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SOLUTION OF A SYSTEM OF LINEAR EQUATIONS WITH GIVEN ERROR SETS FOR COEFFICIENTS

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1. FORMULATION OF THE PROBLEM

Let a system of linear equations over the field of real numbers be given,

$$(1)\qquad \begin{array}{c} a_{10}x_0 + \ldots + a_{1n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m0}x_0 + \ldots + a_{mn}x_n = 0 \end{array}$$

or in the matrix form

$$(2)\qquad A\mathbf{x} = 0\,, \quad A \quad \text{is of type} \quad m \times (n + 1)\,.$$

Let K be a fixed set of matrices of type $m \times (n + 1)$ with real-valued coefficients. Our problem consists in finding the set

$$R(K) = \{\mathbf{x} \in E^{n+1} : \text{there exists } A \in K \text{ such that } A\mathbf{x} = 0\}\,.$$

We shall reformulate the problem to give it a more operative form.

Order all elements of a matrix A in a column (in an arbitrary ordering) and denote this column-vector by $\boldsymbol{a}$ ($\boldsymbol{a}$ has $m(n + 1)$ components). In this new notation we specify the set K for which we shall solve the problem. Let two vectors $\boldsymbol{a}_*$ and $\boldsymbol{a}^*$ of length $m(n + 1)$ with $\boldsymbol{a}_* \leqq \boldsymbol{a}^*$ and a matrix $C = (c_{ij})$ of type $r \times m(n + 1)$ be given. Then we define

$$(3)\qquad K = \{\boldsymbol{a} \in E^{m(n+1)} : \boldsymbol{a}_* \leqq \boldsymbol{a} \leqq \boldsymbol{a}^* \text{ and } C\boldsymbol{a} = 0\}\,.$$

Hence, to define K we use vectors of length $m(n + 1)$ instead of matrices of type $m \times (n + 1)$. This mapping $A \to \boldsymbol{a}$ is one-to-one.

Our problem is to describe the set

$$(4)\qquad R(K) = \{\mathbf{x} \in E^{m(n+1)} : \text{there exists } \boldsymbol{a} \in K \text{ such that } A\mathbf{x} = 0\}\,.$$

Thus we are interested in finding all solutions of a system of linear equations with inexact data, or more in detail, of equations whose coefficients are not given exactly, but are dispersed in prescribed sets. The condition $C\boldsymbol{a} = 0$ in (3) follows e.g. from the requirement for the sum of errors of the adjusted values to be equal to zero, as it is in the flow separators in various chemical equipments. The requirement of finding the set $R(K)$ can be found also in various types of technical and economical problems.

The problem is solved by Theorem 1 (and its modifications – Theorems 2 and 3). The result presented in Theorem 1 consists in reducing the description of $R(K)$ to solving a system of linear equations with coefficients depending linearly on nonzero vectors $\boldsymbol{\lambda} \geqq 0$. Thus from the computational point of view, it is very simple to determine the vectors of $R(K)$. This is the contribution of the present paper.

In [3] a similar problem is solved. Here the set $R(K)$ is constructed as the meet of a descending sequence of sets $R(L_{\mathscr{D}_j})$, where every $R(L_{\mathscr{D}_j})$ is the union of sets of the form $R(I)$, I interval. Recall that the construction of the set $R(I)$ is known – see [3]–[6].

2. CONSTRUCTION OF $R(K)$

We solve the system

$$C\boldsymbol{a} = 0\,. \tag{5}$$

Suppose that $\boldsymbol{a} = \begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{z}\end{pmatrix}$ and that $\boldsymbol{y}$ is a vector of free indeterminates of (5). Then for a matrix D, we have

$$\boldsymbol{z} = D\boldsymbol{y}\,, \quad \text{where} \quad \boldsymbol{y} = (y_1, \ldots, y_k)'\,. \tag{6}$$

The vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ are subject to restrictions

$$\boldsymbol{y}_* \leqq \boldsymbol{y} \leqq \boldsymbol{y}^* \quad \text{and} \quad \boldsymbol{z}_* \leqq \boldsymbol{z} \leqq z^*\,, \tag{7}$$

respectively, which follows from the requirement $\begin{pmatrix}\boldsymbol{y}_*\\ \boldsymbol{z}_*\end{pmatrix} = \boldsymbol{a}_* \leqq \boldsymbol{a} = \begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{z}\end{pmatrix} \leqq \boldsymbol{a}^* = \begin{pmatrix}\boldsymbol{y}^*\\ \boldsymbol{z}^*\end{pmatrix}$.

