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Aplikace matematiky, Vol. 27 (1982), No. 4, 251–258

Persistent URL: http://dml.cz/dmlcz/103970

SOME METHODICAL REMARKS CONCERNING THE FLOW AROUND ARBITRARY PROFILES

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(Received October 20, 1980)

Two definitions of the flow of fluid around a system of profiles are commonly used. By the first definition, the stream function is constant on the boundary of each profile; by the second one, the normal component of the velocity vector is zero there. The main object of this paper is to prove that, after an appropriate arrangement of the first definition, this definition is more general than the second one, and, moreover, invariant under conformal mappings.

1. Let us commence with some generalities. The closed (i.e., extended) and the open Gaussian plane will be denoted by $\mathbf{S}$ and $\mathbf{E}$, respectively. By a well known theorem (see [3] or [5]),

(1) for each two distinct components A_1, A_2 of a compact set $A \subset \mathbf{S}$ there is always a topological circle[1]) $D \subset \mathbf{S} - A$ such that A_1, A_2 lie in distinct components G_1, G_2 of the set $\mathbf{S} - D$.

Note that, by the Janiszewski Theorem (see [3] or [5]), $G_j \cap \Omega$ are regions, if $\Omega = \mathbf{S} - A$ is a region.

If a topological circle D has the properties from (1), we say that D *separates* A_1 *from* A_2. Further, we say a component A_1 of a compact set $A \subset \mathbf{S}$ is *isolated*, iff there is a topological circle D separating it from all components A_2 of A different from A_1.

Let H be a homeomorphism of a region $\Omega \subset \mathbf{S}$ onto Ω^*. Let $\{z'_n\}$, $\{z''_n\}$ be two sequences of points from Ω such that $\lim z'_n = z' \in \partial\Omega$, $\lim z''_n = z'' \in \partial\Omega$, and that both limits $\lim H(z'_n) = w'$, $\lim H(z''_n) = w''$ exist. By (1) (and the note just after it), we easily see that the points z', z'' belong to two distinct components of $\mathbf{S} - \Omega$, if and only if the points w', w'' belong to two distinct components of $\mathbf{S} - \Omega^*$. This implies the existence of a one-one mapping χ of the system $\mathcal{M}(\Omega)$ of all components

[1]) I.e., a set homeomorphic to the circle $\{z;\ |z| = 1\}$.

of $\mathbf{S} - \Omega$ onto the system $\mathcal{M}(\Omega^*)$ of all components of $\mathbf{S} - \Omega^*$ with the following property:

(2) $$z_n \in \Omega\,,\ \operatorname{Ls} z_n \subset A \in \mathcal{M}(\Omega) \Rightarrow \operatorname{Ls} H(z_n) \subset \chi(A)\,.$$

(Ls z_n denotes the topological limes superíor of the sequence $\{z_n\}$, i.e., the set of all its accumulation points – see [3] or [5].)

We then say that $\chi(A)$ is *the component of* $\mathbf{S} - \Omega^*$ *affixed to the component* A *of* $\mathbf{S} - \Omega$ *by the homeomorphism* H.

By (1), we easily see that

(3) *the component* A^* *of* $\mathbf{S} - \Omega^*$ *affixed to an isolated component* A *of* $\mathbf{S} - \Omega$ *by any homeomorphism* H *of* Ω *onto* Ω^* *is isolated as well.*

As a consequence of the famous Lindelöf's Lemma (see [2] or [5]) the following assertion can be proved:

Theorem 1. *Let* Φ *be meromorphic on a region* $\Omega \subset \mathbf{S}$ *and let a point* $z_0 \in \partial\Omega$ *lie in a continuum* $A \subset \mathbf{S} - \Omega$ *containing more than one point. Suppose there is a* $w_0 \in \mathbf{S}$ *and a neighbourhood* $U(z_0, R)$ *of* z_0 *such that the implication*

(4) $$z_n \in \Omega\,,\ z_n \to z \in \partial\Omega \cap U(z_0, R) \Rightarrow \Phi(z_n) \to w_0$$

holds.

