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EUCLIDEAN SPACE MOTIONS WITH AFFINELY EQUIVALENT TRAJECTORIES

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Let a Euclidean motion $g(t)$, $t \in I$, of the moving space $\bar{E}_3$ in the fixed space E_3 be given. Let us choose orthonormal frames $\bar{\mathcal{R}}(t)$ and $\mathcal{R}(t)$ in $\bar{E}_3$ and E_3, respectively, in such a way that $g(t)\,\bar{\mathcal{R}}(t) = \mathcal{R}(t)$ for all $t \in I$. The pair $(\bar{\mathcal{R}}, \mathcal{R})$ is called a moving frame of the motion $g(t)$. Let us denote

$$\bar{\mathcal{R}}' = \bar{\mathcal{R}}\psi,\ \mathcal{R}' = \mathcal{R}\varphi,\ \varphi - \psi = \omega\,, \quad \varphi + \psi = \eta\,.$$

If the moving frame is changed, we get new matrices $\tilde{\omega}$ and $\tilde{\eta}$ instead of ω and η, where

$$\tilde{\omega} = h^{-1}\omega h\,, \quad \tilde{\eta} = h^{-1}\eta h + 2h^{-1}h'\,,$$

where $h = h(t)$ is the matrix of the change.

Let us write

$$\omega = \begin{pmatrix} 0, & 0 \\ \omega_0, & \omega_1 \end{pmatrix},$$

where ω_0 is a 3-column and ω_1 is a 3×3 skew-symmetric matrix, and similarly for η, φ and ψ.

In what follows we shall suppose that $\omega_1 \neq 0$ for all $t \in I$. (It means that the instantaneous motion is not a translation.) If this is the case, then ω can be given the form

$$\omega_0 = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}, \quad \omega_1 = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & -1 \\ 0, & 1, & 0 \end{pmatrix}, \quad v \geqq 0$$

by a suitable choice of the moving frame and the parameter t of the motion.

In order to simplify some of the computations we shall use the following identification for 4×4 matrices of a special form:

$$\begin{pmatrix} 0, & 0, & 0, & 0 \\ a_{10}, & a_{11}, & a_{12}, & a_{13} \\ a_{20}, & a_{21}, & a_{22}, & -a_{32} \\ a_{30}, & a_{31}, & a_{32}, & a_{22} \end{pmatrix} \leftrightarrow \begin{pmatrix} 0, & 0, & 0 \\ a_{10}, & a_{11}, & a_{12} + ia_{13} \\ a_{20} + ia_{30}, & a_{21} + ia_{31}, & a_{22} + ia_{32} \end{pmatrix}.$$

Then

(1)
$$\omega = \begin{pmatrix} 0, & 0, & 0 \\ v, & 0, & 0 \\ 0, & 0, & \mathrm{i} \end{pmatrix}$$

and

$$H = \begin{pmatrix} 1, & 0, & 0 \\ \lambda, & 0, & 0 \\ 0, & 0, & \exp(\mathrm{i}\alpha) \end{pmatrix}, \quad \lambda, \alpha \in R\,,$$

is the group of all changes of the of the moving frame which preserve ω. Let us further denote

$$\eta = \begin{pmatrix} 0, & 0, & 0 \\ \eta_1, & 0, & -\eta_{21} \\ \eta_2, & \eta_{21}, & \mathrm{i}\eta_{22} \end{pmatrix}, \quad \text{where} \quad \eta_1, \eta_{22} \in R\,, \quad \eta_2, \eta_{21} \in C\,.$$

Changing the moving frame from H we get

$$\tilde{\eta}_{21} = \exp(-\mathrm{i}\alpha)\,\eta_{21}\,, \quad \tilde{\eta}_2 = \exp(-\mathrm{i}\alpha)\,(\eta_2 + \lambda\eta_{21})\,.$$

Now we have two cases to consider:

a) $\eta_{21} \neq 0$. This is the case of a general space motion, axoids are not cylinders. We can change the moving frame in such a way that $\eta_{21} > 0$, $\eta_2 = \mathrm{i}\eta_{30}$, $\eta_{21}, \eta_{30} \in R$. The moving frame is then fixed.

b) $\eta_{21} = 0$, $\eta_2 \neq 0$. This is the case of a cylindrical motion, axoids are cylinders and the motion preserves one direction. We have $\tilde{\eta}_2 = \exp(-\mathrm{i}\alpha)\,\eta_2$, $\tilde{\eta}_1 = \eta_1 + \lambda'$. We can change the moving frame to get $\eta_2 = \mathrm{i}\eta_{30}$, $\eta_{30} \in R$, $\eta_{30} > 0$, $\eta_1 = v$. The moving frame is fixed up to a constant translation along the 1st axis.