By (6) and (7) the set K can be expressed in the following form:

$$K = \left\{\boldsymbol{a} = \begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{z}\end{pmatrix} \in E^{m(n+1)} : \boldsymbol{y}_* \leqq \boldsymbol{y} \leqq \boldsymbol{y}^*,\ \boldsymbol{z}_* \leqq \boldsymbol{z} \leqq \boldsymbol{z}^*, \boldsymbol{z} = D\boldsymbol{y}\right\}. \tag{8}$$

The domain over which the vector $\boldsymbol{y}$ ranges is

$$K_1 = \left\{\boldsymbol{y} \in E^k : \text{there exists } \boldsymbol{a} = \begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{z}\end{pmatrix} \in K\right\}, \tag{9}$$

or by (8),

$$K_1 = \{\boldsymbol{y} \in E^k : \boldsymbol{z}_* \leqq D\boldsymbol{y} \leqq \boldsymbol{z}^*, \boldsymbol{y}_* \leqq \boldsymbol{y} \leqq \boldsymbol{y}^*\} \tag{10}$$

or still in another manner,

$$K_1 = \{\mathbf{y} \in E^k : \mathbf{z}_* \leqq D\mathbf{y} \leqq \mathbf{z}^*\} \cap [\mathbf{y}_*, \mathbf{y}^*] . \tag{11}$$

From (10) or (11) it follows that K_1 is a subset of the interval $[\mathbf{y}_*, \mathbf{y}^*]$ and so K_1 is a bounded convex polyhedral set. Thus, it is the convex hull of its (finitely many) vertices.

We may combine (2) and (6) into a single formula. Indeed, some entries of the matrix A are components of the vector $\mathbf{z}$, the other entries of A are components of the vector $\mathbf{y}$. Replace the components of $\mathbf{z}$ in (2) by means of (6). We obtain

$$\sum_{j=1}^{k} y_j\Big(\sum_{i=0}^{n} v_{ji}^s x_i\Big) = 0\,, \quad s = 1, 2, \ldots, m\,. \tag{12}$$

Denoting

$$V^s = \begin{pmatrix} v_{10}^s & \ldots & v_{1n}^s \\ \ldots & \ldots & \ldots \\ v_{k0}^s & \ldots & v_{kn}^s \end{pmatrix}, \quad s = 1, 2, \ldots, m\,, \tag{13}$$

we can rewrite (12) as

$$\langle V^s \mathbf{x}, \mathbf{y} \rangle = 0\,, \quad s = 1, \ldots, m \tag{14}$$

or

$$\langle V^{s\prime} \mathbf{y}, \mathbf{x} \rangle = 0\,, \quad s = 1, \ldots, m\,, \tag{15}$$

where $\langle\ ,\ \rangle$ means the scalar product.
Thus we have got

$$R(K) = \{\mathbf{x} \in E^{n+1} : \exists \mathbf{y} \in K_1 \quad \text{such that} \quad \langle V^{s\prime}\mathbf{y}, \mathbf{x} \rangle = 0\,,\ s = 1, \ldots, m\}\,. \tag{16}$$

Let $\mathbf{y}^1, \ldots, \mathbf{y}^l$ be all vertices of the set K_1. (Various methods for the calculation of vertices of a convex polyhedral set are known – see e.g. [1] or [7].) An arbitrary element $\mathbf{y} \in K_1$ is a convex combination of vertices of the set K_1, thus

$$\mathbf{y} = \sum_{i=1}^{l} \lambda_i \mathbf{y}^i \quad \text{for suitable } \lambda_i \geqq 0,\ \sum_{i=1}^{l} \lambda_i = 1\,. \tag{17}$$

