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APPROXIMATION AND NUMERICAL SOLUTION OF CONTACT PROBLEMS WITH FRICTION

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INTRODUCTION

In applications we often meet with problems, in which one deformable body comes in contact with another. Contact problems are non-classical in the sense that they cannot be directly formulated as usual boundary value problems. The reason for this is the fact that neither the contact surface nor the distribution of contact forces are known a – priori. It is well known that the mathematical formulation leads to variational inequalities. The aim of the present paper is to study the approximation and the computation of contact problems between an elastic body and a rigid foundation, taking into account the influence of friction on the contact surface. We analyse the simplest model involving friction, namely that with "a given friction". The importance of this model consists in the fact that it serves as an auxiliary problem, by means of which we can approximate contact problems with friction obeying the classical Coulomb law. The main attention heie is paid to the modification of Uzawa's algorithm, using specific features of our problem.

1. SETTING OF THE PROBLEM

Let us assume a plane elastic body subjected to a body force F and to surface tractions on a portion of its boundary, unilateraly supported by a rigid foundation. Our aim is to determine the behaviour of the structure, taking into account the influence of friction between the body and the foundation. We start with the classical formulation of the problem.

Let $\Omega \subset R^2$ be *a bounded polygonal domain with Lipschitz boundary* $\partial\Omega$, which is decomposed as follows:

$$\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_\tau \cup \bar{\Gamma}_0 \cup \bar{\Gamma}_K\,,$$

where Γ_u, Γ_τ, Γ_0, Γ_K are mutually disjoint parts of $\partial\Omega$, Γ_u and Γ_K are non-empty. By a *classical solution* of the pıoblem in question we mean a displacement field

$u = (u_1, u_2)$ satisfying the boundary conditions

$$(1.1) \qquad u = 0 \qquad \text{on} \quad \Gamma_u ,$$

$$(1.2) \qquad u_n = 0 , \quad T_t(u) = 0 \quad \text{on} \quad \Gamma_0 ,$$

$$(1.3) \qquad \tau_{ij} n_j = P_i \qquad \text{on} \quad \Gamma_\tau ,$$ [1]

$$(1.4) \qquad u_n \leqq 0 , \quad T_n(u) \leqq 0 , \quad u_n \, T_n(u) = 0 \quad \text{on} \quad \Gamma_K ,$$

$$(1.5) \qquad |T_t(u)| \leqq g \quad \text{on} \quad \Gamma_K ;$$

$$\text{if} \quad |T_t(u)(x)| < g(x) , \quad \text{then} \quad u_t(x) = 0$$

$$\text{if} \quad |T_t(u)(x)| = g(x) , \quad \text{then} \quad \exists \lambda \geqq 0 \quad u_t(x) = -\lambda \, T_t(u)(x)$$

and the system of equilibrium equations

$$(1.6) \qquad \frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \quad \text{on} \quad \Omega , \quad i = 1, 2 .$$

Here u_n, u_t denote the normal and tangential components of the displacement field u, i.e. $u_n = u \, . \, n$, $u_t = u \, . \, t$; $n = (n_1, n_2)$, $t = (t_1, t_2) = (-n_2, n_1)$ are the outward unit normal and tangential vectors to $\partial\Omega$. Similarly $T_n(u)$ and $T_t(u)$ are the normal and tangential components of the stress vector $T = (T_1, T_2) = (\tau_{1j} n_j, \tau_{2j} n_j)$.

The stress tensor $\tau = (\tau_{ij})_{i,j=1}^2$ and the strain tensor $\varepsilon = (\varepsilon_{ij})_{i,j=1}^2$,

$$\varepsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

are related by means of the generalized Hooke's law

$$\tau_{ij} = \tau_{ij}(u) = c_{ijkl} \varepsilon_{kl}(u) .$$

The elastic coefficients c_{ijkl} are supposed to be *bounded and measurable in* Ω, i.e. $c_{ijkl} \in L^\infty(\Omega)$, satisfying the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klij} \quad \text{a.e.} \quad \text{in } \Omega$$

and the condition of ellipticity

$$\begin{cases} \exists c_0 = \text{const} > 0 \quad \textit{such that} \\ c_{ijkl} e_{ij} e_{kl} \geqq c_0 e_{ij} e_{ij} \quad \forall e_{ij} = e_{ji},\ e_{ij} \in R^1 . \end{cases}$$

In order to give the variational form of our problem, we introduce the following notations.

By $H^k(\Omega)$ ($k \geqq 0$ integer, $H^0(\Omega) = L^2(\Omega)$) we denote the Sobolev space of functions, the derivatives of which up to order k are square integrable in Ω, equipped with the classical norm denoted by $\|\cdot\|_{k,\Omega}$.

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[1]) Throughout the paper, summation convention is used.

Let V be *the space of virtual displacements:*

$$V = \{v \in (H^1(\Omega))^2 \mid v = 0 \text{ on } \Gamma_u, v_n = 0 \text{ on } \Gamma_0\},$$

and

$$K = \{v \in V \mid v_n \leqq 0 \text{ on } \Gamma_K\}$$

the set of admissible displacements.