Then $w_0 \in \mathbf{E}$ *and* $\Phi \equiv w_0$ *on* Ω.[2])

As an easy consequence of Theorem 1 we have the following assertion:

Theorem 2. *Let* K *be a conformal mapping of a region* $\Omega \subset \mathbf{S}$ *onto* Ω^*. *If* A *is a component of* $\mathbf{S} - \Omega$ *containing more than one point and if there is a disc* $U(z_0, R)$ *with* $z_0 \in \partial A$ *and* $U(z_0, R) - \Omega = U(z_0, R) \cap A$[3]), *then the component* A^* *od* $\mathbf{S} - \Omega^*$ *affixed to* A *by* K *also contains more than one point.*

Proof. Suppose that, on the contrary, the component A contains more that one point, but the affixed component A^* only one point w_0. Then (by (2) with $H = K$ and $\chi(A) = A^*$) the implication (4) holds with $\Phi = K$, which leads to the contradictory conclusion that the conformal mapping K is constant.

Remark. No analogue of Theorem 2 holds for homeomorphisms.

2. From this moment on we always suppose that

(5) $\Omega \subset \mathbf{E}$ is a region, $\mathbf{S} - \Omega$ has only a countable number of components, and at most one of these components is non-isolated.

For such a region Ω, we denote by $\mathcal{N}(\Omega)$ and $\mathcal{P}(\Omega)$ the system of all isolated compo-

[2]) Theorem 1 is proved in [5]; a proof of its special case is given in [6].
[3]) Such a disc certainly exists, if A is an isolated component of $\mathbf{S} - \Omega$.

nents of the set $\mathbf{S} - \Omega$ containing exactly one point and more than one point, respectively.

Further, we suppose that a vector field $\boldsymbol{f} = (f_1, f_2)$ on Ω is given with $\operatorname{rot} \boldsymbol{f} \equiv 0$ and $\operatorname{div} \boldsymbol{f} \equiv 0$.[4]) Setting $F_1 = f_1$, $F_2 = -f_2$, the complex velocity $F = F_1 + iF_2$ will be holomorphic on Ω. Each analytic function $\mathscr{F}$ primitive of F is called the *complex potential* of the field $\boldsymbol{f}$.

The complex potential $\mathscr{F}$ admits unrestricted continuation in Ω. We say that the complex potential $\mathscr{F}$ has a *single-valued imaginary part* v, iff the identity $\operatorname{Im} \Phi = v$ holds for each element $[\Phi, z] \in \mathscr{F}$ (see [4]) in some neighbourhood of z. Then, of course, v is a *stream function of the field* $\boldsymbol{f}$. Two complex potential (two stream functions — if they exist) of the same field differ only by a complex (real) additive constant.

Bearing in mind only the case when the profiles are *impermeable*, for each $A \in \mathscr{P}(\Omega)$ we suppose to have a Jordan region R_A (i.e., a region whose boundary is a topological circle) with the following properties:

(6′) $$R_A - \Omega = A\,, \quad \partial R_A \subset \Omega\,,$$

(6″) for some Jordan parametrization φ_A of the topological circle ∂R_A, the identity $\operatorname{Im} \int_{\varphi_A} F = 0$ holds.

Remark. We tacitly suppose that all curves (especially, the curve φ_A from (6″)) in this paper have finite lengths. The identity $\operatorname{Im} \int_{\varphi_A} F = 0$ means that the flow of the field $\boldsymbol{f}$ through ∂R_A is zero.

Note that, by the Janiszewski Theorem, $R_A \cap \Omega = R_A - A$ is a region. ———

By the Cauchy Theorem,

(7) $\operatorname{Im} \int_{\psi} F = 0$ for any Jordan curve ψ loophomotopic to φ_A with respect to $\bar{R}_A - A$[5]);

further, this implies $\operatorname{Im} \int_{\omega} F = 0$ for any closed curve ω in $R_A - A$, and this is equivalent to the existence of a single-valued stream function v_A of the field $\boldsymbol{f}$ on $R_A - A$. R_A being fixed, we call v_A briefly the *local stream function of* $\boldsymbol{f}$ *at* A.

Definition. Supposing Ω as in (5) we say *the field* $\boldsymbol{f}$ *flows around* $A \in \mathscr{P}(\Omega)$, iff there is a constant c_A such that the local stream function v_A at A extended by $v_A = c_A$

[4]) $\boldsymbol{f}$ is the velocity field of a fluid. $\mathscr{N}(\Omega)$ correspond to the set of all (isolated) point-singularities of the field $\boldsymbol{f}$, $\mathscr{P}(\Omega)$ — to the set of all profiles. If both the systems are finite, then each profile is isolated, of course; for a periodical cascade of profiles, the only non-isolated component of $\mathbf{S} - \Omega$ is the set $\{\infty\}$.