A vector $\mathbf{x} \in E^{n+1}$ belongs to $R(K)$ iff for some $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_l)$ with $\boldsymbol{\lambda} \geqq 0, \sum_{i=1}^{l} \lambda_i = 1$ the identity

$$\Big\langle V^{s\prime} \sum_{i=1}^{l} \lambda_i \mathbf{y}^i, \mathbf{x} \Big\rangle = 0\,, \quad s = 1, \ldots, m$$

holds, which can be written in another manner as

$$\sum_{i=1}^{l} \langle \mathbf{x}, \lambda_i V^{s\prime} \mathbf{y}^i \rangle = 0\,, \quad s = 1, \ldots, m\,. \tag{18}$$

Denoting

$$V^{s\prime}\mathbf{y}^i = (t^s_{i0}, \ldots, t^s_{in})' , \quad s = 1, \ldots, m; \ i = 1, \ldots, l$$

(19)
$$T^s = \begin{pmatrix} t^s_{10} \ldots t^s_{1n} \\ \ldots\ldots\ldots \\ t^s_{l0} \ldots t^s_{ln} \end{pmatrix}, \quad s = 1, \ldots, m ,$$

then (18) can be written in the form

(20)
$$\begin{aligned} & t^s_{10}\lambda_1 x_0 + \ldots + t^s_{1n}\lambda_1 x_n + \\ & + \ldots\ldots\ldots\ldots\ldots\ldots + \\ + \ & t^s_{l0}\lambda_l x_0 + \ldots + t^s_{ln}\lambda_l x_n = 0 , \quad s = 1, \ldots, m \end{aligned}$$

or in the matrix form

(21)
$$\langle T^{s\prime}\boldsymbol{\lambda}, \mathbf{x}\rangle = 0 , \quad s = 1, \ldots, m .$$

We arrive at the following conclusion

Theorem 1. *A vector $\mathbf{x} \in E^{n+1}$ belongs to $R(K)$ iff it fulfils*

$$\langle T^{s\prime}\boldsymbol{\lambda}, \mathbf{x}\rangle = 0 , \quad s = 1, \ldots, m$$

for a non-zero vector $\boldsymbol{\lambda} \geqq 0$.

Note that with respect to the homogeneity of equations of the system (21) it was possible to substitute equivalently the requirement $\boldsymbol{\lambda} \neq 0$ for the condition $\sum_{i=1}^{l} \lambda_i = 1$. But sometimes it may be useful to preserve the conditions for $\boldsymbol{\lambda} : \boldsymbol{\lambda} \geqq 0$ and $\sum_{i=1}^{l} \lambda_i = 1$.

Conversely, if we wish to verify whether or not a vector $\mathbf{x} \in E^{n+1}$ belongs to the set $R(K)$ we solve the system (21) with respect to $\boldsymbol{\lambda}$ and we obtain

Theorem 2. *A vector $\mathbf{x} \in E^{n+1}$ belongs to $R(K)$ iff the system*

(22)
$$\langle T^s\mathbf{x}, \boldsymbol{\lambda}\rangle = 0 , \quad s = 1, \ldots, m$$

has a non-zero solution $\boldsymbol{\lambda} \geqq 0$.

Note that by the same argument as in Theorem 1, we replace $\sum_{i=1}^{l} \lambda_i = 1$ equivalently by $\boldsymbol{\lambda} \neq 0$. The same reason (homogeneity of (22)) enables us to put $\boldsymbol{\lambda} \leqq 0$ in place of $\boldsymbol{\lambda} \geqq 0$.

Furthermore, we can substitute another equivalent condition for the condition (22). This is introduced in the following Theorem 3, in which

(23)
$$W^i = (T_i^1\mathbf{x}, \ldots, T_i^m\mathbf{x})' , \quad i = 1, \ldots, l ,$$

where T^s_i is the i-th row of the matrix T^s.

Theorem 3. *The system* (22) *has a solution* $0 \neq \boldsymbol{\lambda} \geqq 0$ *iff no solution* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)'$ *of the system*

$$\langle W^i, \boldsymbol{\mu} \rangle < 0, \; i = 1, \ldots, l \tag{24}$$

exists.

The assertion follows from Th. 22.2[2].

Example.

Let a system of linear equations

$$\begin{aligned} a_{10}x_0 + a_{11}x_1 + a_{12}x_2 &= 0\,, \\ a_{20}x_0 + a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned} \tag{2}$$

be given, whose coefficients fulfil the inequalities

(*) $-1 \leqq a_{21} \leqq 1$, $-1 \leqq a_{22} \leqq -\frac{1}{2}$, $0 \leqq a_{ij} \leqq 1$ otherwise.