Finally, let us set

$$a(u, v) = \int_\Omega \tau_{ij}(u)\, \varepsilon_{ij}(v) \,\mathrm{d}x,$$

$$L(v) = \int_\Omega F_i v_i \,\mathrm{d}x + \int_{\Gamma_\tau} P_i v_i \,\mathrm{d}s,$$

$$\mathscr{J}_0(v) = \tfrac{1}{2} a(v, v) - L(v),$$

$$j(v) = \int_{\Gamma_K} g|v_t| \,\mathrm{d}s,$$

$$\mathscr{J}(v) = \mathscr{J}_0(v) + j(v),$$

where $F = (F_1, F_2) \in (L^2(\Omega))^2$, $P = (P_1, P_2) \in (L^2(\Gamma_\tau))^2$, $g \in L^\infty(\Gamma_K)$, $g \geqq 0$ a.e. on Γ_K.

Definition 1.1. By a *variational solution* of the contact problem with a given friction we mean a function $u \in K$ such that

$$(\mathscr{P}) \qquad \mathscr{J}(u) \leqq \mathscr{J}(v) \quad \forall v \in K.$$

Using a classical result of calculus of vaıiations (see [1], [2]), one obtains the following result:

Theorem 1.1. *Under the above mentioned hypotheses there exists a unique solution u of $(\mathscr{P})$.*

The problem $(\mathscr{P})$ can be equivalently characterized by the relation

$$(1.7) \qquad u \in K : a(u, v - u) + j(v) - j(u) \geqq L(v - u) \quad \forall v \in K.$$

Applying Green's formula to (1.7) one can prove formal equivalence between (1.1) to (1.6) and $(\mathscr{P})$.

2. THE APPROXIMATION OF $(\mathscr{P})$

Let all the assumptions and notations of Section 1 be satisfied. Moreover, let us suppose that Ω *is a polygonal domain*. Let $\{\tau_h\}$, $h \to 0_+$ be a *regular family* of triangulations of $\bar{\Omega}$, satisfying the usual requirements on the mutual position of triangles $T_i \in \tau_h$.

With every τ_h we associate the space

(2.1) $$V_h = \{v \mid v \in (C(\bar{\Omega}))^2,\ v/T \in (P_1)^2,\ v = 0 \text{ on } \Gamma_u,\ v_n = 0 \text{ on } \Gamma_0,\ \forall T \in \tau_h\}\,,$$

where P_1 is the set of all linear polynomials.

Denote by A_j, $j = 1, 2, \ldots, k$, the vertices of Ω. By $n^- = (n_1^-, n_2^-)$, $n^+ = (n_1^+, n_2^+)$ we denote the outward unit normal vector to $\overline{A_{j-1}A_j}$, $\overline{A_jA_{j+1}}$, respectively (see Fig. 1).

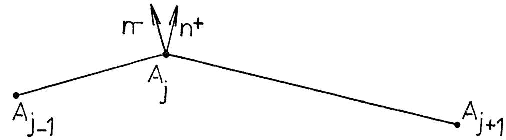

Fig. 1.

Let $a_1, a_2, \ldots, a_m$ by the nodes of τ_h, lying on $\bar{\Gamma}_K$ and define

(2.2) $$K_h = \{v \mid v \in V_h,\ v_n(a_i) \leqq 0 \text{ for } a_i \neq A_j,\ j = 1, 2, \ldots, k;$$
$$(v_1 n_1^- + v_2 n_2^-)(a_i) \leqq 0,\ (v_1 n_1^+ + v_2 n_2^+)(a_i) \leqq 0, \text{ if } a_i = A_j$$
$$\text{for some } j = 1, 2, \ldots, k;\ i = 1, 2, \ldots, m\}\,.$$

Definition 2.1. An element $u_h \in K_h$ is said to be *the approximation of the contact problem with a friction, if*

($\mathscr{P}_h$) $$\mathscr{J}(u_h) \leqq \mathscr{J}(v) \quad \forall v \in K_h\,.$$

It is readily seen that for every $h \in (0, 1)$, there exists a unique solution u_h of $(\mathscr{P}_h)$.

The question arises, what is the relation between u and u_h.

Theorem 2.1. *Let us suppose that $\bar{\Gamma}_K \cap \bar{\Gamma}_u = \emptyset$, $\bar{\Gamma}_K \cap \bar{\Gamma}_0 = \emptyset$ and let there exist a finite number of boudary points $\bar{\Gamma}_\tau \cup \bar{\Gamma}_K$, $\bar{\Gamma}_u \cup \bar{\Gamma}_\tau$, $\bar{\Gamma}_\tau \cup \bar{\Gamma}_0$. Then*

(2.3) $$u_h \to u \quad in \quad (H^1(\Omega))^2,\ h \to 0_+\,.$$

Proof. If the assumptions concerning Γ_u, Γ_τ, Γ_K, Γ_0 are satisfied, then the system of $\{K_h\}$, $h \in (0, 1)$ is dense in K in the following sense:

(2.4) $$\forall v \in K\,,\quad \exists v_h \in K_h \quad \text{such that} \quad v_h \to v \quad (\text{see [3]})\,.$$

As $K_h \subset K$, $\forall h \in (0, 1)$, (2.4) yields (2.3).