Remark. If $\eta_{21} = \eta_2 = 0$, we get $\eta = 0$ in a suitable moving frame. The motion is any space motion with fixed axis. We leave this case out as a trivial one.

In accordance with the usual notation we shall write

(2)
$$\varphi = \begin{pmatrix} 0, & 0, & 0 \\ \mu_2, & 0, & -\varkappa_1 \\ \mathrm{i}\mu_1, & \varkappa_1, & \mathrm{i}\varkappa_2 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0, & 0, & 0 \\ \bar{\mu}_2, & 0, & -\bar{\varkappa}_1 \\ \mathrm{i}\bar{\mu}_1, & \bar{\varkappa}_1, & \mathrm{i}\bar{\varkappa}_2 \end{pmatrix},$$

all entries being real.

Then $\varkappa_1 = \bar{\varkappa}_1$, $\mu_1 = \bar{\mu}_1$, $\varkappa_2 - \bar{\varkappa}_2 = 1$, $\mu_2 - \bar{\mu}_2 = v$. In the general case a) we have $\varkappa_1 > 0$, for cylindrical motions in case b) we have $\varkappa_1 = 0$, $\mu_1 > 0$, $\mu_2 = v\,(\bar{\mu}_2 = 0)$.

Let us suppose for a moment that the matrices φ and ψ are written without the use of complex numbers. The operator Ω_k of the k-th derivative of the trajectory of a point $\bar{X} \in \bar{E}_3$ is defined by $X^{(k)} = \mathcal{R}\Omega_k X$, where $\bar{X} = \bar{\mathcal{R}}X$. For Ω_k we have (see [2])

(3)
$$\Omega_{k+1} = \varphi\Omega_k - \Omega_k\psi + \Omega'\,, \quad \Omega_1 = \omega\,.$$

If we write

$$\Omega_k = \begin{pmatrix} 0, & 0 \\ \vartheta_k, & \theta_k \end{pmatrix},$$

we get

(4) $$\vartheta_{k+1} = \varphi_1 \vartheta_k + \theta_k \psi_0 + \vartheta'_k, \quad \theta_{k+1} = \varphi_1 \theta_k - \theta_k \psi_1 + \theta'_k.$$

Unfortunately, formulas (3) and (4) do not preserve their form in the case of a general motion if we use the complex number identification mentioned above. On the other hand, we have

Lemma 1. *Formulas* (3) *and* (4) *remain valid in the cylindrical case if the notation from* (1) *and* (2) *is used.*

Proof. In the cylindrical case ω, φ and ψ have the form

$$\delta = \begin{pmatrix} 0, & 0, & 0 \\ a, & 0, & 0 \\ b, & 0, & c \end{pmatrix},$$

where $a \in R$, b, $c \in C$. Such matrices form an associative algebra over R and so our lemma follows from the well known representation of complex numbers by

$$a + ib \leftrightarrow \begin{pmatrix} a, & -b \\ b, & a \end{pmatrix}.$$

Definition 1. *Let G be a Lie group. We say that a motion $g(t) \subset G$ splits, if there exist nontrivial subgroups G_1 and G_2 in G such that*

a) $G_1 \cap G_2 = e$,

b) $g_1 g_2 = g_2 g_1$ for all $g_1 \in G_1$ and $g_2 \in G_2$,

c) $g(t) = g_1(t)\, g_2(t)$, where $g_1(t) \subset G_1$, $g_2(t) \subset G_2$.

We call $g_i(t)$ the factor of $g(t)$ in G_i for $i = 1, 2$.

Remark. The factors are unique for given G_1 and G_2. If $g(t)$ splits, it belongs to the direct product of G_1 and G_2.

Lemma 2. *Any cylindrical motion in E_3 splits into a Euclidean plane motion and a translation.*

Proof. The Frenet formulas for the moving frame split into two parts which integrate separately as can be seen from the proof of Lemma 1.

The plane motion from Lemma 2 will be called the plane factor of the cylindrical motion.

Definition 2. *We say that all trajectories of a given Euclidean space motion $g(t)$ are affinely equivalent if there exists a space curve $X(t)$ such that for any* *trajectory* $Y(t) = g(t)\,\bar{Y}$ *od* $\bar{Y} \in \bar{E}_3$ *there is an affine mapping* m *of the space* E_3 (m *need not be regular*) *such that* $m[X(t)] = Y(t)$ *for all* $t \in I$.

