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GENERALIZED METHOD OF LEAST SQUARES COLLOCATION

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1. INTRODUCTION

The least square collocation is a statistical method currently used in geodesy, particularly in physical geodesy, for processing continuous phenomena observed at discrete points suitable located in the region of investigation (e.g. the gravitational field of the Earth). The least square collocation algorithm published in [7] was derived under the conditions of regularity of the model used and is a subject for the present generalization.

The aim of the paper is to show two different approaches based on two different systems of conditions for deriving the algorithm for the generalized method of least squares collocation.

Obviously the procedure can be used in other regions of research as well.

2. FORMULATION OF THE PROBLEM

Let $v(\cdot)$ be a stochastic function defined on a finite, countable or uncountable set of points of the Euclidean space $\mathscr{R}^3$. The covariance function $R_v(\cdot, \cdot\cdot) = \operatorname{cov}\{v(\cdot), v(\cdot\cdot)\}$, $Q, Q' \in S$, is assumed to be known and the mean value function $m(\cdot) = \mathrm{E}[v(\cdot)]$, $Q \in S$, is only assumed to belong to a known class $\mathscr{M}$. Let $v(\cdot)$, $Q \in S$, be the sum of an uncorrelated useful signal $\xi(\cdot)$ and of a noise $\eta(\cdot)$ for which $\mathrm{E}[v(\cdot)] = \mathrm{E}[\xi(\cdot)] = m(\cdot)$, $\mathrm{E}[\eta(\cdot)] = 0$, $Q \in S$.

The Hilbert random function $v(\cdot)$, $Q \in S$, is observed at points $P_1, \ldots, P_N$ suitably located in S; evidently $\{P_1, \ldots, P_N\} \subset S$. Our knowledge on the input observed random function is given in terms of an N-dimensional column vector $\boldsymbol{v} = (v(P_1), \ldots \ldots, v(P_N))'$ ($'$ indicates transposition of the vector).

The problem is to determine an unbiased linear statistical estimator of the mean value (the trend component) of the function $v(\cdot)$ and an unbiased optimal linear estimator of the useful component $\xi(\cdot)$ at an arbitrary point $Q \in S$.

The formulated problem is solved by means of stochastic functions, where the geometrical point of view is emphasized, and in the framework of the universal model [2].

3. SOLUTION BY MEANS OF STOCHASTIC FUNCTIONS

The following assumptions are introduced:

1. Knowledge of the $N \times N$ dimensional covariance matrix $\mathbf{R}_v = \operatorname{cov}\{\boldsymbol{v}, \boldsymbol{v}'\} = (\mathbf{R}_v(P_1), \ldots, \mathbf{R}_v(P_N))$; $\mathbf{R}_v(P_k) = (R_v(P_1, P_k), \ldots R_v(P_N, P_k))'$ is the k^{th} column of the matrix $\mathbf{R}_v$; no assumptions on the regularity of the covariance matrix are made.
2. Knowledge of N-dimensional column vectors $\mathbf{R}_v(Q) = \operatorname{cov}\{\boldsymbol{v}, v(Q)\}$ and $\mathbf{R}_{v\xi}(Q) = \operatorname{cov}\{\boldsymbol{v}, \xi(Q)\}$, $Q \in S$.
3. Knowledge of the class $\mathcal{M}$ of vectors of the mean values $\boldsymbol{m} = (m(P_1), \ldots, m(P_n))'$ of the random vector $\boldsymbol{v}$ ($\mathcal{M} \subset \mathcal{H}(\mathbf{R}_v)$).

$\mathcal{H}(\mathbf{R}_v)$ indicates the reproducing kernel Hilbert space spanned by the columns of the covariance matrix $\mathbf{R}_v : \mathcal{H}(\mathbf{R}_v) = \{\boldsymbol{x} : \boldsymbol{x} = (x(P_1), \ldots, x(P_N))' = \mathbf{R}_v \boldsymbol{a}, \boldsymbol{a} \in \mathcal{R}^N\}$. The inner product in $\mathcal{H}(\mathbf{R}_v)$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathbf{R}_v^-} = \boldsymbol{x}' \mathbf{R}_v^- \boldsymbol{y}$, where $\mathbf{R}_v^-$ is the generalized inversion (g-inversion) [9] of the covariance matrix. The g-inversion is used because no assumption on the regularity of $\mathbf{R}_v$ is made. The kernel $\mathbf{R}_v$ has the reproducing property $(\langle \boldsymbol{x}, \mathbf{R}_v(P_k) \rangle_{\mathbf{R}_v^-} = \boldsymbol{a}' \mathbf{R}_v' \mathbf{R}_v^- \mathbf{R}_v(P_k) = \boldsymbol{a}' \mathbf{R}_v(P_k) = \mathbf{R}_v'(P_k) \boldsymbol{a} = x(P_k))$.

Let $\mathcal{M}_0$ be the subspace of the space $\mathcal{H}(\mathbf{R}_v)$ generated by the elements of the class $\mathcal{M}$ with the inner product defined by $\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle_{\mathbf{R}_v^-} = \boldsymbol{m}_1' \mathbf{R}_v^- \boldsymbol{m}_2$ (in accordance with the assumption $\mathcal{M} \subset \mathcal{H}(\mathbf{R}_v)$).

Further, it is assumed that each $\boldsymbol{m} \in \mathcal{M}$ has the form $\boldsymbol{m} = \mathbf{W}\boldsymbol{\alpha}$, where $\boldsymbol{\alpha} \in \mathcal{R}^n$ ($n \leqq N$) and $\mathbf{W}$ is an $N \times n$ dimensional matrix

$$\mathbf{W} = \begin{bmatrix} w_1(P_1), w_2(P_1), \ldots, w_n(P_1) \\ w_1(P_2), w_2(P_2), \ldots, w_n(P_2) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ w_1(P_N), w_2(P_N), \ldots, w_n(P_N) \end{bmatrix}, \tag{3.1}$$

the N dimensional columns of which are formed by the values of known functions $w_1(\cdot), w_2(\cdot), \ldots, w_n(\cdot)$, $Q \in S$, at the points $P_k \in \{P_1, P_2, \ldots, P_N\}$; i.e. $\mathcal{M}_0 = \{\boldsymbol{m} : \boldsymbol{m} = \mathbf{W}\boldsymbol{\alpha}, \boldsymbol{\alpha} \in \mathcal{R}^n\}$. The situation is especially simple when the columns of the matrix $\mathbf{W}$ are elements of the orthonormal basis of the Hilbert space $\mathcal{M}_0$ considered.

Further, let $\mathcal{L}^2\{v(P_k) : P_k \in \{P_1, \ldots, P_N\}\}$ be the Hilbert space spanned by the random vector $\boldsymbol{v}$ (evidently $\mathcal{L}^2\{v(P_k) : P_k \in \{P_1, \ldots, P_N\}\} \subset \mathcal{L}^2(\Omega, \mathcal{S}, \mathcal{P})$; thus the inner product in $\mathcal{L}^2\{v(P_k) : P_k \in \{P_1, \ldots, P_N\}\}$ has the form $\langle \alpha_1, \alpha_2 \rangle = \mathrm{E}[\alpha_{1c}\alpha_{2c}]$).

It may be proved that between the reproducing kernel Hilbert space $\mathcal{H}(\mathbf{R}_v)$ and

$\mathscr{L}^2\{v(P_k) : P_k \in \{P_1, \ldots, P_N\}\}$ there exists an isometric isomorphism F with the property $F\{\mathbf{R}_v(P_k)\} = v(P_k)$; this property and the reproducing one imply $\langle \boldsymbol{m}, \boldsymbol{h} \rangle_{\mathbf{R}_v^-} = \mathrm{E}_{\boldsymbol{m}}[F\{\boldsymbol{h}\}]$ for an arbitrary $\boldsymbol{m} \in \mathscr{M} \subset \mathscr{H}(\mathbf{R}_v)$ and each $\boldsymbol{h} \in \mathscr{H}(\mathbf{R}_v)$ (the index at the expectation operator indicates that the mean value is computed under the assumption that $\boldsymbol{m} \in \mathscr{M}_0$ is the true mean value).

Now it is easy to solve our problem using the geometry of the Hilbert space considered and the isometric isomorphism F.