[5]) For the definition of the loophomotopy see [1].

onto ∂A is continuous on $R_A \cap \Omega \cup \partial A$.[6]) We say that *the field* $\mathbf{f}$ *flows around* $\partial\Omega$ (and $\mathbf{S} - \Omega$), iff it flows around each $A \in \mathscr{P}(\Omega)$.

The main theoretical and practical advantage of this definition is its invariance under conformal mapping of Ω into $\mathbf{E}$; it is very interesting and substantial that *no special boundary properties are imposed on the conformal mapping.*

Theorem 3. *Let K be a conformal mapping of the region Ω (with the above properties) onto a region $\Omega^* \subset \mathbf{E}$. Let $\mathscr{F}$ be a complex potential of a field $\mathbf{f}$ on Ω and let $\mathscr{F}^* = \mathscr{F} \circ K_{-1}$.*[7])

Then the field $\mathbf{f}$ flows around $\partial\Omega$, if and only if the field $\mathbf{f}^*$ with the complex potential $\mathscr{F}^*$ flows around $\partial\Omega^*$.

Proof. Ω^* evidently has analogous properties as Ω. By Theorem 2, the component A^* of $\mathbf{S} - \Omega^*$ affixed by K to any component $A \in \mathscr{P}(\Omega)$ of $\mathbf{S} - \Omega$ belongs to $\mathscr{P}(\Omega^*)$.

Let us suppose the field $\mathbf{f}$ flows around $\partial\Omega$ and let $A^* \in \mathscr{P}(\Omega^*)$ be arbitrary; let $A \in \mathscr{P}(\Omega)$ be the component affixed to A^* by the mapping K_{-1}. Use the above notations R_A, φ_A, v_A, c_A omitting the subscript A, and set $D^* = K(\partial R)$. As may be easily seen, the region $K(R \cap \Omega)$ is part of a component R^* of the set $\mathbf{S} - D^*$; R^* is a Jordan region containing A^*, $R^* - \Omega^* = A^*$, $\partial R^* = D^* \subset \Omega^*$. The function $v^* = v \circ K_{-1}$ (defined on $R^* - A^*$) is a local stream function at A^* of the field $\mathbf{f}^*$.

If $z_n^* \in R^* \cap \Omega^*$ are arbitrary points converging to a point $z^* \in A^*$, we have $\operatorname{Ls} K_{-1}(z_n^*) \subset \partial A$ by (2). As v is continuous on $R \cap \Omega \cup \partial A$ and equal to c on ∂A, it follows that $\operatorname{Ls} v^*(z_n^*) = \operatorname{Ls} v(K_{-1}(z_n^*)) = \{c\}$. Therefore, $\lim v^*(z_n^*) = c$ (for each sequence of points $z_n^* \in R^* \cap \Omega^*$ tending to any point $z^* \in \partial A^*$). Thus, setting $v^* = c$ on ∂A^*, we extend the function v^* continuously.

The field $\mathbf{f}^*$ flows around A^*; as $A^* \in \mathscr{P}(\Omega^*)$ was arbitrary, $\mathbf{f}^*$ flows around $\partial\Omega^*$. The conditions of the theorem being symmetrical in Ω, $\mathscr{F}$ and Ω^*, $\mathscr{F}^*$ there is no need of proof of the reverse implication; Theorem 3 holds.

3. Now let us proceed to another definition of the flow around $\partial\Omega$. Let Ω have the hitherto properties and suppose moreover that

(8′) every $A \in \mathscr{P}(\Omega)$ is a closure of a Jordan region with $\partial A \subset \mathbf{E}$

and that

(8″) $\lambda_A : \langle 0, V_A \rangle \to A$ is a Jordan parametrization of ∂A, the parameter $s \in \langle 0, V_A \rangle$ being the length of the curve $\lambda_A \mid \langle 0, s \rangle$.[8])

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[6]) We see at once that the definition is independent of the choice of v_A and also of R_A; in the following, both R_A and v_A are supposed to be fixed.