Definition 3. *A Euclidean space motion is called a 3-Darboux motion if all its trajectories are affinely equivalent and at least one of them is not a plane curve.*

Lemma 3. *If* $g(t)$ *is a 3-Darboux Euclidean space motion, then there exist unique functions* $\alpha_i(t)$, $i = 1, 2, 3$, *such that*

$$\Omega_4(t) = \sum_{i=1}^{3} \alpha_i(t)\, \Omega_i(t)\,. \tag{5}$$

Proof. The existence of functions $\alpha_i(t)$ was proved in [2]. If those functions are not unique, we get $0 = \sum_{i=1}^{3} \beta_i(t)\, \Omega_i(t)$ for some functions $\beta_i(t)$ and all trajectories are plane curves.

Lemma 4. *Any 3-Darboux motion in* E_3 *is cylindrical. Moreover, if* $v \neq 0$, *then* (5) *is also sufficient.*

Proof. Direct computation from (4) gives

$$\Theta_1 = \omega_1,\ \Theta_2 = \begin{pmatrix} 0, & 0, & \varkappa_1 \\ 0, & -1, & 0 \\ -\varkappa_1, & 0, & -1 \end{pmatrix},$$

$$\Theta_3 = \begin{pmatrix} 0, & \varkappa_1(1 - \bar{\varkappa}_2), & \varkappa_1' \\ \varkappa_1(\varkappa_2 + 1), & 0, & \varkappa_1^2 + 1 \\ -\varkappa_1', & -\varkappa_1^2 - 1, & 0 \end{pmatrix}, \quad (\Theta_4)_{11} = -3\varkappa_1^2\,.$$

This shows the impossibility of solving (5) for $\varkappa_1 \neq 0$. Further, let (5) be satisfied. Then all trajectories are solutions of the differential equation $Y^{IV} - \sum_{i=1}^{3} \alpha_i(t)\, Y^{(i)} = 0$. So if at least one of the trajectories is not a plane curve, all trajectories are affinely equivalent. If all trajectories are plane curves, then (see [1]) either the motion is the 2-Darboux motion and the functions $\alpha_i(t)$ are not unique, or $v = 0$.

Lemma 5. *Any Euclidean space motion which has infinitely many points with straight line trajectories is a cylindrical motion.*

Proof. From (4) we have

$$\vartheta_1 = \omega_0,\ \vartheta_2 = \begin{pmatrix} v' \\ \psi \\ 0 \end{pmatrix}, \quad \text{where} \quad \psi = \varkappa_1 v + \mu_1\,. \quad \text{Denote} \quad \vartheta_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Let x_i, $i = 1, 2, 3$, be coordinates in $\bar{E}_3$ with respect to $\bar{\mathcal{R}}$. Then the set of all points in $\bar{E}_3$ such that their trajectory has an inflexion point at a given instant is given by the equation $X' \times X'' = 0$.

Computation gives

$$(6) \qquad x_2^2 + x_3^2 + \varkappa_1 x_1 x_3 - \psi x_2 = 0\,,$$

$$v\varkappa_1 x_1 + v x_3 + v' x_2 + \varkappa_1 x_2 x_3 = 0\,,$$

$$\varkappa_1 x_3^2 + v' x_3 - v x_2 + v\psi = 0\,.$$

Let $\varkappa_1 v \neq 0$. Then (6) determines a curve in E_3 which can be parametrized by x_3:

$$x_2 = v^{-1}(\varkappa_1 x_3^2 + v' x_3 + v\psi)\,, \quad x_1 = -v^{-1}\varkappa_1^{-1}(v' x_2 + v x_3 + \varkappa_1 x_2 x_3)\,.$$

As infinitely many points satisfying (6) have straight line trajectories, equations $X' \times X''' = 0$ must have infinitely many solutions satisfying (6). Two of these equations are

$$(7) \qquad \varkappa_1(\varkappa_2 + 1)\, x_1 x_2 - \varkappa_1' x_1 x_3 + b x_2 + c x_3 = 0\,,$$

$$\varkappa_1(1 - \bar{\varkappa}_2)\, x_2 x_3 + v\varkappa_1(\varkappa_2 + 1)\, x_1 + \varkappa_1' x_3^2 + (\varkappa_1^2 + 1)\, v x_3 + a x_3 + v b = 0\,.$$

Now we substitute from (6) and look at the terms with the highest power in x_3. We get $\varkappa_2 = -1$ and $\bar{\varkappa}_2 = 1$, which contradicts $\varkappa_2 - \bar{\varkappa}_2 = 1$.

Let finally $v = 0$. From (6) we get $x_3 = 0$, $x_2(x_2 - \psi) = 0$. The equation $X' = -\psi X$, which is satisfied by any point of the moving space, implies

$$x_2' = -\varkappa_1 x_1 + \bar{\varkappa}_2 x_3\,.$$

This equation cannot have infinitely many solutions for x_1 unless $\varkappa_1 = 0$. This completes the proof.