All unbiased linear estimators $\hat{\xi}^{(N)}(Q)$ of the random variable $\xi(Q) \in \mathscr{L}^2(\Omega, \mathscr{S}, \mathscr{P})$ form a linear variety $G_{\mathscr{L}}^{(N)} = \{\hat{\xi}^{(N)}(Q) : \mathrm{E}_{\boldsymbol{m}}[\hat{\xi}^{(N)}(Q)] = \mathrm{E}_{\boldsymbol{m}}[\xi(Q)] = m(Q)$, $G_{\mathscr{L}}^{(N)} \subset \mathscr{L}^2\{v(P_k) : P_k \in \{P_1, \ldots, P_N\}\}$.

The optimal ($\equiv$ efficient $\equiv$ unbiased and with minimal dispersion) linear estimator of $\xi(Q)$ is that element $\hat{\xi}_{\mathrm{opt}}^{(N)}(Q) \in G_{\mathscr{L}}^{(N)}$ for which the distance $\varrho_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})}(\hat{\xi}_{\mathrm{opt}}^{(N)}(Q)$, $\xi(Q) = \min$.

As $\varrho^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})}(\hat{\xi}_{\mathrm{opt}}^{(N)}(Q), \xi(Q)) = \varrho^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})}(\xi^*(Q), \xi(Q)) + \varrho^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})}(\xi^*(Q), \hat{\xi}_{\mathrm{opt}}^{(N)}(Q))$, where $\xi^*(Q) = P_{\mathscr{L}^2\{v(P_k)\}}(\xi(Q))$ is a projection of the element $\xi(Q) \in \mathscr{L}^2(\Omega, \mathscr{S}, \mathscr{P})$ on the Hilbert space $\mathscr{L}^2\{v(P_k) : P_k \in \{P_1, \ldots, P_N\}\}$, the problem to minimize the distance $\varrho_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})}(\hat{\xi}_{\mathrm{opt}}^{(N)}(Q),\ \xi(Q))$ is transfered to the problem to minimize the distance $\varrho_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})}(\xi^*(Q), \hat{\xi}_{\mathrm{opt}}^{(N)}(Q)) = \varrho_{\mathscr{L}^2\{v(P_k)\}}(\xi^*(Q), \hat{\xi}_{\mathrm{opt}}^{(N)}(Q)) = \varrho_{\mathscr{L}^2\{v(P_k)\}}(F\{\mathbf{R}_{v\xi}(Q)\}$, $F\{\boldsymbol{h}_{\mathrm{opt}}^{(N)}\}) = \varrho_{\mathscr{H}(\mathbf{R}_v)}(\mathbf{R}_{v\xi}(Q), \boldsymbol{h}_{\mathrm{opt}}^{(N)})$ (it may be proved that $\mathbf{R}_{v\xi^*}(P) = \mathbf{R}_{v\xi}(P) \in \mathscr{H}(\mathbf{R}_v)$ and the reproducing property of the kernel $\mathbf{R}_v$ together with the isometric isomorphism F implies that $F(\mathbf{R}_{v\xi}(P)\} = \xi^*(P))$.

The isometric isomorphism $F : \mathscr{H}(\mathbf{R}_v) \to \mathscr{L}^2\{v(P_k), P_k \in \{P_1, \ldots, P_N\}\}$ enables us to solve the problem in the reproducing kernel Hilbert space. The set of inverse images of elements of the linear variety $G_{\mathscr{L}}^{(N)} \subset \mathscr{L}^2\{v(P_k), P_k \in \{P_1, \ldots, P_N\}\}$ forms the corresponding linear variety $G_{\mathscr{H}}^{(N)} = \{\boldsymbol{h}^{(N)} : F\{\boldsymbol{h}^{(N)}\} = \hat{\xi}^{(N)}(P)\}$, $G_{\mathscr{H}}^{(N)} \subset \mathscr{H}(\mathbf{R}_v)$. The variety $G_{\mathscr{H}}^{(N)}$ is uniquely determined by one of its elements and by the Hilbert space $\mathscr{M}_0$. One unbiased linear estimator of the random variable $\xi(Q)$ is $F\{\mathbf{R}_v(Q)\}$ (because $\mathrm{E}_{\boldsymbol{m}}[F\{\mathbf{R}_v(Q)\}] = \langle \boldsymbol{m}, \mathbf{R}_v(Q) \rangle_{\mathbf{R}_v^-} = m(Q) \Rightarrow \mathrm{E}_{\boldsymbol{m}}[\hat{\xi}(Q)] = m(Q) = \mathrm{E}_{\boldsymbol{m}}[\xi(Q)]$ for an arbitrary $\boldsymbol{m} \in \mathscr{M}$) and that is why $G_{\mathscr{H}}^{(N)} = \{\boldsymbol{h}^{(N)} : P_{\mathscr{M}_0}\boldsymbol{h}^{(N)} = P_{\mathscr{M}_0}\, \mathbf{R}_v(Q)\}$; $P_{\mathscr{M}_0}$ is an $\mathbf{R}_v^-$-projector of elements of the Hilbert space $\mathscr{H}(\mathbf{R}_v)$ on its subspace $\mathscr{M}_0$.

The projector $P_{\mathscr{M}_0}$ has the form $P_{\mathscr{M}_0} = \mathbf{W}(\mathbf{W}'\mathbf{R}_v^-\mathbf{W})^-\,\mathbf{W}'\mathbf{R}_v^-$, $\mathbf{W}$ is the matrix (3.1).

Applying the relations

$$\hat{\xi}_{\mathrm{opt}}^{(N)}(Q) = F\{\mathbf{R}_{v\xi}(Q) + P_{\mathscr{M}_0}[\mathbf{R}_v(Q) - \mathbf{R}_{v\xi}(Q)]\}$$

and

$$\hat{m}_{\mathrm{opt}}^{(N)}(Q) = F\{P_{\mathscr{M}_0}\, \mathbf{R}_v(Q)\}$$

derived in [5] we get

$$\hat{\boldsymbol{m}}_{\mathrm{opt}}^{(N)} = \mathbf{W}(\mathbf{W}'\mathbf{R}_v^-\mathbf{W})^-\,\mathbf{W}'\mathbf{R}_v^-\,v \tag{3.2}$$

and

$$\hat{\xi}_{\mathrm{opt}\,c}^{(N)}(Q) = F\{\mathbf{R}_{v\xi}(Q) - P_{\mathscr{M}_0}\,\mathbf{R}_{v\xi}(Q)\} = \tag{3.3}$$
$$= [v' - v'\mathbf{R}_v^-\mathbf{W}(\mathbf{W}'\mathbf{R}_v^-\mathbf{W})^-\,\mathbf{W}']\,\mathbf{R}_v^-\,\mathbf{R}_{v\xi}(Q) = [v' - (\hat{\boldsymbol{m}}_{\mathrm{opt}}^{(N)})']\,\mathbf{R}_v^-\mathbf{R}_{v\xi}(Q)\,.$$

Relations (3.2) and (3.3) represent a generalization of relations (2.36), (2.38) and (2.35), respectively, from [7].

The fundamental statistical characteristics of the derived statistical estimators are

$$\mathrm{D}[\hat{m}^{(N)}_{\mathrm{opt}}(Q)] = \|\hat{m}^{(M)}_{\mathrm{opt}}(Q)\|^2_{\mathscr{L}^2\{v(P_k)\}} = \|P_{\mathscr{M}_0}\, \mathbf{R}_v(Q)\|^2_{\mathscr{H}(\mathbf{R}_v)} =$$
$$= \mathbf{R}'_v(Q)\, \mathbf{R}_v^- \mathbf{W}(\mathbf{W}'\mathbf{R}_v^-\mathbf{W})^- \mathbf{W}'\mathbf{R}_v^- \ \mathbf{R}_v(Q)\,;$$

$$\mathrm{D}[\hat{\xi}^{(N)}_{\mathrm{opt}\,c}(Q)] = \|\hat{\xi}^{(N)}_{\mathrm{opt}\,c}(Q)\|^2_{\mathscr{L}^2\{v(P_k)\}} = \|\mathbf{R}_{v\xi}(Q) - P_{\mathscr{M}_0}\, \mathbf{R}_{v\xi}(Q)\|^2_{\mathscr{H}(\mathbf{R}_v)} =$$
$$= \mathbf{R}'_{v\xi}(Q)\, \mathbf{R}_v^-[\mathbf{I} - \mathbf{W}(\mathbf{W}'\mathbf{R}_v^-\mathbf{W})^- \mathbf{W}'\mathbf{R}_v^-]\, \mathbf{R}_{v\xi}(Q)\,,$$