If some additional smoothness assumptions concerning u are satisfied, then the following rate of convergence result can be established (see [4]):

Theorem 2.2. *Let us suppose that $u \in (H^2(\Omega))^2 \cap K$, $T(u) \in (L^\infty(\Gamma_K))^2$, $u \in (W^{1,\infty}(\Gamma_K))^2$, let the number of points from Γ_K, where u_n, u_t change from $u_n < 0$ to $u_n = 0$ and* *from* $u_t \neq 0$ *to* $u_t = 0$ *,respectively, be finite. Then*

$$\|u - u_h\|_{1,\Omega} = \mathcal{O}(h), \quad h \to 0_+ .$$

Under weaker assumptions, lower order of convergence can be derived, namely:

$$\textit{if} \quad u \in (H^2(\Omega))^2 \cap K \quad \textit{and} \quad T_n(u) \in L^2(\Gamma_K), \quad \textit{then}$$

$$\|u - u_h\|_{1,\Omega} = \mathcal{O}(h^{3/4}), \quad h \to 0_+ .$$

3. SADDLE-POINT FORMULATION OF CONTACT PROBLEMS WITH A GIVEN FRICTION

Formulation $(\mathscr{P})$ is very simple from the theoretical point of view, but it is not suitable for practical computations. The reason for this is the fact that the direct application of finite elements to $(\mathscr{P})$ leads to a non-differentiable optimization problem. To avoid this difficulty, we proceed as follows (see [5]):

Let us define

(3.1) $$\Lambda = \{\mu \mid \mu \in L^2(\Gamma_K),\ |\mu| \leqq 1 \text{ a.e. on supp } g,\ \mu = 0 \text{ on } \Gamma_K \setminus \text{supp } g\} .$$

Clearly

$$j(v) = \sup_{\mu \in \Lambda} \int_{\Gamma_K} \mu g v_t \, \mathrm{d}s .$$

Now the problem $(\mathscr{P})$ can be equivalently formulated:

$$\begin{cases} \text{find} \quad u \in K \quad \text{such that} \\ \mathscr{J}(u) = \min\limits_{v \in K} \ \sup\limits_{\mu \in \Lambda} \{\frac{1}{2} a(v, v) - L(v) + \int_{\Gamma_K} \mu g v_t \, \mathrm{d}s\} . \end{cases}$$

Let us set

$$\mathscr{L}(v, \mu) = \tfrac{1}{2} a(v, v) - L(v) + \int_{\Gamma_K} \mu g v_t \, \mathrm{d}s .$$

Now, instead of $(\mathscr{P})$, we shall study the problem of finding a saddle-point (w, λ) of $\mathscr{L}$ on $K \times \Lambda$, i.e.:

(P) $$\mathscr{L}(w, \mu) \leqq \mathscr{L}(w, \lambda) \leqq \mathscr{L}(v, \lambda) \quad \forall v \in K, \quad \forall \mu \in \Lambda .$$

The relation between $(\mathscr{P})$ and **(P)** is given by

Theorem 3.1. *There exists a unique saddle-point of* $\mathscr{L}$ *on* $K \times \Lambda$. *Moreover,*

$$w = u, \quad \lambda g = -T_t(u),$$

where $u \in K$ *is the solution of* $(\mathscr{P})$.

Proof. As the bilinear form a is V-elliptic due to Korn's inequality and Λ

is a bounded, convex subset of $L^2(\Gamma_K)$, the existence of a saddle-point (w, λ) is a direct consequence of [6], *Prop.* 6.2.4.

If (w, λ) is such a saddle-point, then (see [6], *Prop.* 6.1.6)

$$\frac{\partial \mathscr{L}}{\partial v}(w, \lambda)(v - w) \geqq 0 \quad \forall v \in K ,$$

$$\frac{\partial \mathscr{L}}{\partial \mu}(w, \lambda)(\mu - \lambda) \leqq 0 \quad \forall \mu \in \Lambda .$$

Applying Green's formula to the first inequality we have:

$$g\lambda = -T_t(w) \quad \text{on} \quad \operatorname{supp} g . \tag{3.2}$$

As $T_t(w) = 0$ on $\Gamma_K \setminus \operatorname{supp} g$, the previous identity is true on the whole Γ_K.

Moreover, the pair $(w, \lambda) \in K \times \Lambda$, being a saddle-point of $\mathscr{L}$ on $K \times \Lambda$, satisfies

$$\mathscr{J}(w) = \sup_{\mu \in \Lambda} \mathscr{L}(w, \mu) = \mathscr{L}(w, \lambda) = \inf_{v \in K} \sup_{\mu \in \Lambda} \mathscr{L}(v, \mu) = \inf_{v \in K} \mathscr{J}(v) = \mathscr{J}(u) .$$

As the solution u of $(\mathscr{P})$ is unique, necessarily $w = u$. The uniqueness of λ on $\operatorname{supp} g$ follows from (3.2).

Hence, the contact problem with a given friction can be approximated, starting from the saddle-point formulation $(\boldsymbol{P})$. This approach has some advantages. *First* of all, the minimization problem for a non-differentiable functional is replaced by the saddle-point formulation for a functional, which is smooth in all components. *Secondly*, the Lagrange multiplier λ has a "nice" physical meaning and the saddle-point formulation makes it possible to approximate it independently of the displacement field u. *Finally*, the approximation of our problem, which is based on the saddle-point formulation, offers a very effective algorithm for its numerical realization (see below).

