where $t \ll a$. Expanding the nine coefficients, equating them to zero and neglecting small quantities of higher order we find the condition of neutral equilibrium:

$$C_1 + C_2 k_1 = C_5 q_3 \,, \tag{7}$$

where C_1, C_2 and C_5 are given by

$$C_1 = A_6(1 - A_3 A_{10})\,\lambda^4 \,, \tag{8a}$$

$$\begin{aligned} C_2 = {} & [A_9\lambda^4 + 2A_{11}\lambda^2 m^2 + A_{12}m^4]\,[A_6\lambda^4 + 2A_{13}\lambda^2 m^2 + A_1 m^4] - \\ & - A_6(A_3A_9 + A_{10})\,\lambda^6 - 2\lambda^4 m^2[A_8 + A_{10} - A_5 - A_3(A_5A_8 + A_6A_7)] - \\ & - \lambda^2 m^4[2A_1A_8 + 4A_{12}A_{13} + A_4(A_5 + A_6 - A_{10}) - 2A_1A_{12}m^6 + \\ & + [3A_1A_7 + A_4A_6 + 2A_{12}A_{13}]\,\lambda^2 m^2 + A_1A_{12}m^4 \,, \end{aligned} \tag{8b}$$

(8c) $$C_5 = 2\lambda m[(A_1 m^4 + A_6\lambda^4 + 2A_{13}\lambda^2 m^2) - A_1 m^2 - \\ - (1 + A_3A_{10} - 2A_2A_{10} - A_6)\lambda^2] \,.$$

It is evident that neither λ nor m can be zero, because in both the cases $C_5 = 0$ and hence $q_3 = \infty$. It is also without interest to consider negative values of λ or m. When both are negative nothing is changed in the equation (8c) while if either λ or m above is negative, the buckling mode (5) is altered so that the modal lines (Fig. 3) become right-handed screws. One would expect that the buckling load T must be applied in the opposite sense, and this is exactly what happens. In the equation (7) the left hand side remains the same while C_5 and hence q_3 change signs.

The discussion of the buckling formula (7) is now restricted to positive values λ and to integers m. We may solve it for q_3, differentiating the expression with respect to λ and m, and putting the first partial derivative equal to zero. This would yield two algebraic equations for λ and m, and their solutions (or one of them) would lead to the smallest possible q_3. This procedure, however, is rather tiresome and may be avoided. By some trial computations we may find out that any $m > 2$ yields a higher buckling load than does $n = 2$, and that λ must be chosen rather small, $\lambda \ll 1$, to obtain low q_3.

With this idea in mind, we now investigate separately the two cases $m = 1$ and $m = 2$.

For $m = 2$, the equation (7) yields

$$q_3 = \frac{A_6(1 - A_3A_{10})\lambda^4 + k_1[A_6A_9\lambda^8 + A_1\lambda^6 + A_{15}\lambda^4 + A_{16}\lambda^2 + 144A_1A_{12}]}{4[12A_1 + A_{17}\lambda^2 + A_6\lambda^4]}$$

where

(10) $$A_{14} = 2[4(A_6A_{11} + A_4A_{10}] \,,$$

$$A_{15} = 16A_1A_9 + 17A_6A_{12} + 64A_{13}A_{11} + 3A_7 - 8(A_8 + A_{10} - A_5) + \\ + 8A_3(A_5A_8 + A_6A_7) + 128A_1A_{12}[2A_1A_8 + 4A_{12}A_{13} + A_4(A_5 + A_6 - A_{10})] \,,$$

$$A_{16} = 128(A_1A_{11} + A_{12}A_{13}) - 4(3A_1A_7 + A_4A_6 + 2A_{12}A_{13}) \,,$$

$$A_{17} = 8A_3 - 1 + A_{10}(2A_3 - A_3) + A_8 \,.$$

Neglecting λ^2 as compared to unity, we have

(11) $$q_3 = \frac{A_6(1 - A_3A_{10})\lambda^3}{48A_1} + \frac{3A_{12}}{\lambda} k_1 \,.$$

Now it is easy to find from

$$\frac{\partial q_3}{\partial \lambda} = \frac{A_6(1 - A_3A_{10})\lambda}{16A_1} - \frac{3A_{12}}{\lambda^2} k_1 = 0$$

that

$$(12) \qquad \lambda^4 = \frac{48A_1A_{12}}{A_6(1 - A_3A_{10})} k_1$$

yields the lowest possible value of q_3 and

$$q_{3|\min} = 2k_1 \sqrt[4]{\left(\frac{4(1 - A_3A_{10}) A_{12}^3}{3A_1k_1}\right)} = 2 \sqrt[4]{\left(\frac{A_6(1 - A_3A_{10}) A_{12}^3}{3A_1}\right)} \left(\frac{K_x}{D_x}\right)^{3/2} \frac{1}{a^{3/2}} .$$

Using the last of the equations (3) we may now return to the real shear load T and find the critical value,

$$(13b) \qquad T_{\text{cr.}} = 2 \sqrt[4]{\left(\frac{A_6(1 - A_3A_{10}) A_{12}^3}{3A_1}\right)} \frac{K_x^{3/4} D_x^{1/4}}{a^{3/2}} .$$

The total torque applied to the tube is given by

$$M = T 2\pi a a .$$

The critical value for this torque is

$$M_{\text{cr.}} = 4\pi a^{1/2} \sqrt[4]{\left(\frac{A_6(1 - A_3A_{10}) A_{12}^3}{3A_1}\right)} .$$

All these results have been derived for an infinitely long cylinder of an anisotropic shell. Since they do not contain any wave length, we are tempted to apply them to cylinders of finite length.