$$\mathrm{D}[\hat{\xi}^{(N)}_{\mathrm{opt}}(Q) - \xi(Q)] = \|\hat{\xi}^{(N)}_{\mathrm{opt}}(Q) - \xi(Q)\|^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})} =$$
$$= \|\xi(Q)\|^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})} + \|\hat{\xi}^{(N)}_{\mathrm{opt}}(Q)\|^2_{\mathscr{L}^2\{v(P_k)\}} - 2\langle\hat{\xi}^{(N)}_{\mathrm{opt}}(Q), \xi(Q)\rangle_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})} =$$
$$= \|\xi(Q)\|^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})} + \|\hat{\xi}^{(N)}_{\mathrm{opt}}(Q)\|^2_{\mathscr{L}^2\{v(P_k)\}} -$$
$$- 2\langle\hat{\xi}^{(N)}_{\mathrm{opt}}(Q), P_{\mathscr{L}^2\{v(P_k)\}}\, \xi(Q)\rangle_{\mathscr{L}^2\{v(P_k)\}} = \|\xi(Q)\|^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})} +$$
$$+ \|\mathbf{R}_{v\xi}(Q) + P_{\mathscr{M}_0}[\mathbf{R}_v(Q) - \mathbf{R}_{v\xi}(Q)]\|^2_{\mathscr{H}(\mathbf{R}_v)} - 2\langle\mathbf{R}_{v\xi}(Q) +$$
$$+ P_{\mathscr{M}_0}[\mathbf{R}_v(Q) - \mathbf{R}_{v\xi}(Q)], \mathbf{R}_{v\xi}(Q)\rangle_{\mathscr{H}(\mathbf{R}_v)} = \|\xi(Q)\|^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})} + (\mathbf{R}_{v\xi}(Q) +$$
$$+ P_{\mathscr{M}_0}[\mathbf{R}_v(Q) - \mathbf{R}_{v\xi}(Q)])'\, \mathbf{R}_v^-(P_{\mathscr{M}_0}[\mathbf{R}_v(Q) - \mathbf{R}_{v\xi}(Q)] - \mathbf{R}_{v\xi}(Q)) =$$
$$= \|\xi(Q)\|^2_{\mathscr{L}^2(\Omega,\mathscr{S},\mathscr{P})} - \mathbf{R}'_{v\xi}(Q)\, \mathbf{R}_v^-[\mathbf{I} - \mathbf{W}(\mathbf{W}'\mathbf{R}_v^-\mathbf{W})^- \mathbf{W}'\mathbf{R}_v^-]\, \mathbf{R}_{v\xi}(Q) +$$
$$+ \mathbf{R}'_v(Q)\, \mathbf{R}_v^- \mathbf{W}(\mathbf{W}'\mathbf{R}_v^-\mathbf{W})^- \mathbf{W}'\mathbf{R}_v^-[\mathbf{R}_v(Q) - 2\mathbf{R}_{v\xi}(Q)]\,.$$

If the domain of definition S is a bounded and closed region of the Euclidean space, then the stochastic functions $v(\cdot)$, $\xi(\cdot)$ and $\eta(\cdot)$ are continuous in quadratic mean on S and if the input function $v(\cdot)$ is observed at points of a finite set $\{P_1, \ldots P_N\}$ which for $N \to \infty$ is dense in S, then the matrix relations (3.2), (3.3) converge to their theoretical values $\hat{\xi}_{\mathrm{opt}}(Q) = F\{R_{v\xi}(\cdot, Q) + P_{\mathscr{M}_0}[R_v(\cdot, Q) - R_{v\xi}(\cdot, Q)]\}$ and $\hat{m}_{\mathrm{opt}}(Q) = F\{P_{\mathscr{M}_0}\, R_v(\cdot, Q)\}$ derived under the assumption that the continuous realization of the input field is known [5]. We have [10]

$$\lim_{N\to\infty} \|\hat{\xi}^{(N)}_{\mathrm{opt}}(Q) - \hat{\xi}_{\mathrm{opt}}(Q)\|_{\mathscr{L}^2\{v(Q),Q\in S\}} = 0\,,$$
$$\lim_{N\to\infty} \|m^{(N)}_{\mathrm{opt}}(Q) - \hat{m}_{\mathrm{opt}}(Q)\|_{\mathscr{L}^2\{v(Q),Q\in S\}} = 0\,,$$

where $\mathscr{L}^2\{v(Q), Q \in S\}$ is the Hilbert space spanned by the Hilbert random function $v(\cdot)$, $Q \in S$. This important theorem proves that all the derived expressions are suitable forms for solving the collocation problem in the case when the observed phenomenon is continuous.

4. SOLUTION IN THE FRAMEWORK OF THE UNIVERSAL MODEL

The random function is again considered as a superposition of the signal $\xi(\cdot)$ and the noise $\eta(\cdot)$ which are uncorrelated. When the measurement (i.e. the realization of the random variables $v(P)$) is carried out at points $P_1, \dots, P_N$ we obtain a model $\boldsymbol{v} = \boldsymbol{\xi} + \boldsymbol{\eta}$, where $\boldsymbol{\zeta} = (\zeta(P_1), \dots, \zeta(P_N))'$, $\zeta = v, \xi, \eta$.

In contradiction to the preceding model we do not assume the mean value $\boldsymbol{m} = \mathrm{E}_{\boldsymbol{m}}[\boldsymbol{\xi}]$ to be an element of the subspace $\mathscr{M}_0 \subset \mathscr{H}(\mathbf{R}_v)$ but we express it in the form $\boldsymbol{m} = \mathbf{A}\boldsymbol{\Theta}$, where $\mathbf{A}$ is a known matrix and $\boldsymbol{\Theta} \in \mathscr{R}^n$ is an unknown vector. It is necessary to remark that the matrix $\mathbf{A}$ determines the subspace $\mathscr{M}_0 = \mathscr{M}(\mathbf{A})$ in which the mean value $\boldsymbol{m} = \mathrm{E}_{\boldsymbol{m}}[\boldsymbol{\xi}]$ of the vector lies, however, in this case the inclusion $\mathscr{M}(\mathbf{A}) \subset \mathscr{M}(\mathbf{R}_v)$ is not necessarily fulfilled. The aim is to determine an estimator of the vector $\boldsymbol{\Theta}$ and an estimator of its linear unbiasedly estimable functionals of its components, respectively, and further an estimator of the signal $\xi(Q)$ of the field $v(\cdot)$ at the point $Q \bar{\in} \{P_1, \dots, P_N\}$ and an estimator of the centred signal $\xi_c(Q) = \xi(Q) - \boldsymbol{a}'(Q)\,\boldsymbol{\Theta}$. We assume that the mean value of the random variable $\xi(Q)$ can be expressed as a known combination – given by the vector $\boldsymbol{a}(Q)$ – of the unknown vector $\boldsymbol{\Theta}$.

This problem is a certain generalization of the problem given by H. Moritz in [7].

Analogously as in the previous model it is assumed that the covariance matrix $\mathbf{R}_v$ of the vector $\boldsymbol{v}$ and the covariances $\operatorname{cov}\{\boldsymbol{v}, \xi(Q)\} = \operatorname{cov}\{\boldsymbol{\xi}, \xi(Q)\}$ are known.

In the solution of the first problem it is necessary to take into account that it is possible to calculate the estimate of the quantity $\boldsymbol{p}'\boldsymbol{\Theta}$ from the realization of the vector $\boldsymbol{v}$ if, and only if $\boldsymbol{p} \in \mathscr{M}(\mathbf{A}')$. This fact is corroborated by the following simple consideration. The quantity $\boldsymbol{p}'\boldsymbol{\Theta}$ is estimable (unbiasedly) if, and only if, there exists a vector $\boldsymbol{L} \in \mathscr{R}^n$ which satisfies the relations: $\forall\{\boldsymbol{\Theta} \in \mathscr{R}^n\}\ \mathrm{E}_{\boldsymbol{\Theta}}[\boldsymbol{L}'\boldsymbol{v}] = \boldsymbol{L}'\mathbf{A}\boldsymbol{\Theta} = \boldsymbol{p}'\boldsymbol{\Theta} \Leftrightarrow \mathbf{A}'\boldsymbol{L} = \boldsymbol{p} \Leftrightarrow \boldsymbol{p} \in \mathscr{M}(\mathbf{A}')$.