Denote by F the set of all points in $\bar{E}_3$ given by

$$(8) \qquad |X', X'', X'''| = 0\,.$$

F is the set of all points such that their trajectory has at a given instant zero torsion.

Lemma 6. *If a Euclidean space motion has the property that all points of F have plane trajectories, then this motion is cylindrical.*

Proof. If all points of F have plane trajectories, then the equation

$$(9) \qquad |X', X'', X^{\mathrm{IV}}| = 0$$

must be true for all points of F.

Let $\varkappa_1 \neq 0$. As (8) contains the term $\varkappa_1(2 - \varkappa_2)\, x_2^3 + \varkappa_1^2(1 - 2\varkappa_2)\, x_1 x_2 x_3$ and (9) the term $-3\varkappa_1^3 x_1^2 x_3$, they both give cubic surface in E_3, as can be shown by computation. As each point of F must also satisfy (9), it must be true for their intersections with the plane at infinity as well. Let x_1, x_2, x_3 denote the homogeneous coordinates in the plane at infinity. The intersection of F with this plane has the equation

$$[\varkappa_1' x_3 + \varkappa_1(1 - \varkappa_2)\, x_2]\,(x_2^2 + x_3^2) + 3\varkappa_1^2 x_1 x_2 x_3 = 0.$$

(We get it from (8) by neglecting all lower degree terms). If this intersection is an irreducible cubic curve then the intersection of (9) with the plane at infinity must be the same and so the corresponding equation must be a multiple, but this is impossible because of the term $-3\varkappa_1^3 x_1^2 x_2$ which does not appear in (8).

In case the intersection of F with the plane at infinity is reducible, we get a contradiction with $\varkappa_1 \neq 0$ by considering all possible cases.

Remark. Any 3-Darboux motion has the property from Lemma 6 because we have $X^{IV} = \sum_{i=1}^{3} \alpha_i X^{(i)}$ and so $|X', X'', X'''| = 0$ implies $|X', X'', X^{IV}| = 0$.

Lemma 7. *Let $g(t)$ be a cylindrical motion. Let us write*

$$\Omega_k = \begin{pmatrix} 0, & 0, & 0 \\ a_k, & 0, & 0 \\ b_k, & 0, & c_k \end{pmatrix}, \quad a_k \in R, \quad b_k, c_k \in C.$$

Then

(10) $$a_k = v^{(k-1)}, \quad c_k = \mathrm{i}^k, \quad b_{k+1} = \mathrm{i}\varkappa_2 b_k - \mathrm{i}^{k+1}\mu_1 + b_k', \; k \geqq 1, \quad b_1 = 0$$

$$\tilde{\Omega}_k = \begin{pmatrix} 0, & 0 \\ b_k, & c_k \end{pmatrix}$$

at the same time gives the k-th derivative for the plane factor of $g(t)$.

Proof. We know that $a_1 = v$, $b_1 = 0$, $c_1 = \mathrm{i}$. The result follows from (3) by computation using Lemma 1.

Lemma 8. *The 3-Darboux Euclidean space motions are characterized by the existence of a function $\alpha_1(t)$ such that*

(11) $$\begin{aligned} \mu_1'(2\varkappa_2 + 1) + \mu_1\varkappa_2' &= \alpha_1\mu_1(\varkappa_2 + 1), \\ \mu_1'' - \mu_1\varkappa_2(1 + \varkappa_2) &= \alpha_1\mu_1', \\ v_1''' + v' &= \alpha_1(v'' + v), \end{aligned}$$

where $v \neq 0$ and $\varkappa_2 + 1 \neq 0$ or $v'' + v \neq 0$.

Proof. First we compute from (10):

$$b_1 = 0, \quad b_2 = \mu_1, \quad b_3 = \mu_1' + \mathrm{i}\mu_2(\varkappa_2 + 1),$$

$$b_3 = \mu_1'' - \mu_1 - \mu_1\varkappa_2(\varkappa_2 + 1) + \mathrm{i}[(2\varkappa_2 + 1)\mu_1' + \varkappa_2'\mu_1],$$

Equations (5) have the form

$$v''' = \alpha_3 v'' + \alpha_2 v' + \alpha_1 v,$$

$$1 = -\mathrm{i}\alpha_3 - \alpha_2 + \mathrm{i}\alpha_1,$$

$$\mu_1'' - \mu_1 - \mu_1\varkappa_2(\varkappa_2 + 1) = \alpha_3\mu_1' + \alpha_2\mu_1 ,$$

$$(2\varkappa_2 + 1)\,\mu_1' + \varkappa_2'\mu_1 = \alpha_3\mu_1(\varkappa_2 + 1) .$$

From them we get $\alpha_3 = \alpha_1$, $\alpha_2 = -1$ and this gives (11). Further, if $v'' + v = 0$ and $\varkappa_2 + 1 = 0$, then also $\mu_1' = 0$ and α_1 is not uniquely determined. If $v = 0$ all trajectories are plane curves.