However, such a cylinder usually has some kind of stiffening at the end, say a bulkhead requiring $w = 0$. Any such condition is in contradiction to the equation (5) and the additional constraint imposed by the bulkhead will increase the buckling load. One may expect that the difference is not too big if the cylinder is rather long.

With $m = 1$, the equation (7) yields

$$(14) \qquad q_3 = \frac{A_6(1 - A_3A_{10}) + k_1\lambda^3[A_6A_9\lambda^2 + 2A_9A_{13} + 2A_6(A_{11} - A_{10})]}{2[A_6\lambda^2 + (A_1A_6 - A_2A_5 + (2A_2 - A_3) A_{10}) + A_6]} .$$

Now, neglecting λ^2 as compared with unity we may drop k_1 terms entirely and we get

$$(15) \qquad q_3 = \frac{A_6(1 - A_3A_{10})}{2[A_1A_6 - A_2A_5 + A_6 + (2A_2 - A_3) A_{10}]} \lambda .$$

If we can choose λ arbitrarily, we may choose it as small as we like and thus make q_3 approach zero. This shows that there is no finite buckling load for the infinite shell unless we prevent the buckling mode with $m = 1$. In this mode the axis of the tube is deformed to a steep helical curve, while the circular cross-sections remain circular and normal to the deformed axis. Since every such cross-section rotates about one of its diameters, this mode may be excluded by preventing such a rotation of the terminal cross-sections of a long cylinder.

4. PARTICULAR CASE

To get the corresponding results for the isotropic case we put

$$t_2 = 0\,, \quad t_1 = t\,, \quad E_1 = E_2 \frac{E}{1 - \nu^2}\,, \tag{16}$$

$$E_\nu = \frac{E\nu}{1 - \nu^2}\,, \quad G = \frac{E}{2(1 + \nu)}\,, \quad (\nu = \text{Poisson's ratio}).$$

Substituting (16) in the equations (11), (12), (13a), (13b), (13c) and (15) we get, respectively,

$$q_3 = \frac{1 - \nu^2}{48} \lambda^3 + \frac{3k_1}{\lambda}\,, \tag{17}$$

$$\lambda^4 = \frac{48k_1}{1 - \nu^2}\,, \tag{18}$$

$$q_3\big|_{\min} = 2k_1 \sqrt[4]{\left(\frac{1 - \nu^2}{3k_1}\right)} = \frac{\sqrt[4]{(1 - \nu^2)}}{3\sqrt{2}} \left(\frac{t}{a}\right)^{3/2}, \tag{19}$$

$$T_{\text{cr.}} = \frac{1}{3\sqrt{(2)}\,(1 - \nu^2)^{3/4}} \frac{Et^{5/2}}{a^{3/2}}\,, \tag{20}$$

$$M_{\text{cr.}} = \frac{\pi\sqrt{(2)}}{3(1 - \nu^2)^{3/4}} E \,.\, \sqrt{(t^5 a)}\,, \tag{21}$$

$$q_3 = \frac{1 - \nu^2}{2} \lambda\,. \tag{22}$$

The equations (17)–(22) are exactly the same as those by Flügge[1] in the case of isotropic shells.

Acknowledgement. In conclusion the author wishes to thank Dr. S. Basuli, Professor in Mathematics, Tripura Engineering College, Tripura, India for his kind help and guidance in the preparation of this paper.

References

[1] *W. Flügge:* Stresses in shells. Springer Verlag, New York, 1967.

[2] *J. Singer:* Buckling of orthotropic and stiffened conical shells. Collected papers on instability of shell structures. N. A. S. A. T. N. D-1510, p. 463.

[3] *J. Singer*, *R. Fersh-Scher:* Buckling of orthotropic conical shell under external pressure. Vol. XV, Aeronautical Quarterly, 1964, pp. 151—168.

[4] *A. De*: Buckling of anisotropic shells I. Apl. mat. 28 (1983), 120—128.

Souhrn

STABILITA ANIZOTROPNÍCH SKOŘEPIN II

ANUKUL DE

Cílem článku je podat řešení diferenciálních rovnic pro problém stability anizotropních válcových skořepin se smykovým zatížením v případě torze dlouhé trubky. Jsou rovněž nalezeny kritické hodnoty pro smykové zatížení a celkový kroutící moment. Jako zvláštní případ jsou odvozeny výsledky pro izotropní případ.

Author's address: Prof. *Anukul De*, Department of Mathematics, Calcutta University, Post-Graduate Centre, Agartala, Tripura-799004, India.