As the relations $\mathrm{E}_{\boldsymbol{\Theta}}[\boldsymbol{v}] = \mathrm{E}_{\boldsymbol{\Theta}}[\boldsymbol{\xi}] + \mathrm{E}_{\boldsymbol{\Theta}}[\boldsymbol{\eta}] = \mathbf{A}\boldsymbol{\Theta}$ are valid and the aim is to determine the estimator of the quantity $\boldsymbol{p}'\boldsymbol{\Theta}$, $\boldsymbol{p} \in \mathscr{M}(\mathbf{A}')$, it is possible to utilize the results of the paper [2]. There it is shown that the unbiased estimator with the minimal dispersion is given by the relation $\widehat{(\boldsymbol{p}'\boldsymbol{\Theta})} = \boldsymbol{p}'[(\mathbf{A}')^-_{m(\mathbf{R}_v)}]'\,\boldsymbol{v}$. When several functionals $f_1(\boldsymbol{\Theta}) = \boldsymbol{p}'\boldsymbol{\Theta}$, $f_2(\boldsymbol{\Theta}) = \boldsymbol{p}_2'\boldsymbol{\Theta}, \dots, f_r(\boldsymbol{\Theta}) = \boldsymbol{p}_r'\boldsymbol{\Theta}$, $\boldsymbol{p}_i \in \mathscr{M}(\mathbf{A}')$, $i = 1, \dots, r$, are given then even the random vector $(\boldsymbol{p}_1, \dots, \boldsymbol{p}_r)'\,[(\mathbf{A}')^-_{m(\mathbf{R}_v)}]'\,\boldsymbol{v}$ is the joint efficient estimator of the vector $(\boldsymbol{p}_1, \dots, \boldsymbol{p}_r)'\,\boldsymbol{\Theta}$.

The second aim is to find out a random variable in the form $\boldsymbol{L}'\boldsymbol{v}$ which satisfies the following conditions: $\forall\{\boldsymbol{\Theta} \in \mathscr{R}^n\}\ \mathrm{E}_{\boldsymbol{\Theta}}[\boldsymbol{L}'\boldsymbol{v}] = \boldsymbol{L}'\mathbf{A}\boldsymbol{\Theta} = \boldsymbol{a}'(Q)\,\boldsymbol{\Theta}\ \&\ \mathrm{E}[(\boldsymbol{L}'\boldsymbol{v} - \xi(Q))^2] = \min$. We use the method of indefinite Lagrange multipliers. An auxiliary Lagrange function has the form $\phi(\boldsymbol{L}) = \mathrm{E}[(\boldsymbol{L}'\boldsymbol{v} - \xi(Q)^2] + 2\boldsymbol{\lambda}'(\mathbf{A}'\boldsymbol{L} - \boldsymbol{a}(Q))$, where $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers and $\mathrm{E}[(\boldsymbol{L}'\boldsymbol{v} - \xi(Q))^2] = \mathrm{D}[\boldsymbol{L}'\boldsymbol{v}] + \mathrm{D}[\xi(Q)] - 2\operatorname{cov}\{\boldsymbol{L}'\boldsymbol{v}, \xi(Q)\}$ (with respect to the assumption $\mathrm{E}_{\boldsymbol{\Theta}}[\boldsymbol{L}'\boldsymbol{v}] = \mathrm{E}_{\boldsymbol{\Theta}}[\xi(Q)] = \boldsymbol{a}'(Q)\,\boldsymbol{\Theta}$). Further, $\mathrm{D}[\boldsymbol{L}'\boldsymbol{v}] = \boldsymbol{L}'\mathbf{R}_v\boldsymbol{L}$, $\mathrm{D}[\xi(Q)] = \operatorname{cov}(\xi(Q), \xi(Q)\}$ and $\operatorname{cov}\{\boldsymbol{L}'\boldsymbol{v}, \xi(Q)\} = \boldsymbol{L}'\operatorname{cov}\{\boldsymbol{v}, \xi(Q)\} = \boldsymbol{L}'(\operatorname{cov}\{\xi(P_1), \xi(Q)\}, \operatorname{cov}\{\xi(P_2), \xi(Q)\}, \dots, \operatorname{cov}\{(P_N), \xi(Q)\})'$. The auxiliary function $\phi(\cdot)$ can be investigated in the form $\phi(\mathbf{L}) = \boldsymbol{L}'\mathbf{R}_v\boldsymbol{L} - 2\boldsymbol{L}'\operatorname{cov}\{\boldsymbol{v}, \xi(Q)\} + 2\boldsymbol{\lambda}'(\mathbf{A}'\boldsymbol{L} - \boldsymbol{a}(Q))$; $(\frac{1}{2}\,\partial\phi(\mathbf{L})/\partial\mathbf{L} =)\ \mathbf{R}_v\boldsymbol{L} - \operatorname{cov}\{\boldsymbol{v}, \xi(Q)\} +$
$\mathbf{A}'\boldsymbol{L}$

$\left.\begin{array}{l} + \mathbf{A}\lambda = \mathbf{0} \\ = \mathbf{a}(Q) \end{array}\right\} \Rightarrow \begin{bmatrix} \mathbf{R}_v, & \mathbf{A} \\ \mathbf{A}', & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L} \\ \lambda \end{bmatrix} = \begin{bmatrix} \operatorname{cov}\{\mathbf{v}, \xi(Q)\} \\ \mathbf{a}(Q) \end{bmatrix}$. Using the theory of "Pandora-Box" matrix [3] we obtain $\begin{bmatrix} \mathbf{L} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1, & \mathbf{C}_2 \\ \mathbf{C}_3, & -\mathbf{C}_4 \end{bmatrix} \begin{bmatrix} \operatorname{cov}\{\mathbf{v}, \xi(Q)\} \\ \mathbf{a}(Q) \end{bmatrix}$, where $\mathbf{C}_2 \in \in (\mathscr{A}')^-_{m(\mathbf{R}_v)}$, $\mathbf{C}'_3 \in (\mathscr{A}')^-_{m(\mathbf{R}_v)}$, thus $\mathbf{L} = \mathbf{C}_1 \operatorname{cov}\{\mathbf{v}, \xi(Q)\} + \mathbf{C}_2\, \mathbf{a}(Q)$. The symbol $(\mathscr{A}')^-_{m(\mathbf{R}_v)}$ denotes the class of the g-inversions of the matrix $\mathbf{A}'$ which are denoted by the symbol $(\mathbf{A}')^-_{m(\mathbf{R}_v)}$ and which satisfy $\mathbf{A}'(\mathbf{A}')^-_{m(\mathbf{R}_v)}\, \mathbf{A}' = \mathbf{A}'$ & $[(\mathbf{A}')^-_{m(\mathbf{R}_v)}\, \mathbf{A}']' \,.\, \mathbf{R}_v = \mathbf{R}_v(\mathbf{A}')^-_{m(\mathbf{R}_v)}\, \mathbf{A}'$ [9].

The estimate $\mathbf{L}'\mathbf{v}$ of the quantity $\xi(Q)$ is given by the relation $\hat{\xi}(Q) = \mathbf{L}'\mathbf{v} = \mathbf{a}'(Q) \,.\, \mathbf{C}'_2\mathbf{v} + [\operatorname{cov}\{\mathbf{v}, \xi(Q)\}]'\, \mathbf{C}'_1\mathbf{v}$. The first term $\mathbf{a}'(Q)\, \mathbf{C}'_2\mathbf{v}$ is the unbiased estimator of the trend component $\mathbf{a}'(Q)\, \boldsymbol{\Theta}$ with the minimal dispersion given by $\mathrm{D}[\mathbf{a}'(Q)\, \mathbf{C}'_2\mathbf{v}] = = \mathbf{a}'(Q)\, \mathbf{C}'_2\mathbf{R}_v\mathbf{C}_2\, \mathbf{a}(Q) = \mathbf{a}'(Q)\, \mathbf{C}_4\, \mathbf{a}(Q)$ (the last identity is useful for the numerical check of the computation and its validity is proved in [3]). For the second term $[\operatorname{cov}\{\mathbf{v}, \xi(Q)\}]'\, \mathbf{C}'_1\mathbf{v}$ we have $[\operatorname{cov}\{\mathbf{v}, \xi(Q)\}]'\, \mathbf{C}_1\mathbf{v} = [\operatorname{cov}\{\mathbf{v}, \xi(Q)\}]'\, \mathbf{C}'_1(\mathbf{v} - \mathbf{A} \,.\, [(\mathbf{A}')^-_{m(\mathbf{R}_v)}]'\, \mathbf{v})$. The last relation follows from the assumption $\operatorname{cov}\{\mathbf{v}, \xi(Q)\} \in \in \mathscr{M}(\mathbf{R}_v)$ and from the identity $\mathbf{R}_v\mathbf{C}'_1\mathbf{A} = \mathbf{0}$ (see [3], relation 4.1). As $\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{R}_v)}]'\, \mathbf{v} = = \mathbf{A}\hat{\boldsymbol{\Theta}}$ (this vector is invariant with respect to the choice of the g-inversion $(\mathbf{A}')^-_{m(\mathbf{R}_v)}$), the quantity $\hat{\xi}_c(Q) = [\operatorname{cov}\{\mathbf{v}, \xi(Q)\}]'\, \mathbf{C}'_1(\mathbf{v} - \mathbf{A}\hat{\boldsymbol{\Theta}})$ can be declared an estimator of the centred signal $\xi_c(Q) = \xi(Q) - \mathbf{a}'(Q)\, \boldsymbol{\Theta}$.