Remark. The case $\mu_1 = 0$ was left out; $\varkappa_2 + 1 = 0$, $v'' + v = 0$, $\mu_1' = 0$ gives the 2-Darboux space motion.

Lemma 9. *The plane factor of a 3-Darboux space motion is characterized by the differential equation*

$$(12) \qquad [\mu_1'' - \mu_1\varkappa_2(1 + \varkappa_2)]\,\mu_1(\varkappa_2 + 1) = \mu_1'[(2\varkappa_2 + 1)\,\mu_1' + \varkappa_2'\mu_1] .$$

Proof. It follows from Lemma 8 immediately.

Remark. We get an interesting special case for $\varkappa_2 = -1$. Then $\mu_1 = \text{const.}$ and the plane factor is the elliptical motion in plane and v can be chosen arbitrarily provided $v'' + v \neq 0$. For instance, if we choose v in such a way that $v''' + v' = 0$ ($\alpha_1 = 0$), we get a 3-Darboux motion with all trajectories affinely equivalent to the helix. We also see that any cylindrical space motion which has the elliptical plane motion as its plane factor is a 3-Darboux space motion (except the 2-Darboux motion in space).

The plane factor of a 3-Darboux space motion satisfies the equation $\tilde{\Omega}_4 = \sum_{i=1}^{3}\alpha_i\tilde{\Omega}_i$ and if $\varkappa_2 + 1 \neq 0$, then the functions $\alpha_1, \alpha_2, \alpha_3$ are uniquely determined. We shall call such motions 3-Darboux plane motions. This means that equation (12) with the condition $\varkappa_2 + 1 \neq 0$ characterizes all 3-Darboux plane motions.

The 3-Darboux plane motions have the following geometrical characterization: For any 3-Darboux plane motion $g(t)$ there exists a space curve $K(t)$ such that any trajectory of $g(t)$ is an affine image of $K(t)$. $K(t)$ lies on a quadratic cylinder. To see it, it is enough to notice that all trajectories of 3-Darboux motions satisfy the differential equation $X^{\mathrm{IV}} + X'' = \alpha_1(X''' + X')$.

Theorem 1. *The* 3-*Darboux motions in plane are all plane motions with exactly one straight line trajectory.*

To prove Theorem 1 we shall need the following two lemmas:

Lemma 10. *Let*

$$(13) \qquad y_i' = f_i(t, y_j) , \quad i, j = 1, \ldots, n$$

be a system of n ordinary differential equations and let $F_j(t, y)$ be functions of

$n + 1$ variables such that

$$F_j(t, y) = 0 \quad \textit{for} \quad j = 1, \ldots, n \quad \textit{implies} \quad \frac{\partial F_j}{\partial t} + \sum_{i=1}^{n} \frac{\partial F_j}{\partial y_i} f_i(t, y) = 0 \quad \textit{for}$$

$j = 1, \ldots, n$

and rank $(\partial F_j/\partial y_i) = n$. *Then any functions* $y_i(t)$, $i = 1, \ldots, n$, *which satisfy* $F_j(t, y_i(t)) = 0$ for $j = 1, \ldots, n$, *satisfy* (13) as well.

Proof. Let $F_j(t, y_i(t)) = 0$ for some $y_i(t)$. Then

$$\frac{\partial F_i}{\partial t} + \sum_{i=1}^{n} \frac{\partial F_j}{\partial y_i} \cdot y_i'(t) = 0 \text{ and also } \frac{\partial F_j}{\partial t} + \sum_{i=1}^{n} \frac{\partial F_j}{\partial y_i} f_i(t, y(t)) = 0 .$$

Subtraction gives

$$\sum_{i=1}^{n} \frac{\partial F_j}{\partial y_i} [y_i' - f_i(t, y(t))] = 0 \quad \text{and so} \quad y_i' = f_i(t, y) .$$

Remark. Equations

$$\frac{\partial F_j}{\partial t} + \sum_{i=1}^{n} \frac{\partial F_j}{\partial y_i} f_i = 0$$

will be called differential consequences of (13) and $F_j = 0$

Lemma 11. *Let $g(t)$ be an affine motion in an affine space $\mathscr{A}_n$, let Ω_k be given by* (3). *Let $|X^{(k)}, X^{(l)}|$ denote all subdeterminants of order 2 of coordinates of the vectors $X^{(k)}$ and $X^{(l)}$ in $\mathscr{R}$ for any point $X \in \mathscr{A}_n$. Then for $X' \neq 0$ and $k \geqq 2$ the condition $|X', X^{(k+1)}| = 0$ is the differential consequence of $X' = -\psi X$ and $|X', X^{(j)}| = 0$, $j = 2, \ldots, k$.*