Remark 3.1. In the approach, described above, only the dualization of the non-differentiable term is used. In order to release the geometrical constraint $v \in K$, another set of Lagrange multipliers has to be introduced. This formulation, involving two fields of Lagrange multipliers, together with its approximation by finite elements, is analysed in [7]. The approach, presented in the present paper, is suitable for the approximation of the so-called semicoercive cases, when the bilinear form is a V/P-elliptic only (P denotes the set of rigid body motions).

4. APPROXIMATION OF $(\boldsymbol{P})$

The general idea of the approximation of the saddle-point formulation is the following:

Let $\{\mathscr{K}_h\}$, $\mathscr{K}_h \subset V$, $\{\Lambda_h\}$, $\Lambda_h \subset L^2(\Gamma_K)$ be the families of sets approximating K

and Λ, respectively. Instead of $(\mathbf{P})$, we shall consider the following problem:

$(\mathfrak{P}_h)$ find $(w_h, \lambda_h) \in \mathscr{K}_h \times \Lambda_h$ such that

$$\mathscr{L}(w_h, \mu_h) \leqq \mathscr{L}(w_h, \lambda_h) \leqq \mathscr{L}(v_h, \lambda_h) \quad \forall v_h \in \mathscr{K}_h, \quad \forall \mu_h \in \Lambda_h .$$

One can formulate conditions concerning $\mathscr{K}_h$, Λ_h under which (w_h, λ_h) tends to (w, λ) in the corresponding norms, where (w, λ) is the solution of $(\mathbf{P})$ (see [8]). Moreover, the first component w_h minimizes the functional

$$\mathscr{I}_0(v_h) + j_h(v_h) \quad \text{over} \quad \mathscr{K}_h ,$$

where

$$j_h(v_h) = \sup_{\mu_h \in \Lambda_h} \int_{\Gamma_K} g \mu_h v_{ht} \, ds .$$

In what follows we use the following choice of $\mathscr{K}_h$, Λ_h:

$$\mathscr{K}_h = K_h , \quad \Lambda_h = \Lambda , \tag{4.1}$$

with K_h and Λ given by (2.2) and (3.1), respectively. In this case

$$j_h(v_h) = j(v_h) \quad \forall v_h \in V_h , \quad h \in (0, 1)$$

and $w_h \in K_h$ coincides with the solution u_h of $(\mathscr{P}_h)$.

Remark 4.1. In (4.1) the family $\{\Lambda_h\}$ does not depend on h. Nevertheless, some other constructions of Λ_h are possible. One can use for example "finite" dimensional approximations" of Λ, when Λ_h is a closed, convex and bounded subset of some finite-dimensional space L_h (see [8]).

Let us assume *a regular family* $\{\tau_h\}$, $h \to 0_+$, *of triangulations* of $\bar{\Omega}$. We shall study what happens if $h \to 0_+$. According to Theorem 2.1 we have

$$w_h = u_h \to u , \quad h \to 0_+ ,$$

provided all the assumptions of Theorem 2.1 are satisfied. Let us study the behaviour of the second component λ_h. (It should be noted that the second component λ_h depends on h, even if $\Lambda_h = \Lambda$ are independent of h.)

Theorem 4.1. *Let all the assumptions of Theorem* 2.1 *be satisfied. Then*

$$\lambda_h \rightharpoonup \lambda , \quad h \to 0_+ , \quad \text{weakly in} \quad L^2(\Gamma_K) ,$$

where λ is the second component of the saddle-point of $\mathscr{L}$ on $K \times \Lambda$.

Proof. From the definition of Λ it follows that the sequence $\{\lambda_h\}$, $h \in (0, 1)$, is bounded in $L^2(\Gamma_K)$. Hence there exists a subsequence $\{\lambda_{h'}\} \subset \{\lambda_h\}$ and an element $\chi \in \Lambda$ such that

$$\lambda_{h'} \rightharpoonup \chi , \quad h' \to 0_+ , \quad \text{in} \quad L^2(\Gamma_K) . \tag{4.2}$$

We shall show that $\chi = \lambda$. Following the definition of $(\mathscr{P}_{h'})$ we have

$$(4.3) \qquad a(w_{h'}, v_{h'} - w_{h'}) + \int_{\Gamma_K} g\lambda_{h'}(v_{h't} - w_{h't})\,\mathrm{d}s \geqq L(v_{h'} - w_{h'}),$$

$$\int_{\Gamma_K} g(\mu - \lambda_{h'})\,w_{h't}\,\mathrm{d}s \leqq 0 \quad \forall \mu \in \Lambda, \quad \forall v_{h'} \in K_{h'}.$$

As noted before,

$$(4.4) \qquad w_{h'} \to u, \quad h' \to 0_+, \quad \text{in} \quad (H^1(\Omega))^2.$$

Let $v \in K$ be an arbitrarily chosen element. Then there exists a sequence $\{v_{h'}\}$, $v_{h'} \in K_{h'}$, such that

$$(4.5) \qquad v_{h'} \to v \quad \text{in} \quad (H^1(\Omega))^2 \quad \text{see (2.4)}).$$

Passing to the limit for $h' \to 0_+$ in (4.3), taking into account (4.4), (4.5) as well as the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Gamma_K)$, we obtain the following system of inequalities:

$$a(u, v - u) + \int_{\Gamma_K} g\,\chi(v_t - u_t)\,\mathrm{d}s \geqq L(v - u) \quad \forall v \in K,$$

$$\int_{\Gamma_K} g(\mu - \chi)\,u_t\,\mathrm{d}s \leqq 0 \quad \forall \mu \in \Lambda,$$

which means that (u, χ) is a saddle point of $\mathscr{L}$ on $K \times \Lambda$. Since the saddle point is unique (Theorem 3.1),

$$\chi = \lambda = -g\,T_t(u).$$

As λ is unique, the whole sequence $\{\lambda_h\}$ tends weakly to λ in the $L^2(\Gamma_K)$-norm.