Further, the quantities $\mathrm{D}[\hat{\xi}(Q)]$, $\mathrm{D}[\hat{\xi}_c(Q)]$, $\mathrm{E}[(\hat{\xi}(Q) - \xi(Q))^2]$ and $\mathrm{E}[(\hat{\xi}_c(Q) - - \xi_c(Q))^2]$ characterizing the quality of the estimators are determined. As $\hat{\xi}(Q) = \mathbf{a}'(Q)\, \hat{\boldsymbol{\Theta}} + \xi_c(Q)$ and the random variables $\mathbf{a}'(Q)\, \hat{\boldsymbol{\Theta}}$, $\hat{\xi}_c(Q)$ are uncorrelated (this follows from the identity $\mathbf{A}\mathbf{C}'_2\mathbf{R}_v(\mathbf{I} - \mathbf{C}_2\mathbf{A}') = \mathbf{0}$), the relation $\mathrm{D}[\hat{\xi}(Q)] = = \mathrm{D}[\mathbf{a}'(Q)\, \hat{\boldsymbol{\Theta}}] + \mathrm{D}[\hat{\xi}_c(Q)]$ holds for the dispersion $\mathrm{D}[\hat{\xi}(Q)]$; the relation $\mathrm{D}[\mathbf{a}'(Q)\, \hat{\boldsymbol{\Theta}}] = \mathbf{a}'(Q)\, \mathbf{C}_4\mathbf{a}(Q)$ was mentioned above. The formula $\mathrm{D}[\hat{\xi}_c(Q)] = = \mathrm{D}[(\operatorname{cov}\{\mathbf{v}, \xi(Q)\})'\, \mathbf{C}_1\mathbf{v}] = (\operatorname{cov}\{\mathbf{v}, \xi(Q)\})'\, \mathbf{C}'_1\mathbf{R}_v\mathbf{C}_1 \operatorname{cov}\{\mathbf{v}, \xi(Q)\}$ is valid for the dispersion $\mathrm{D}[\hat{\xi}_c(Q)]$. The assumption $\operatorname{cov}\{\mathbf{v}, \xi(Q)\} \in \mathscr{M}(\mathbf{R}_v)$ and the relations (4.1) and (3.10) from [3] imply the following relation: $(\operatorname{cov}\{\mathbf{v}, \xi(Q)\})'\, \mathbf{C}'_1\mathbf{R}_v\mathbf{C}_1 \operatorname{cov}\{\mathbf{v}, \xi(Q)\} = (\operatorname{cov}\{\mathbf{v}, \xi(Q)\})'\, \mathbf{C}_1 \operatorname{cov}\{\mathbf{v}, \xi(Q)\}$.

Analogous relations are obtained for the covariances $\operatorname{cov}\{\hat{\xi}(Q), \xi(Q)\}$ and $\operatorname{cov}\{\hat{\xi}_c(Q), \xi_c(Q)\}$:

$$\operatorname{cov}\{\hat{\xi}(Q), \xi(Q)\} = \mathbf{a}'(Q)\, \mathbf{C}'_2 \operatorname{cov}\{\mathbf{v}, \xi(Q)\} + (\operatorname{cov}\{\mathbf{v}, \xi(Q)\})'\, \mathbf{C}_1 \operatorname{cov}\{\mathbf{v}, \xi(Q)\}\,,$$

$\operatorname{cov}\{\hat{\xi}_c(Q), \xi_c(Q)\} = (\operatorname{cov}\{\mathbf{v}, \xi(Q)\})'\, \mathbf{C}_1 \operatorname{cov}\{\mathbf{v}, \xi(Q)\}$. The quality of the estimators $\hat{\xi}(Q)$ and $\hat{\xi}_c(Q)$ is judged by means of the quantities:

$$\begin{aligned} (4.1) \qquad \mathrm{E}[(\hat{\xi}(Q) - \xi(Q))^2] &= \mathrm{D}[\hat{\xi}(Q)] - 2\operatorname{cov}\{\hat{\xi}(Q), \xi(Q)\} + \mathrm{D}[\xi(Q)] = \\ &= \mathbf{a}'(Q)\, \mathbf{C}_4\, \mathbf{a}(Q) - (\operatorname{cov}\{\mathbf{v}, \xi(Q)\})'\, \mathbf{C}_1 \operatorname{cov}\{\mathbf{v}, \xi(Q)\} - 2\mathbf{a}'(Q)\, \mathbf{C}'_2 \,. \\ &\quad .\operatorname{cov}\{\mathbf{v}, \xi(Q)\} + \mathrm{D}[\xi(Q)] \end{aligned}$$

and

$$(4.2) \qquad \mathrm{E}[(\hat{\xi}_c(Q) - \xi_c(Q))^2] = \mathrm{D}[\hat{\xi}_c(Q)] - 2\operatorname{cov}\{\hat{\xi}_c(Q), \xi_c(Q)\} + \\ + \mathrm{D}[\xi_c(Q)] = (\operatorname{cov}\{\boldsymbol{v}, \xi(Q)\})' \mathbf{C}_1 \operatorname{cov}\{\boldsymbol{v}, \xi(Q)\} - \\ - 2(\operatorname{cov}\{\boldsymbol{v}, \xi(Q)\})' \mathbf{C}_1 \operatorname{cov}\{\boldsymbol{v}, \xi(Q)\} + \mathrm{D}[\xi(Q)] = \mathrm{D}[\xi(Q)] - \\ - (\operatorname{cov}\{\boldsymbol{v}, \xi(Q)\})' \mathbf{C}_1 \operatorname{cov}\{\boldsymbol{v}, \xi(Q)\},$$

respectively.

For the dispersion of the trend component, as was mentioned above, we have

$$(4.3) \qquad \mathrm{D}[\boldsymbol{a}'(Q)\,\hat{\boldsymbol{\Theta}}] = \boldsymbol{a}'(Q)\,\mathbf{C}_4\,\boldsymbol{a}(Q).$$

5. CONCLUSION

In Parts 3 and 4 of the paper two different approaches to the solution of the least squares collocation problem are given under a certain generalization of the original formulation given in [7]. Now it can be easily proved that in the regular case both solutions given in the present paper are identical and at the same time they are identical with the solution given in [7].

Conditions of regularity: The rank $R(\mathbf{R}_v)$ of the covariance matrix $\mathbf{R}_v$ is N, i.e. the matrix $\mathbf{R}_v$ is regular and $R(\mathbf{A}_{N,n}) = n \leqq N$, i.e. the columns of the matrix $\mathbf{A}$ are linearly independent. In the case of regularity the following relations are valid automatically: $\operatorname{cov}\{\boldsymbol{v}, \xi(Q)\} \in \mathscr{M}(\mathbf{R}_v)$, $\boldsymbol{a}(Q) \in \mathscr{M}(\mathbf{A}')$. For a comparison we should remark that $m(Q)$ (the denotation from Part 3) $= \boldsymbol{a}'(Q)\,\boldsymbol{\Theta}$ (the denotation from Part 4) and analogously $\mathbf{R}_{v\xi}(Q) = \operatorname{cov}\{\boldsymbol{v}, \xi(Q)\}$, $\mathscr{M}(\mathbf{W}) = \mathscr{M}(\mathbf{A})$, $\mathbf{P}_{\mathscr{M}_0} = \mathbf{W}(\mathbf{W}'\mathbf{R}_v^{-1}\mathbf{W})^{-1}\,\mathbf{W}'\mathbf{R}_v^{-1} = \mathbf{A}(\mathbf{A}'\mathbf{R}_v^{-1}\mathbf{A})^{-1}\,\mathbf{A}'\mathbf{R}_v^{-1}$ (a different expression for the $\mathbf{R}_v^{-}$-projector) etc.