Furthermore, suppose these coefficients fulfil the equations

$$\begin{aligned} a_{10} \quad - 2a_{12} \qquad\quad - 3a_{21} \qquad &= 0\,, \\ a_{11} - a_{12} + 2a_{20} \qquad\qquad &= 0\,, \\ a_{12} \qquad + 2a_{21} \qquad &= 0\,, \\ a_{20} + 2a_{21} - a_{22} &= 0\,. \end{aligned} \tag{5}$$

Evidently, the first four columns of the matrix C of the system (5) are linearly independent. Thus the components of the vector $\mathbf{y}$ are a_{21}, a_{22}. The components a_{20}, a_{12}, a_{11}, a_{10} of the vector $\mathbf{z}$ follow from (5):

$$\begin{aligned} a_{20} &= -2a_{21} + a_{22}\,, \\ a_{12} &= -2a_{21}\,, \\ a_{11} &= 2a_{21} - 2a_{22}\,, \\ a_{10} &= -\,a_{21}\,. \end{aligned} \tag{6}$$

The set K_1 (see (10)) is a convex polyhedral set with vertices

$$\mathbf{y}^1 = (-\tfrac{1}{2}, -\tfrac{1}{2})'\,, \quad \mathbf{y}^2 = (-\tfrac{1}{4}, -\tfrac{1}{2})'\,, \quad \mathbf{y}^3 = (-\tfrac{1}{2}, -1)'$$

(which are the intersection points of the straight lines

$$2a_{21} - a_{22} = 0\,, \quad a_{21} = -\tfrac{1}{2}\,, \quad a_{22} = -\tfrac{1}{2}\,.$$

In more complicated cases, it is necessary to use a computer; an algorithm is given, e.g. in [1] or [7]).

Therefore K_1 is the family of all vectors of the form

$$\lambda_1(-\tfrac{1}{2}, -\tfrac{1}{2})' + \lambda_2(-\tfrac{1}{4}, -\tfrac{1}{2})' + \lambda_3(-\tfrac{1}{2}, -1)'\,.$$

where $\lambda_i \geqq 0$, $\sum_{i=1}^{3} \lambda_i = 1$.

Evidently, we can omit the condition $\sum_{i=1}^{3} \lambda_i = 1$ (to simplify the numerical computation).

It follows that

$$(17) \qquad \begin{aligned} a_{21} &= -\tfrac{1}{2}\lambda_1 - \tfrac{1}{4}\lambda_2 - \tfrac{1}{2}\lambda_3 , \\ a_{22} &= -\tfrac{1}{2}\lambda_1 - \tfrac{1}{2}\lambda_2 - \lambda_3 . \end{aligned}$$

The coeffiicients a_{ij} for the remaining ij follow from (6):

$$(17') \qquad \begin{aligned} a_{20} &= \tfrac{1}{2}\lambda_1 . \\ a_{12} &= \lambda_1 + \tfrac{1}{2}\lambda_2 + \lambda_3 , \\ a_{11} &= \tfrac{1}{2}\lambda_2 + \lambda_3 , \\ a_{10} &= \tfrac{1}{2}\lambda_1 + \tfrac{1}{4}\lambda_2 + \tfrac{1}{2}\lambda_3 . \end{aligned}$$

By Theorem 1, a "feasible" matrix A of the system (2) (in other words a vector $\boldsymbol{a} \in K$, see (8)) corresponds to every choice of $\boldsymbol{\lambda}$, $0 \neq \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)' \geqq 0$ – and all elements of K can be found in this way. The entries of the matrix A ($\equiv$ the components of the vector $\boldsymbol{a}$) are given in (17) and (17'). Consequently, (2) will assume the form

$$(20) \qquad \begin{aligned} (\tfrac{1}{2}\lambda_1 + \tfrac{1}{4}\lambda_2 + \tfrac{1}{2}\lambda_3) x_0 + (\tfrac{1}{2}\lambda_2 + \lambda_3) x_1 + (\lambda_1 + \tfrac{1}{2}\lambda_2 + \lambda_3) x_2 &= 0 , \\ \tfrac{1}{2}\lambda_1 x_0 + (-\tfrac{1}{2}\lambda_1 - \tfrac{1}{4}\lambda_2 - \tfrac{1}{2}\lambda_3) x_1 + (-\tfrac{1}{2}\lambda_1 - \tfrac{1}{2}\lambda_2 - \lambda_3) x_2 &= 0 . \end{aligned}$$