Remark 4.2. As the embedding of $L^2(\Gamma_K)$ into $H^{-1/2}(\Gamma_K)$ is compact, (4.2) implies the strong convergence of λ_h to λ in the $H^{-1/2}(\Gamma_K)$-norm.

For the numerical realization of $(\mathfrak{P}_h)$ one can use *the classical Uzawa's algorithm* (see [6] or [1]).

Uzawa's algorithm proceeds as follows:

we choose an element $\lambda_0 \in \Lambda$;
if $\lambda_k \in \Lambda$ is known, we look for $u_h^{(k)} \in K_h$ satisfying

$$\mathscr{L}(u_h^{(k)}, \lambda_k) = \min_{v \in K_h} \left\{ \mathscr{J}_0(v) + \int_{\Gamma_K} \lambda_k g v_t\,\mathrm{d}s \right\},$$

after which we replace λ_k by λ_{k+1} as follows:

$$\lambda_{k+1} = \Pi_\Lambda(\lambda_k + \varrho g u_{ht}^{(k)}),$$

where Π_Λ denotes the projection of $L^2(\Gamma_K)$ on Λ. ϱ is a positive parameter satisfying $\varrho_1 \leqq \varrho \leqq \varrho_2$, $\varrho_1, \varrho_2 > 0$.

It is readily seen that the projection Π_Λ is given by the following relations:

$$\Pi_\Lambda\,\mu(x) = \begin{cases} -1, & \text{if} \quad \mu(x) < -1 \\ \mu(x), & \text{if} \quad |\mu(x)| \leqq 1 \\ 1, & \text{if} \quad \mu(x) > 1, \quad x \in \Gamma_K . \end{cases}$$

It is well known (see [6]) that there exist $\varrho_1, \varrho_2 > 0$ such that for any choice $\varrho \in (\varrho_1, \varrho_2)$, the components $u_h^{(k)}$ tend to u_h (i.e. to the first component of the solution of $(\mathbf{P}_h)$).

5. NUMERICAL REALIZATION OF CONTACT PROBLEMS WITH A GIVEN FRICTION

The present section deals with implementation of Uzawa's algorithm on computers.

Each iterative step of Uzawa's method consists of two substeps. The first is the problem of quadratic programming, the second is the computation of the projection Π_Λ. The latter substep is very simple. The Lagrange multipliers λ will be defined by their values at a finite number of points from Γ_K. These points will coincide with the nodes of the quadrature formula used for the approximation of $\int_{\Gamma_K} g\lambda u_t \,\mathrm{d}s$. Therefore we define the set

$$\Lambda_E = \{z \in R^Q \mid |z_i| \leqq 1\} .$$

z_i will be interpreted as the values of $\lambda \in \Lambda$ at the nodes of the chosen quadrature formula. The projection Π_Λ is now replaced by "pointwise" projection Π_{Λ_E} of R^Q on Λ_E, the i-th component of which is given by

$$(\Pi_{\Lambda_E}\mu)_i = \begin{cases} -1 & \text{if} \quad \mu_i < -1 , \\ \mu_i & \text{if} \quad |\mu_i| \leqq 1 , \\ 1 & \text{if} \quad \mu_i > 1 . \end{cases}$$

As was already mentioned, the first part of each iterative step is the minimization problem for a quadratic function over a convex set given by linear constraints. Expressed in the matrix form, we look for $x^* \in K_E$ such that

$$\mathscr{G}(x^*) \leqq \mathscr{G}(x) \quad \forall x \in K_E , \tag{5.1}$$

where

$$\mathscr{G}(x) = \tfrac{1}{2}(x, Cx)_{R^m} - (\mathscr{F}, x)_{R^m} ,$$

$$K_E = \{x \in R^m \mid B'x \leqq 0,\ B''x = 0\} .$$

Here C denotes the stiffness matrix, $\mathscr{F} \in R^m$ is a vector obtained by the integration of the linear term of $\mathscr{L}(v_h, \lambda_h)$. B', B'' are rectangular matrices of the ranges $n_1 \times m$ and $n_2 \times m$, respectively. If we suppose that Γ_K is composed from one straight line segment we see that each row contains at least one and at most two non-zero elements and at most one non-zero element in each column.

Elements of B', B'' are formed by components of the outward unit normal vector with respect to Γ_K and to Γ_0, respectively. The set of equality constraints can be empty, if the conditions on Γ_0 are respected during the construction of the matrix C.