Proof. Let $|X', X^{(j)}| = |\Omega_1 X_1, \Omega_j X| = 0$ for $j = 2, \ldots, k$. The differentiation of $|\Omega_1 X, \Omega_k X|$ and substitution from $X' = -\psi X$ gives

$$\begin{aligned} &|\Omega_1' X - \Omega_1 \psi X, \Omega_k X| + |\Omega_1 X, \Omega_k' X - \Omega_k \psi X| = \\ &= |\Omega_2 X - \varphi \Omega_1 X, \Omega_k X| + |\Omega_1 X, \Omega_{k+1} X - \varphi \Omega_k X| = \\ &= |\Omega_2 X, \Omega_k X| - [|\varphi \Omega_1 X, \Omega_k X| + |\Omega_1 X, \varphi \Omega_k X|] + |\Omega_1 X, \Omega_{k+1} X| = \\ &= |\Omega_1 X, \Omega_{k+1} X| = |X', X^{(k+1)}| , \end{aligned}$$

because $\Omega_1 X \neq 0$ and $|\Omega_1 X, \Omega_2 X| = 0$ and $|\Omega_1 X, \Omega_k X| = 0$ implies $|\Omega_2 X, \Omega_k X| = 0$ and $|X', X^{(k)}| = 0$ implies $|AX', X^{(k)}| + |X', AX^{(k)}| = 0$ for any matrix A.

Proof of Theorem 1: Let $g(t)$ be a plane motion with one straight line trajectory. Then the point $x = 0$, where $x = x_2 + \mathrm{i}x_3$, cannot have a straight line trajectory because we must have

(14) $$x' = -\mathrm{i}\mu_1 - \mathrm{i}\bar{\varkappa}_2 x$$

and so $x = 0$ implies $\mu_1 = 0$. Equations $|X', X''| = |X', X^{\text{III}}| = |X', X^{\text{IV}}| = 0$ must then have common nonzero solution. They are

$$x_2^2 + x_3^2 - \mu_1 x_2 = 0\,, \tag{15}$$

$$x_2 \mu_1' + x_3 \mu_1(\varkappa_2 + 1) = 0\,,$$

$$x_2[\mu_1'' - \mu_1 \varkappa_2(\varkappa_2 + 1)] + x_3[(2\varkappa_2 + 1)\,\mu_1' + \varkappa_1' \mu_1] = 0\,.$$

Equations (15) may have a nonzero solution only if the determinant of the second and the third of them is zero, but this is (12).

Conversely, let a 3-Darboux motion in plane be given and so let (12) be satisfied. We know from Lemma 11 that each equation in (15) is a differential consequence of the preceding ones and (14). The common solution of the first two of them satisfies also the third one and so the assumptions of Lemma 10 are satisfied. This means that this common solution satisfies (14) and so it gives a point in the moving plane. The trajectory of this point consists of inflexion points only and so it lies on a straight line ($x' \neq 0$ for all t).

Remark. In the course of proof of Theorem 1 we have also proved the following statement: If a plane motion has a inflexion point of order 3 ($X' \neq 0$, $|X', X''| = |X', X'''| = |X', X^{\text{IV}}| = 0$) for any t, then this point is fixed in the moving plane and has a straight line trajectory.

Let us denote $\int \varkappa_2 \, \mathrm{d}t = k$. Then we have

Theorem 2. *The 3-Darboux Euclidean space motions are characterized by*

$$v = C_1 \cos t + C_2 \sin t + C_3 \exp\left(\int \tan k \, \mathrm{d}t\right), \tag{16}$$

$$\mu_1 = C_4 \exp\left[\int (\varkappa_2 + 1) \tan k \, \mathrm{d}t\right], \quad (\mu_1' = \mu_1(\varkappa_2 + 1) \tan k)\,, \tag{17}$$

where in the case $\varkappa_2 = -1$, v *may be any function such that* $v(t)$, $\cos t$, $\sin t$ *are linearly independent.* C_1, C_2, C_3 *and* $C_4 > 0$ *are real constants.*

Plane motions with one straight line trajectory are characterized by (17) *with* $\varkappa_2 + 1 \neq 0$.

Proof. We substitute in (11) and (12).