Put $\boldsymbol{\lambda} = (2, 4, 2)'$. Then

$$a_{10} = 3 , \quad a_{11} = 4 , \quad a_{12} = 6 , \quad a_{20} = 1 , \quad a_{21} = -3 , \quad a_{22} = -5 .$$

(Check: The coefficients $a_{10}, \ldots, a_{22}$ have to fulfil the equations (5) and, divided by 8, the inequalities (*).)

The system (2) with these coefficients

$$\begin{aligned} 3x_0 + 4x_1 + 6x_2 &= 0 , \\ x_0 - 3x_1 - 5x_2 &= 0 \end{aligned}$$

has the solutions

$$x_0 : x_1 : x_2 = -2 : 21 : -13 .$$

Thus

$$\boldsymbol{x} = (-2, 21, -13)' \in R(K) .$$

Conversely, if we wish to verify whether or not a vector $\boldsymbol{x}$ belongs to $R(K)$, we aply Theorem 2. By this theorem, a solution $\boldsymbol{\lambda}$, $0 \neq \boldsymbol{\lambda} \geqq 0$, corresponds to every $\boldsymbol{x} \in R(K)$.

First, we apply Theorem 2 to $\boldsymbol{x} = (-2, 21, -13)'$ and find such a $\boldsymbol{\lambda}$. Using (20) we obtain

$$(22)\qquad -(\tfrac{1}{2}\lambda_1 + \tfrac{1}{4}\lambda_2 + \tfrac{1}{2}\lambda_3)\,2 + (\tfrac{1}{2}\lambda_2 + \lambda_3)\,21 - (\lambda_1 + \tfrac{1}{2}\lambda_2 + \lambda_3)\,13 = 0\,,$$
$$-\tfrac{1}{2}\lambda_1\,2 + (-\tfrac{1}{2}\lambda_1 - \tfrac{1}{4}\lambda_2 - \tfrac{1}{2}\lambda_3)\,21 - (-\tfrac{1}{2}\lambda_1 - \tfrac{1}{2}\lambda_2 - \lambda_3)\,13 = 0$$

or

$$-4\lambda_1 + \lambda_2 + 2\lambda_3 = 0\,,$$
$$-4\lambda_1 + \lambda_2 + 2\lambda_3 = 0\,.$$

Clearly, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)' = (2, 4, 2)'$ is one solution of this system.

Then, we apply Theorem 2 to $\boldsymbol{x} = (4, 4, 2)'$. By the previous procedure, we obtain

$$(22)\qquad 2\lambda_3 = -\lambda_1 - \lambda_2\,, \quad 4\lambda_3 = -\lambda_1 - 2\lambda_2\,.$$

Thus there exists no solution $\boldsymbol{\lambda}$, $0 \neq \boldsymbol{\lambda} \geqq 0$ and so $\boldsymbol{x} \,\bar{\in}\, R(K)$.

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Souhrn

ŘEŠENÍ SYSTÉMU LINEÁRNÍCH ROVNIC S NEPŘESNĚ ZADANÝMI KOEFICIENTY

FRANTISEK ŠIK

V práci je nalezena metoda pro stanovení všech řešení systému lineárních rovnic s intervalově zadanými koeficienty a za dodatečného předpokladu, že tyto koeficienty splňují zadaný systém homogenních lineárních rovnic. Vektor $\boldsymbol{x}$ je řešením této úlohy, právě když jistý systém homogenních lineárních rovnic tvaru $\langle B^s\boldsymbol{\lambda}, \boldsymbol{x}\rangle = 0$, $s = 1, \ldots, m$, je splněn pro vhodný nenulový vektor $\boldsymbol{\lambda} \geqq 0$. Věta 2 a 3 jsou modifikace věty 1. Práci uzavírá jednoduchý příklad.

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