The minimization problem (5.1) can be realized by means of *the conjugate gradient method*, which proceeds as follows (see [9]):

(i) choose $x_0 \in K_E$;

(ii) set $\mathscr{I}(x_0) = \{1, 2, \ldots, n_2\} \cup \{i \mid (b_i', x_0)_{R^m} = 0\}$,
where b_i' is the i-th row of B';

(iii) form a matrix $A_{\mathscr{I}(x_0)}$, the rows of which are the vectors b_i' and b_i'', $i \in \mathscr{I}(x_0)$, and calculate

$$\mathscr{G}'(x_0) = Cx_0 - \mathscr{F},$$

$$P_{\mathscr{I}(x_0)} = A^{\mathsf{T}}_{\mathscr{I}(x_0)}\left(A_{\mathscr{I}(x_0)}A^{\mathsf{T}}_{\mathscr{I}(x_0)}\right)^{-1} A_{\mathscr{I}(x_0)}\,,$$

$$u(x_0) = -\left(A_{\mathscr{I}(x_0)}A^{\mathsf{T}}_{\mathscr{I}(x_0)}\right)^{-1} A_{\mathscr{I}(x_0)}\,\mathscr{G}'(x_0),$$

where $A^{\mathsf{T}}_{\mathscr{I}(x_0)}$ is the transpose of $A_{\mathscr{I}(x_0)}$.

(iv) We distinguish two cases:

(I) $(I - P_{\mathscr{I}(x_0)})\,\mathscr{G}'(x_0) \neq 0$. Then set

$p_1 = -(I - P_{\mathscr{I}(x_0)})\,\mathscr{G}'(x_0)$ and calculate

$$(*)\quad \alpha_{k+1} = -\frac{(\mathscr{G}'(x_k), p_{k+1})_{R^m}}{(p_{k+1}, Cp_{k+1})_{R^m}}\,,$$

$$\bar{\alpha}'_{k+1} = \min_{i=1,\ldots,n_1} \frac{-(b_i', x_k)_{R^m}}{(b_i', p_{k+1})_{R^m}}\,,$$

$$\bar{\alpha}''_{k+1} = \min_{i=1,\ldots,n_2} \frac{-(b_i'', x_k)_{R^m}}{(b_i'', p_{k+1})_{R^m}}\,,$$

where the minimum is taken for i satisfying

$$(b_i', p_{k+1})_{R^m} > 0 \quad\text{and}\quad (b_i'', p_{k+1})_{R^m} > 0.$$

Define

$$\bar{\alpha}_{k+1} = \min\left(\bar{\alpha}'_{k+1}, \bar{\alpha}''_{k+1}\right).$$

If $\alpha_{k+1} \leqq \bar{\alpha}_{k+1}$ set

$$x_{k+1} = x_k + \alpha_{k+1}p_{k+1},$$

$$p_{k+2} = -(I - P_{\mathscr{I}(x_0)})\,\mathscr{G}'(x_{k+1}) + \frac{\|(I - P_{\mathscr{I}(x_0)})\,\mathscr{G}'(x_{k+1})\|^2}{\|(I - P_{\mathscr{I}(x_0)})\,\mathscr{G}'(x_k)\|^2}\,p_{k+1}\,,$$

$k = 0, 1, \ldots.$

If x_{k+1} minimizes $\mathscr{G}(x)$ on the convex set given by

$$(b'_i, x) = 0\,, \quad i \in \mathscr{J}(x_0) - \{1, \ldots, n_2\}\,,$$
$$(b''_i, x) = 0\,, \quad i = 1, 2, \ldots, n_2\,,$$

set $x_0 = x_{k+1}$ and go to (v).

Otherwise go to (*).

If $\alpha_{k+1} > \bar{\alpha}_{k+1}$ set

$x_{k+1} = x_k + \bar{\alpha}_{k+1} p_{k+1}$

and go to (ii).

(II) If $(I - P_{\mathscr{J}(x_0)})\, \mathscr{G}'(x_0) = 0$ go to (v).

(v) If $(u(x_0))_i \geqq 0 \quad \forall i \in \mathscr{J}(x_0) - \{1, 2, \ldots, n_2\}$,
then x_0 is the desired minimum of $\mathscr{G}$ on the set K_E.
If there exists $i \in \mathscr{J}(x_0) \setminus \{1, 2, \ldots, n_2\}$ such that
$(u(x_0))_i < 0$, set $\mathscr{J}(x_0) - \{i\} \to \mathscr{J}(x_0)$ and go to (iii).

Clearly, this step is a limiting factor of the efficiency of Uzawa's algorithm. However, certain features of our problem enable us to modify it in such a way that the outcoming version repiesents an economic tool for the numerical realization.

The above mentioned features are the following ones:

- the stiffness matrix is the same during the whole iterative process;
- the numbei of constrained variables is small as compared with the number of all variables;
- only a few components on the right hand side change, namely the normal and tangential tractions along Γ_K, which are related to constrained variables.

Taking into account all these circumstances, the substructuring technique for the minimization of $\mathscr{G}$ over K_E may be used (see [10]), where this method is applied to the frictionless contact problem).