Remark. α_1 in (11) is given by

$$\alpha_1 = (2\varkappa_2 + 1) \tan k + \varkappa_2'(\varkappa_2 + 1)^{-1}\,.$$

Theorem 3. *Trajectories of a 3-Darboux space motion are affinely equivalent to a cylindrical curve. The plane factor of a 3-Darboux space motion has one straight line trajectory. Conversely, for any plane motion with a straight line trajectory there exist infinitely many 3-Darboux space motions having it as their plane factors.*

Proof. The trajectory of any point can be written in the form $X = A_0 + f_1 \cos t + f_2 \sin t + f_3 \int \exp\left(\int \tan k \, dt\right) dt$, where A_0 is a fixed point and f_i, $i = 1, 2, 3$, are fixed vectors in E_3. The rest follows from Theorem 2.

Theorem 4. *Euclidean space motions in E_3 which have infinitely many points with straight line trajectories and at least one non-planar trajectory are all 3-Darboux space motions given by (14) and (15) with $C_1 = C_2 = 0$ ($\varkappa_2 \neq -1$).*

Proof. Let a space motion have infinitely many points with straight line trajectories. Then it is cylindrical according to Lemma 5. Further, equations $X' \times X'' = X' \times X''' = 0$ must have infinitely many solutions. Computation gives

$$X' = \begin{bmatrix} v \\ \mathrm{i}x \end{bmatrix}, \quad X'' = \begin{bmatrix} v' \\ \mu_1 - x \end{bmatrix}, \quad X''' = \begin{bmatrix} v'' \\ \mu_1 + \mathrm{i}(\varkappa_2 + 1)\mu_1 - \mathrm{i}x \end{bmatrix}$$

and we get the following equations:

$$x_2^2 + x_3^2 - \mu_1 x_2 = 0\,,$$

$$v x_3 + v' x_2 = 0\,,$$

$$v\mu_1 - v x_2 + v' x_3 = 0 \quad (\text{from } X' \times X'' = 0)\,,$$

$$\mu_1(\varkappa_2 + 1)\, x_3 + \mu_1' x_2 = 0\,,$$

$$\mu_1 v(\varkappa_2 + 1) - (v'' + v)\, x_2 = 0\,,$$

$$v\mu_1' + (v'' + v)\, x_3 = 0 \quad (\text{from } X' \times X''' = 0)\,.$$

We immediately see from the first equation that $x_2 = 0$ implies $x_3 = 0$ which gives $\mu_1 = 0$ from (14). So we may suppose $x_2 \neq 0$. Now we shall consider all possible cases:

a) $\varkappa_2 + 1 = 0$. Then $\mu_1' = 0$, $v'' + v = 0$ and all trajectories are plane curves.

b) $\varkappa_2 + 1 \neq 0$, $v = 0$. Trajectories are plane curves as well.

c) $\varkappa_2 + 1 \neq 0$, $v \neq 0$. Then

$$x_2 = \left[v^2 + (v')^2\right]^{-1} v^2 \mu_1\,, \quad x_3 = -\left[v^2 + (v')^2\right]^{-1} v v' \mu_1$$

and we get the following equations:

$$v'\mu_1(\varkappa_2 + 1) = v\mu_1' \quad \text{and} \quad (\varkappa_2 + 1)\left[v^2 + (v')^2\right] = (v'' + v)\, v\,.$$

Denote $\sigma = v^{-1} v'$. Then we get

$$\sigma = \mu_1'\left[\mu_1(\varkappa_2 + 1)\right]^{-1} \quad \text{and} \quad \sigma' = (\sigma^2 + 1)\,\varkappa_2\,.$$

So $\sigma = \tan k$, $v = C_3 \exp\left(\int \tan k \, dt\right)$ and $\mu_1' = \mu_1(\varkappa_2 + 1) \tan k$, which gives (17).

This proves the first part of the theorem.

Conversely, let a 3-Daiboux motion with $C_1 = C_2 = 0$, $C_3 \neq 0$ be given. Then $v' = v \tan k$, $\mu_1' = \mu_1(\varkappa_2 + 1) \tan k$. Equation $vx_4 + v'x_2 = 0$ changes to the second equation from (15). Similarly, $\mu_1 v(\varkappa_2 + 1) = (v'' + v) x_2$ is a consequence of the first two equations from (15). As $X' = -\psi X$ splits into $x_1' = 0$ and (14) and we have got (15) with (12) satisfied, the statement follows from Theorem 1.

Remark. Points with straight line trajectories have coordinates $x_1 =$ const., $x_2 = \mu_1 \cos^2 k$, $x_3 = -\mu_1 \sin k \cos k$.

In the end we shall characterize all 3-Darboux motions as motions which have as much plane trajectories as possible (nontrivially).

Theorem 5. *The 3-Darboux Euclidean space motions are characterized by the following property: F is not the whole space, each point of F has a planar trajectory and each point not in F has not a planar trajectory.*

Remark. The property just stated can also be expressed as follows: If any trajectory has a planar point, then it is planar and there is at least one non-planar trajectory.