In the regular case the following relation is valid for the "Pandora-Box" matrix:

$$\begin{bmatrix} \mathbf{R}_v, & \mathbf{A} \\ \mathbf{A}', & 0 \end{bmatrix}^{-} =$$

$$= \begin{bmatrix} \mathbf{C}_1, & \mathbf{C}_2 \\ \mathbf{C}_3, & -\mathbf{C}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_v^{-1} - \mathbf{R}_v^{-1}\mathbf{A}(\mathbf{A}'\mathbf{R}_v^{-1}\mathbf{A})^{-1}\,\mathbf{A}\mathbf{R}_v^{-1}, & \mathbf{R}_v^{-1}\mathbf{A}(\mathbf{A}'\mathbf{R}_v^{-1}\mathbf{A})^{-1} \\ (\mathbf{A}'\mathbf{R}_v^{-1}\mathbf{A})^{-1}\,\mathbf{A}'\mathbf{R}_v^{-1} \quad , & -(\mathbf{A}'\mathbf{R}_v^{-1}\mathbf{A})^{-1} \end{bmatrix}.$$

If in Part 4 in the formulae for $\boldsymbol{a}'(Q)\,\hat{\boldsymbol{\Theta}}$, $\hat{\xi}(Q)$, $\mathrm{D}[\boldsymbol{a}'(Q)\,\hat{\boldsymbol{\Theta}}]$, $\mathrm{E}[(\hat{\xi}(Q) - \xi(Q))^2]$, $\mathrm{E}[(\xi_c(Q) - \hat{\xi}_c(Q))^2]$ etc. we substitute successively $\mathbf{R}_v^{-1} - \mathbf{R}_v^{-1}\mathbf{A}(\mathbf{A}'\mathbf{R}_v^{-1}\mathbf{A})^{-1}\,.\,\mathbf{A}'\mathbf{R}_v^{-1}$, $\mathbf{R}_v^{-1}\mathbf{A}(\mathbf{A}'\mathbf{R}_v^{-1}\mathbf{A})^{-1}$, $(\mathbf{A}'\mathbf{R}_v^{-1}\mathbf{A})^{-1}$ for the matrices $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_4$ and if we compare the relations obtained in this way with the corresponding relations in Part 3 we easily verify that the results are identical and correspond to the relations obtained by H. Moritz [7].

Definition 1. *The triple* $(\Omega, \mathscr{S}, \mathscr{P})$ *denotes a probability space* (Ω *is a set of elementary events,* $\mathscr{S}$ *is a nonempty class of random events and probability* $\mathscr{P}$ *is a real, positive, countably additive function defined on* $\mathscr{S}$). *The symbol* $\mathscr{L}^2(\Omega, \mathscr{S}, \mathscr{P})$ *denotes a Hilbert space of random variables of the probability space* $(\Omega, \mathscr{S}, \mathscr{P})$ *whose inner product is* $\langle \alpha_1, \alpha_2 \rangle = \mathrm{E}[\alpha_{1c}\alpha_{2c}] = \mathrm{E}[(\alpha_1 - \mathrm{E}[\alpha_1])(\alpha_2 - \mathrm{E}[\alpha_2])]$, *its metric* ϱ *being induced by the norm*: $\varrho^2(\alpha_1, \alpha_2) = \|\alpha_1 - \alpha_2\|^2 = \langle \alpha_1 - \alpha_2, \alpha_1 - \alpha_2 \rangle$.

Definition 2. *A stochastic function* $v(\cdot)$, $Q \in S$, S *being a bounded and closed region,* $S \subset \mathscr{R}^3$, *is continuous in quadratic mean on* S, *if for each* $Q' \in S$, $\lim_{Q \to Q'} \mathrm{E}[(v(Q) - v(Q'))^2] = 0$, *where the limit is taken over all points* Q *of* S.

Definition 3. *The Hilbert space* $\mathscr{H}(K)$ *of functions defined on* S *is a reproducing kernel Hilbert space (with a reproducing kernel* $(K(\cdot, \cdot\cdot), (P, Q) \in S \times S)$, *if* $\forall\{P \in S\}\ K(\cdot, P) \in \mathscr{H}(K)\ \&\ \forall\{P \in S\}\ \{g(\cdot) \in \mathscr{H}(K)\}\ \langle g(\cdot), K(\cdot, P)\rangle = g(P)$ *(reproducing property).*

In our case $S = \{P_1, \ldots, P_N\}$ and each function $g(\cdot) \in \mathscr{H}(K)$ has the form of a column vector $\mathbf{g} = (g(P_1), \ldots, g(P_N))'$.

Definition 4. *An isometric isomorfism of two Hilbert spaces* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *means that there exists a one-one linear mapping* $F: \mathscr{H}_1 \to \mathscr{H}_2$ *for which* $\langle u, v\rangle_{\mathscr{H}_1} = \langle F(u), F(v)\rangle_{\mathscr{H}_2}$ for each $u, v \in \mathscr{H}_1$.

Lemma 1. *Let* $x \in \mathscr{H}$ *and let* $\mathscr{H}'$ *be a Hilbert subspace of the Hilbert space* $\mathscr{H}$. *For an arbitrary* $x \in \mathscr{H}$ *there exists a unique element* $x^* \in \mathscr{H}'$ *for which* $\varrho_{\mathscr{H}}(x, x^*) = \min\{\varrho_{\mathscr{H}}(x, z), z \in \mathscr{H}'\}$. *The element* x^* *is the projection of* $x \in \mathscr{H}$ *on* $\mathscr{H}' \subset \mathscr{H}$; *the element* $x - x^* \in \mathscr{H}$ *is orthogonal to each element* $z \in \mathscr{H}'$: $\langle x - x^*, z\rangle = 0$.

Proof. See [6, p. 93].

Lemma 2. *Let* $\mathbf{A}$ *be an arbitrary* $N \times s$ *dimensional matrix and let* $\mathscr{M}(\mathbf{A})$ *denote the subspace of* $\mathscr{R}^N$ *generated by columns of the matrix* $\mathbf{A}$. *Then the subspace* $\mathscr{M}(\mathbf{A})$ *and* $\operatorname{Ker}(\mathbf{A}') = \{\mathbf{x} : \mathbf{x} \in \mathscr{R}^N, \mathbf{A}'\mathbf{x} = \mathbf{0}\}$ *are mutually orthogonal complements (in the Euclidean metric of the space* $\mathscr{R}^N$ *given by the inner product* $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}'\mathbf{y}$).

Proof. The set $\mathscr{N} \subset \mathscr{R}^N$ of elements orthogonal to $\mathscr{M}(\mathbf{A})$ is $\mathscr{N} = \{\mathbf{y} : \mathbf{y} \in \mathscr{R}^q, \forall\{\mathbf{u} \in \mathscr{R}^s\}\ \langle \mathbf{Au}, \mathbf{y}\rangle = 0\}$; the condition $\forall\{\mathbf{u} \in \mathscr{R}_s\}\ \langle \mathbf{u}, \mathbf{A}'\mathbf{y}\rangle = \langle \mathbf{Au}, \mathbf{y}\rangle = 0$ is equivalent to the condition $\mathbf{A}'\mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{y} \in \operatorname{Ker}(\mathbf{A}')$, thus $\mathscr{N} = \operatorname{Ker}(\mathbf{A}')$. Let $\mathscr{M}$ be a set of elements ortogonal to $\mathscr{N} = \operatorname{Ker}(\mathbf{A}')$; $\mathscr{M} = \{\mathbf{x} : \mathbf{x} \in \mathscr{R}^N, \forall\{\mathbf{k} \in \operatorname{Ker}(\mathbf{A}')\}, \langle \mathbf{k}, \mathbf{x}\rangle = 0\}$. The following equivalence obviously holds: $\forall\{\mathbf{k} \in \operatorname{Ker}(\mathbf{A}')\}\ \langle \mathbf{k}, \mathbf{x}\rangle = 0 \Leftrightarrow \mathbf{x} \in \mathscr{M}(\mathbf{A})$; thus $\mathscr{M} = \mathscr{M}(\mathbf{A})$.