Let us suppose that unknowns are arranged in such a way that all constrained variables are listed last. Due to these assumptions, we may split a vector $x \in R^m$ and write

$$x = (x_1, x_2) \in R^{m-k} \times R^k\,, \quad k = n_1 + n_2\,,$$

where x_1, x_2 correspond to the free and constrained vaiiables, respectively. The closed convex set K_E can be now written as

$$K_E = \{x \in R^m,\ x = (x_1, x_2),\ x_1 \in R^{m-k},\ x_2 \in R^k,\ B'x_2 \leqq 0,\ B''x_2 = 0\}\,.$$

Similarly, we can split the vector $\mathscr{F}$ and decompose the matrix C;

$$C = \begin{pmatrix} C_{11}, & C_{12} \\ C_{21}, & C_{22} \end{pmatrix},$$

where C_{11}, C_{22} are square matrices of the range $m - k$ and k, respectively, and C_{12}, C_{21} are rectangular matrices $(m - k) \times k$ and $k \times (m - k)$, respectively. Since C is symmetric, we have

$$C_{21} = C_{12}^{\mathrm{T}} .$$

Let $x^* \in K_E$ be a solution of (5.1), i.e. $x^* = (x_1^*, x_2^*) \in K_E$ fulfils

$$(5.2) \qquad (Cx^*, y - x^*)_{R^m} \geqq (\mathscr{F}, y - x^*)_{R^m} \quad \forall y \in K_E .$$

Let $y = (y_1, y_2) \in K_E$ be a vector of the form $y_1 = x_1^* \pm z_1$, $z_1 \in R^{m-k}$ arbitrary, $y_2 = x_2^*$. Substituting y into (5.2) we get

$$(5.3) \qquad C_{11}x_1^* + C_{12}x_2^* = \mathscr{F}_1$$

which implies

$$(5.4) \qquad x_1^* = -C_{11}^{-1}C_{12}x_2^* + C_{11}^{-1}\mathscr{F}_1 .$$

Now we choose $y = (y_1, y_2)$ in such a way that $y_1 = x_1^*$, $y_2 = z_2$, where

$$z_2 \in \tilde{K}_E = \{z \in R^k \mid B'z \leqq 0,\ B''z = 0\} .$$

Clearly $y \in K_E$.

Inserting it into (5.2) we ariiwe at

$$(5.5) \qquad (z_2 - x_2^*, C_{21}x_1^* + C_{22}x_2^*)_{R^k} \geqq (z_2 - x_2^*, \mathscr{F}_2)_{R^k} \quad \forall z_2 \in \tilde{K}_E .$$

Taking into account (5.4), we deduce that $x_2^* \in \tilde{K}_E$ is the solution of the variational inequality

$$(z_2 - x_2^*, \tilde{C}x_2^*)_{R^k} \geqq (\mathfrak{F}_2, z_2 - x_2^*)_{R^k} \quad \forall z_2 \in \tilde{K}_E$$

with

$$\tilde{C} = C_{22} - C_{21}C_{11}^{-1}\, C_{12} \quad \text{and} \quad \mathfrak{F}_2 = \mathscr{F}_2 - C_{21}C_{11}^{-1}\mathscr{F}_1 ,$$

or equivalently: x_2^* is a minimizer of

$$\mathfrak{G}(z) = \tfrac{1}{2}(z, \tilde{C}z)_{R^k} - (\mathfrak{F}_2, z)_{R^k} , \quad z \in R^k$$

over the closed convex set $\tilde{K}_E$. It is readily seen $\tilde{C}$ is symmetric and it can be shown that it is obtained from C by the elimination of all unconstrained unknowns. Hence $\tilde{C}$ is positive definite.

NUMERICAL EXAMPLES

In order to illustrate the approach discussed above, let us consider the Signorini problem for an elastic body given by *Fig.* 2, subjected to a *horizontal surface traction* $P_1 = 1000$ only. The decomposition of the boundary $\partial\Omega$ into Γ_u, Γ_0, Γ_τ and Γ_K as well as the triangulation τ_h of $\bar{\Omega}$ are clear from the figure.

We discuss two cases:

(i) the frictionless problem, corresponding to $g \equiv 0$ on Γ_K;

(ii) the contact problem with a given friction and $g = 150$ along Γ_K.

The values of normal and tangential components of displacements u_n, u_t, respectively, at the points lying on Γ_K are given in *Tables* 1, 2. The distribution of normal stresses for both cases is shown in *Fig.* 3. Finally, the computed values of tangential components of stresses are compared with the product $g\lambda$ in the case (ii) (*Fig.* 4).

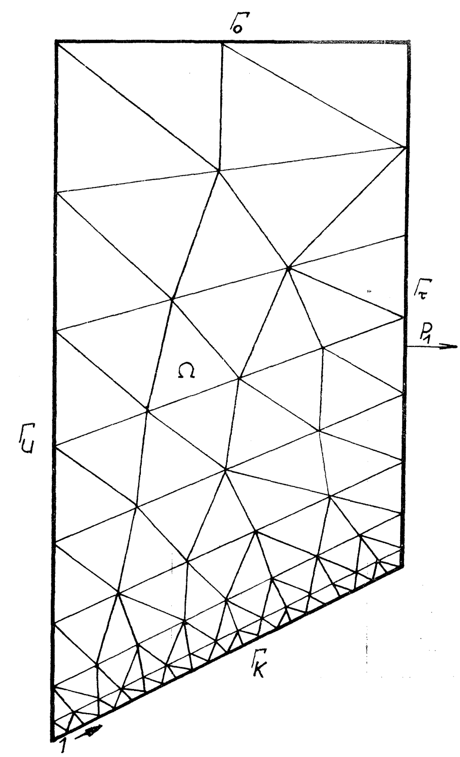

Fig. 2.