Proof. According to Lemmas 4 and 6 all motions in question are cylindrical. Equation (8) for cylindrical motions is

$$(x_2^2 + x_3^2)(v'' + v) + x_2[v'\mu_1' - \mu_1 v'' - v(\varkappa_2 + 2)\mu_1] + x_3[v\mu_1' + v'(\varkappa_2 + 1)\mu_1] + \\ + v\mu_1^2(\varkappa_2 + 1) = 0 .$$

Let the motion have the property from the theorem. F is trivial for $v = 0$. So $v \neq 0$. Then F is trivial only for $v'' + v = 0$, $\varkappa_2 + 1 = \mu_1' = 0$, which gives the 2-Darboux motion. So we may suppose that F is nontrivial. If each point of F has a plane trajectory then (9) must be a consequence of (8), (9) is

$$(x_2^2 + x_3^2)(v''' + v') + x_2(v'a - \mu_1 v''' - bv) + x_3(av + bv' + \mu_1 v) + v\mu_1 b = 0 ,$$

where

$$a = \mu_1'' - \mu_1 - \mu_1\varkappa_2(\varkappa_2 + 1) , \quad b = (2\varkappa_2 + 1)\mu_1' + \varkappa_2'\mu_1 .$$

As F is nontrivial, (9) must be a multiple of (8) with some coefficient λ. If $\varkappa_2 + 1 \neq 0$, we get $\lambda = b[\mu_1(\varkappa_2 + 1)]^{-1}$ from the absolute term. Further,

$$av + v'b + v\mu_1 = [v\mu_1' + v'(\varkappa_2 + 1)\mu_1] ,$$

which gives $a + \mu_1 = \lambda\mu_1'$ and this is (12). Finally, we have

$$v'a - \mu_1 v''' - bv = \lambda[v'\mu_1' - \mu_1 v'' - v(\varkappa_2 + 1)\mu_1 - v\mu_1] ,$$

which is satisfied, and $v''' + v' = \lambda(v'' + v)$.

This means that the motion is a 3-Darboux motion in virtue of Lemmas 8 and 9. If $\varkappa_2 + 1 = 0$, we have $b = 0$, $\mu_1' = 0$, $a = -\mu_1$ and the only equation is $v''' + v' = \lambda(v'' + v)$. The motion is again a 3-Darboux motion.

Conversely, let a 3-Darboux motion be given. The trajectory $X(t)$ of a point may be written as in the proof of Theorem 3, so $|X', X'', X'''| = |f_1, f_2, f_3|\, h(t)$ for some function $h(t)$ and the statement follows.

Example. Let us find all space motions which have trajectories affinely equivalent to the helix. We have two cases:

a) $\varkappa_2 + 1 \neq 0$. Then we see from the proof of Theorem 2 that $\exp \int \tan k \, dt =$ $=$ const. and so $\tan k = 0$, $k = 0$. We get $\varkappa_2 = 0$, $\bar{\varkappa}_2 = -1$, $\mu_1 =$ const., $v =$ $= C_1 \cos t + C_2 \sin t + C_3$. The plane factor is a cycloidal motion with the fixed centrode a straight line, the moving centrode a circle. The expression of such a motion may be written as

$$g(t) = \begin{pmatrix} 1, & 0, & 0, & 0 \\ C_1 \cos t + C_2 \sin t + C_3 t, & 1, & 0, & 0 \\ r(t - \sin t), & 0, & \cos t, & \sin t \\ r(1 - \cos t) & 0, & -\sin t, & \cos t \end{pmatrix}$$

b) $\varkappa_2 + 1 = 0$. This case was discussed above.

References

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Souhrn

EUKLIDOVSKÉ PROSTOROVÉ POHYBY S AFINNĚ EKVIVALENTNÍMI TRAJEKTORIEMI

ADOLF KARGER

V článku se studují euklidovské pohyby v prostoru mající tu vlastnost, že trajektorie každého bodu je afinním obrazem dané prostorové křivky. Ukazuje se, že všechny takové pohyby jsou cylindrické tj. se rozkládají se na pohyb v rovině a na translaci ve směru k této rovině kolmém. Jejich trajektorie jsou cylindrické křivky a odpovídající rovinný faktor má jednu přímkovou trajektorii.

Dále je dokázáno, že prostorové pohyby mající nekonečně mnoho přímkových trajektorií jsou speciálním případem zkoumaných pohybů. Uvažované pohyby jsou nakonec charakterizovány jako pohyby s touto vlastností: Má-li trajektorie nějakého bodu plochý bod (nulovou torzi), leží celá v rovině a ne všechny trajektorie jsou rovinné.

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