Lemma 3. *Let* $\mathbf{A}$ *be an arbitrary* $N \times s$ *dimensional matrix; then* $\mathscr{M}(\mathbf{AA}') = \mathscr{M}(\mathbf{A})$.

Proof. By Lemma 2 the spaces $\mathscr{M}(\mathbf{AA}')$ and $\mathscr{M}(\mathbf{A})$ coincide if, and only if, $\operatorname{Ker}(\mathbf{A}') = \operatorname{Ker}(\mathbf{AA}')$. But $\mathbf{x} \in \operatorname{Ker}(\mathbf{A}') \Rightarrow \mathbf{A}'\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{AA}'\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} \in \operatorname{Ker}(\mathbf{AA}')$; conversely $\mathbf{x} \in \operatorname{Ker}(\mathbf{AA}') \Rightarrow \mathbf{AA}'\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x}'\mathbf{AA}'\mathbf{x} = 0 \Rightarrow \mathbf{A}'\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} \in \operatorname{Ker}(\mathbf{A}')$; thus $\mathscr{M}(\mathbf{AA}') = \mathscr{M}(\mathbf{A})$.

Let $\mathbf{R}_\nu$ be an $N \times N$ dimensional symmetric and positively semidefinite matrix. The number $\|\mathbf{x}\|_{\mathbf{R}_\nu^-} = (\mathbf{x}'\mathbf{R}_\nu^-\mathbf{x})^{1/2}$ is the $\mathbf{R}_\nu^-$-norm of the element $\mathbf{x} \in \mathscr{M}(\mathbf{R}_\nu)$; the number $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{R}_\nu^-} = \mathbf{x}'\mathbf{R}_\nu^-\mathbf{y}$ is the inner product in $\mathscr{M}(\mathbf{R}_\nu)$ (evidently the symbols $\mathscr{M}(\mathbf{R}_\nu)$ and $\mathscr{H}(\mathbf{R}_\nu)$ denote the same Hilbert space).

Lemma 4. *The matrices* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *satisfy the conditions*

(i) $\mathscr{M}(\mathbf{B}) \subset \mathscr{M}(\mathbf{A})$

(ii) $\mathscr{M}(\mathbf{B}) \subset \mathscr{M}(\mathbf{A})\ \&\ \mathscr{M}(\mathbf{C}) \subset \mathscr{M}(\mathbf{A}')$;

then (i) *implies that* $\mathbf{AA}^-\mathbf{B} = \mathbf{B}$ *and* (ii) *implies that the matrix* $\mathbf{C}'\mathbf{A}^-\mathbf{B}$ *is independent of the choice of the g-inversion of the matrix* $\mathbf{A}$.

Proof. $\mathscr{M}(\mathbf{B}) \subset \mathscr{M}(\mathbf{A})$ if, and only if, there exists a matrix $\mathbf{E}$ such that $\mathbf{B} = \mathbf{AE}$; then $\mathbf{AA}^-\mathbf{B} = \mathbf{AA}^-\mathbf{AE} = \mathbf{AE} = \mathbf{B}$. Analogously, $\mathscr{M}(\mathbf{C}) \subset \mathscr{M}(\mathbf{A}')$ if, and only if, there exists a matrix $\mathbf{F}$ such that $\mathbf{C} = \mathbf{A}'\mathbf{F}$; then $\mathbf{C}'\mathbf{A}^-\mathbf{B} = \mathbf{F}'\mathbf{AA}^-\mathbf{AE} = \mathbf{F}'\mathbf{AE}$, which proves the second property.

Lemma 5. *For an arbitrary symmetric matrix* $\mathbf{A}$ *both* $\mathbf{A}^-$ and $(\mathbf{A}^-)'$ *are its g-inversions (it means that there exists a symmetric version of the g-inversion, e.g.* $(\mathbf{A}^- + (\mathbf{A}^-)')/2)$.

Proof. The definition of g-inversion: $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$ yields $(\mathbf{AA}^-\mathbf{A})' = \mathbf{A}' \Rightarrow \Rightarrow \mathbf{A}'(\mathbf{A}^-)'\,\mathbf{A}' = \mathbf{A}' \Rightarrow \mathbf{A}(\mathbf{A}^-)'\,\mathbf{A} = \mathbf{A}$.

Definition 5. *Let* $\mathbf{M}$ *be an* $N \times r$ *dimensional matrix, where* $\mathscr{M}(\mathbf{M}) \subset \mathscr{M}(\mathbf{R}_\nu)$, *then the subspace* $\mathscr{M}(\mathbf{M})^{\perp \mathbf{R}_\nu^-} = \{\mathbf{x} : \mathbf{x} \in \mathscr{M}(\mathbf{R}_\nu),\ \forall\{\mathbf{y} \in \mathscr{M}(\mathbf{M})\}\ \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{R}_\nu^-} = 0\}$ *is* $\mathbf{R}_\nu^-$*-orthogonal in the subspace* $\mathscr{M}(\mathbf{R}_\nu)$ *to its subspace* $\mathscr{M}(\mathbf{M})$ $(\mathscr{M}(\mathbf{M}) \perp_{\mathbf{R}_\nu^-} [\mathscr{M}(\mathbf{M})]^{\perp \mathbf{R}_\nu^-}$.

Lemma 6. *For each element* $\mathbf{x} \in \mathscr{M}(\mathbf{R}_\nu)$ *there exists a unique couple of elements* $\mathbf{x}_1 \in \mathscr{M}(\mathbf{M})$, $\mathbf{x}_2 \in [\mathscr{M}(\mathbf{M})]^{\perp \mathbf{R}_\nu^-}$ *for which* $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$.

Proof. Let $\mathbf{x}$ be an arbitrary element of $\mathscr{M}(\mathbf{R}_\nu)$ and let $\mathbf{x}_1$ be the element of $\mathscr{M}(\mathbf{M})$ for which $\|\mathbf{x} - \mathbf{x}_1\|_{\mathbf{R}_\nu^-} = \min\{\|\mathbf{x} - \mathbf{u}\|_{\mathbf{R}_\nu^-} : \mathbf{u} \in \mathscr{M}(\mathbf{M})\}$. By Lemma 1 there exists a unique element $\mathbf{x}_1 \in \mathscr{M}(\mathbf{M})$ and a unique element $\mathbf{x}_2 = \mathbf{x} - \mathbf{x}_1$ $\mathbf{x}_2 \in [\mathscr{M}(\mathbf{M})]^{\perp \mathbf{R}_\nu^-}$.

Definition 6. *The element* $\mathbf{x}_1$ *from Lemma* 2 *corresponding to the element* $\mathbf{x} \in \mathscr{M}(\mathbf{R}_\nu)$ *is the* $\mathbf{R}_\nu^-$*-projection of the element* $\mathbf{x}$ *on the subspace* $\mathscr{M}(\mathbf{M})$.

Because of linearity of the mapping $\mathbf{x} \to \mathbf{x}_1$, there exists such an $N \times N$ dimensional matrix $\mathbf{P}$ that $\mathbf{x}_1 = \mathbf{Px}$.

Definition 7. *The matrix* $\mathbf{P}$ *is the* $\mathbf{R}_v^-$*-projector of the subspace* $\mathscr{M}(\mathbf{R}_v) \subset \mathscr{R}^N$ *on the subspace* $\mathscr{M}(\mathbf{M}) \subset \mathscr{M}(\mathbf{R}_v)$.

Lemma 7. *The* $N \times N$ *dimensional matrix* $\mathbf{P}$ *is the* $\mathbf{R}_v^-$*-projector of* $\mathscr{M}(\mathbf{R}_v)$ on $\mathscr{M}(\mathbf{M}) \subset \mathscr{M}(\mathbf{R}_v)$ *if, and only if,*

(i) $\mathscr{M}(\mathbf{PR}_v) = \mathscr{M}(\mathbf{M})$,

(ii) $\mathbf{PR}_v = \mathbf{R}_v\mathbf{P}'$,

(iii) $\mathbf{PR}_v = \mathbf{P}^2\mathbf{R}_v$.