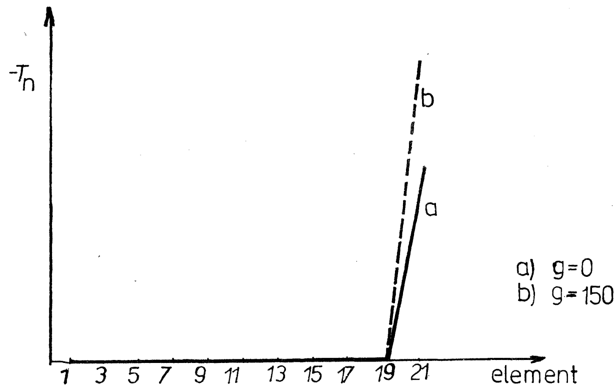

Fig. 3.

Tab. 3

Element No	Normal stresses for $g = 0$	Normal stresses for $g = 150$
1	8·81651	— 81·10406
3	— 1·57262	— 0·18515
5	8·35742	2·67252
7	— 2·61975	— 5·95142
9	3·37544	5·14847
11	3·03627	2·64455
13	9·48375	12·35480
15	— 0·96811	— 3·79457
17	3·90454	2·30423
19	10·67592	8·22925
21	—287·76611	—435·79272

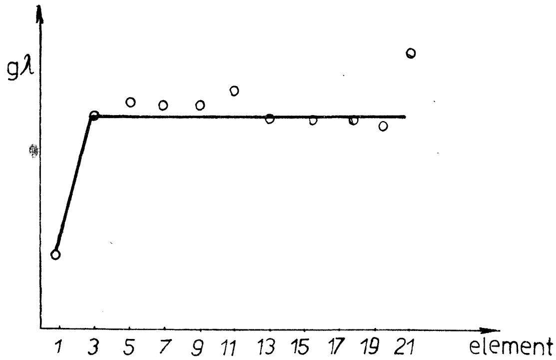

Fig. 4.

Tab. 4

Element No	Tangential stresses	$g\lambda$
1	— 56·96169	—68·2362
3	—149·17897	—150·0
5	—157·75708	—150·0
7	—157·75020	—150·0
9	—158·73059	—150·0
11	—162·73120	—150·0
13	—148·44574	—150·0
15	—149·89653	—150·0
17	—145·13040	—150·0
19	—142·27200	—150·0
21	—191·31741	—150·0

Tab. 1

Node	Normal component of displacement	Tangential component of displacement
1	0·0	0·0
2	−0·708244 E-04	0·124510 E-03
3	−0·140124 E-03	0·244349 E-03
4	−0·208867 E-03	0·359382 E-03
5	−0·269836 E-03	0·471946 E-03
6	−0·323546 E-03	0·580739 E-03
7	−0·367647 E-03	0·690516 E-03
8	−0·404030 E-03	0·799063 E-03
9	−0·431806 E-03	0·905754 E-03
10	−0·450308 E-03	0·101547 E-02
11	−0·458580 E-03	0·112786 E-02
12	−0·457524 E-03	0·123950 E-02
13	−0·440191 E-03	0·135739 E-02
14	−0·416487 E-03	0·147093 E-02
15	−0·379821 E-03	0·159732 E-02
16	−0·326718 E-03	0·172764 E-02
17	−0·269428 E-03	0·185998 E-02
18	−0·203394 E-03	0·199320 E-02
19	−0·129652 E-03	0·215468 E-02
20	−0·521742 E-04	0·231479 E-02
21	−0·390520 E-09	0·249076 E-02
22	−0·492480 E-09	0·269887 E-02
23	−0·108980 E-09	0·288540 E-02

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Tab. 2

Node	Normal component of displacement	Tangential component of displacement
1	0·0	0·0
2	−0·364244 E-04	0·117746 E-05
3	−0·859590 E-04	0·165844 E-04
4	−0·152534 E-03	0·467666 E-04
5	−0·210771 E-03	0·810665 E-04
6	−0·259828 E-03	0·121596 E-03
7	−0·300198 E-03	0·170359 E-03
8	−0·338336 E-03	0·225292 E-03
9	−0·369280 E-03	0·281407 E-03
10	−0·389108 E-03	0·344504 E-03
11	−0·400242 E-03	0·411434 E-03
12	−0·401480 E-03	0·481490 E-03
13	−0·383156 E-03	0·561246 E-03
14	−0·358959 E-03	0·646560 E-03
15	−0·321456 E-03	0·739529 E-03
16	−0·266279 E-03	0·846110 E-03
17	−0·208209 E-03	0·962450 E-03
18	−0·144343 E-03	0·108998 E-02
19	−0·754524 E-04	0·123141 E-02
20	−0·787149 E-05	0·138532 E-02
21	−0·140865 E-09	0·155774 E-02
22	−0·128434 E-09	0·175705 E-02
23	−0·521430 E-10	0·194423 E-02

Souhrn

APROXIMACE A NUMERICKÉ ŘEŠENÍ KONTAKTNÍ ÚLOHY SE TŘENÍM

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V této práci je studováno numerické řešení kontaktní úlohy s daným třením. Celý problém je vhodným zavedením multiplikátorů převeden na nalezení sedlového bodu Lagrangeovy funkce $\mathscr{L}$ na konvexní množině $K \times \Lambda$. V práci je dále definována aproximace tohoto sedlového bodu, dokázána konvergence a odhadnut řád konvergence. K numerickému řešení je použit Uzawův algoritmus. Na závěr jsou uvedeny konkrétní příklady.

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