Proof. a) Necessity. Let $\mathbf{P}$ be an $\mathbf{R}_v^-$-projector. Then each $\boldsymbol{x} \in \mathscr{M}(\mathbf{R}_v)$ is mapped into $\mathbf{P}\boldsymbol{x} \in \mathscr{M}(\mathbf{M}) \Rightarrow \mathscr{M}(\mathbf{PR}_v) \subset \mathscr{M}(\mathbf{M})$; for each $\boldsymbol{x} \in \mathscr{M}(\mathbf{M})$ we have $\mathbf{P}\boldsymbol{x} = \boldsymbol{x}$, thus $\mathscr{M}(\mathbf{M}) = \mathscr{M}(\mathbf{PM}) = \mathscr{M}(\mathbf{PR}_v\mathbf{E}) \subset \mathscr{M}(\mathbf{PR}_v)$; thus $\mathscr{M}(\mathbf{PR}_v) = \mathscr{M}(\mathbf{M})$ (i); (existence of the matrix $\mathbf{E}$ follows from the assumption $\mathscr{M}(\mathbf{M}) \subset \mathscr{M}(\mathbf{R}_v)$).

Each element $\boldsymbol{x}$ of the subspace $\mathscr{M}(\mathbf{R}_v)$ can be uniquely expressed in the form $\boldsymbol{x}_1 + \boldsymbol{x}_2$, where $\boldsymbol{x}_1 = \mathbf{P}\boldsymbol{x} \in \mathscr{M}(\mathbf{M}) \perp_{\mathbf{R}_v^-} [\mathscr{M}(\mathbf{M})]^{\perp \mathbf{R}_v^-}$; thus the matrix $\mathbf{P}$ must satisfy: $\forall\{\boldsymbol{x}, \boldsymbol{y} \in \mathscr{M}(\mathbf{R}_v)\}\ [(\mathbf{I} - \mathbf{P})\,\boldsymbol{x}'\mathbf{R}_v^-\mathbf{P}\boldsymbol{y} = 0 \Leftrightarrow \forall\{\boldsymbol{u}, \boldsymbol{v} \in \mathscr{R}^N\}\ [(\mathbf{I} - \mathbf{P})\,\mathbf{R}_v\boldsymbol{u}]'\,\mathbf{R}_v^- \,.\, \mathbf{PR}_v\boldsymbol{u} = 0 \Leftrightarrow \mathbf{R}_v(\mathbf{I} - \mathbf{P}')\,\mathbf{R}_v^-\mathbf{PR}_v = \mathbf{0} \Leftrightarrow \mathbf{R}_v\mathbf{R}_v^-\mathbf{PR}_v = \mathbf{R}_v\mathbf{P}'\mathbf{R}_v^-\mathbf{PR}_v$. By Lemma 4 the matrix $\mathbf{R}_v\mathbf{R}_v^-\mathbf{PR}_v$ is equal to $\mathbf{PR}_v$. The matrix $\mathbf{R}_v\mathbf{P}'\mathbf{R}_v^-\mathbf{PR}_v$ is symmetric, by Lemma 4 it is independent of the choice of the g-inversion of the matrix $\mathbf{R}_v$ and by Lemma 5 for each symmetric matrix there exists a symmetric g-inversion. Consequently, the matrix $\mathbf{PR}_v$ is symmetric, which means $\mathbf{PR}_v = \mathbf{R}_v\mathbf{P}'$ (ii).

Utilizing the last identity we obtain $\mathbf{PR}_v = \mathbf{R}_v\mathbf{P}'\mathbf{R}_v^-\mathbf{PR}_v = \mathbf{R}_v\mathbf{P}'\mathbf{R}_v^-\mathbf{R}_v\mathbf{P}' = \mathbf{R}_v\mathbf{P}'\mathbf{P}'$ (Lemma 4). Hence $\mathbf{PR}_v = \mathbf{R}_v\mathbf{P}' = \mathbf{R}_v\mathbf{P}'\mathbf{P}' \Leftrightarrow \mathbf{PR}_v = \mathbf{P}^2\mathbf{R}_v$ (iii).

b) Sufficiency. Let the conditions (i), (ii) and (iii) be satisfied. Then $\mathbf{P}\boldsymbol{x} \in \mathscr{M}(\mathbf{M})$ and $(\mathbf{I} - \mathbf{P})\,\boldsymbol{x} \in [\mathscr{M}(\mathbf{M})]^{\perp \mathbf{R}_v^-}$ for each element $\boldsymbol{x} \in \mathscr{M}(\mathbf{R}_v)$. The sufficiency follows from Definition 7 and Lemma 6.

Lemma 8. *The matrix* $\mathbf{P} = \mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^-\,\mathbf{M}'\mathbf{R}_v^-$ *is the* $\mathbf{R}_v^-$*-projector of the space* $\mathscr{M}(\mathbf{R}_v)$ *on its subspace* $\mathscr{M}(\mathbf{M}) \subset \mathscr{M}(\mathbf{R}_v)$.

Proof. The inclusion $\mathscr{M}(\mathbf{PR}_v) \subset \mathscr{M}(\mathbf{M})$ obviously holds. Lemma 4 implies the following relations: $\mathscr{M}(\mathbf{PR}_v) = \mathscr{M}[\mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^-\,\mathbf{M}'\mathbf{R}_v^-\mathbf{R}_v] = \mathscr{M}[\mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^-\,\mathbf{M}'] \supset \mathscr{M}[\mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^-\,\mathbf{M}'\mathbf{R}_v^-\mathbf{M}] = \mathscr{M}(\mathbf{M})$. The last identity follows from the inclusion $\mathscr{M}(\mathbf{M}') \subset \mathscr{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})$. The inclusion is a consequence of the assumption $\mathscr{M}(\mathbf{M}) \subset \mathscr{M}(\mathbf{R}_v)$, which means that there exists a matrix $\mathbf{E}$ with the property $\mathbf{M} = \mathbf{R}_v\mathbf{E}$. Thus $\mathbf{M}'\mathbf{R}_v^-\mathbf{M} = \mathbf{E}'\mathbf{R}_v\mathbf{E}$ and $\mathscr{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M}) = \mathscr{M}(\mathbf{E}'\mathbf{R}_v\mathbf{E}) = \mathscr{M}(\mathbf{E}'\mathbf{JJ}'\mathbf{E}) = \mathscr{M}(\mathbf{E}'\mathbf{J}) \supset \mathscr{M}(\mathbf{E}'\mathbf{JJ}') = \mathscr{M}(\mathbf{M}') \supset \mathscr{M}(\mathbf{E}'\mathbf{R}_v\mathbf{M}) = \mathscr{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})$ (the matrix $\mathbf{R}_v$ can be expressed in the form $\mathbf{R}_v = \mathbf{JJ}'$ because of its positive definiteness). The condition (i) from Lemma 7 is satisfied.

The above mentioned consideration proves that $\mathbf{PR}_v = \mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^-\,\mathbf{M}'$ is symmetric, thus the condition (ii) of Lemma 7 is satisfied.

Concerning the condition (iii) from Lemma 7 we proceed analogously: $\mathbf{PR}_v =$

$$= \mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^- \mathbf{M}'; \ \mathbf{P}^2\mathbf{R}_v = \mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^- \mathbf{M}'\mathbf{R}_v^-\mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^- \mathbf{M}'\mathbf{R}_v^-\mathbf{R}_v = \mathbf{M} \,.$$
$$. (\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^- \mathbf{M}'\mathbf{R}_v\mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^- \mathbf{M}' = \mathbf{M}(\mathbf{M}'\mathbf{R}_v^-\mathbf{M})^- \mathbf{M}'.$$

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Súhrn

ZOVŠEOBECNENÁ METÓDA KOLOKÁCIE PODĽA NAJMENŠÍCH ŠTVORCOV

LUDMILA KUBÁČKOVÁ, LUBOMÍR KUBÁČEK

Priebeh gravitačného poľa Zeme charakterizujeme jeho trendovou a jeho poruchovou zložku, pričom posledná má náhodný charakter. Obidve zložky zisťujeme meraním hodnôt poľa v N vhodne rozložených bodoch vyšetrovanej oblasti.

Určiť najlepší odhad hodnoty trendovej zložky a hodnoty poruchovej zložky v ľubovoľnom bode vyšetrovanej oblasti z nameraných údajov zaťažených náhodnými chybami sa vo fyzikálnej geodézii nazýva kolokačným problémom.

V práci sú ukázané dve všeobecné riešenia tohoto problému a je dokázaná ich vzájomná ekvivalencia a ekvivalencia s klasickým riešením v prípade regulárneho experimentu. Regularita experimentu sa charakterizuje regularitou kovariančnej matice N-tice náhodných premenných, realizáciou ktorých vzniká súbor nameraných údajov. Táto matica je daná súčtom kovariančnej matice poruchovej zložky a kovariančnej matice náhodných chýb merania.

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