[7]) There are no problems about the composition of the conformal mapping K_{-1} and the analytic function $\mathscr{F}$ admitting unrestricted continuation. (See [4] or [5].)

[8]) The condition $\partial A \subset \mathbf{E}$ may be omitted quite easily; it is sufficient to have an appropriate definition of the tangent vector of a curve going through ∞. (See [5].) But here we suppose the finiteness of length of any Jordan parametrization of ∂A (for each $A \in \mathscr{P}(\Omega)$) which is fully sufficient in practical use.

Suppose further that

(9) the field $\mathbf{f}$ (and consequently also its complex velocity F) admits a (finite) continuous extension onto $\Omega \cup \bigcup_{A \in \mathscr{P}(\Omega)} \partial A$.

Remark. As is easily seen, the curves λ_A satisfy the Lipschitz condition $|\lambda_A(s'') - \lambda_A(s')| \leqq |s'' - s'|$. Hence, a finite derivative λ'_A exists almost everywhere in $\langle 0, V_A \rangle$. ———

Now we say that *the normal component of the field* $\mathbf{f}$ *is zero almost everywhere on* $\partial\Omega$, iff

$$\text{(10)} \qquad \operatorname{Im}\left[F(\lambda_A(s))\,\lambda'_A(s)\right] = 0 \quad \text{almost everywhere on } \langle 0, V_A \rangle$$

for each $A \in \mathscr{P}(\Omega)$.

Theorem 4. *If the normal component of a field* $\mathbf{f}$ *is zero almost everywhere on* $\partial\Omega$, *then* $\mathbf{f}$ *flows around* $\partial\Omega$.

Proof. Let $A \in \mathscr{P}(\Omega)$ be arbitrary but fixed; abbreviate the above notations omitting the subscripts A in $R_A, \ldots, V_A$. By the Cauchy - Goursat Theorem (see [1] or [5]), $\int_\varphi F = \int_\lambda F$ provided both φ and λ have the same orientation, which, of course, may be supposed. As $\int_\lambda F = \int_0^V (F \circ \lambda)\,\lambda'$ and $\operatorname{Im}\left[(F \circ \lambda)\,\lambda'\right] = 0$ almost everywhere by assumption, we have

$$\text{(11)} \qquad \operatorname{Im} \int_\varphi F = 0\,.$$

As we have already seen, this implies the existence of local stream functions on $R \cap \Omega$. One of these functions may be obtained in this way:

Choose a linearly accessible point $a \in \partial\Omega$ (see [5]); without any loss of generality suppose $a = \lambda(0)$. For each $z \in R \cap \Omega$ there is a piecewise linear curve $\omega_z : \langle 0, 1 \rangle \to \mathbf{S}$ such that $\omega_z(0) = a$, $\omega_z(1) = z$, $\omega_z((0, 1\rangle) \subset R \cap \Omega$. Then the function

$$\text{(12')} \qquad v(z) = \operatorname{Im} \int_{\omega_z} F\,, \quad z \in R \cap \Omega\,,$$

is, obviously, a local stream function of $\mathbf{f}$ at A. Extend it onto $R \cap \Omega \cup \partial A$ setting

$$\text{(12'')} \qquad v(z) = 0 \quad \text{for each} \quad z \in \partial A\,.$$

Since $v \mid R \cap \Omega$ and $v \mid \partial A$ are continuous, the continuity of v on $R \cap \Omega \cup \partial A$ will be proved by showing that

$$(13) \qquad \lim_{z \to z_0, z \in R \cap \Omega} v(z) = 0 \quad \text{for each} \quad z_0 \in \partial A .$$

For each $z \in R \cap \Omega$ let $z' = \lambda(s(z))$ be the nearest point of $\partial A = \lambda(\langle 0, V \rangle)$, and let $\mu_z : \langle 0, 1 \rangle \to \mathbf{E}$ be the linear curve connecting z with z'. Applying the Cauchy-Goursat Theorem, we easily prove that

$$(14) \qquad \int_{\omega_z} F + \int_{\mu_z} F = \int_{\lambda | \langle 0, s(z) \rangle} F \quad \text{for each} \quad z \in R \cap \Omega.$$

(If $s(z) = 0$ for some z, write 0 instead of the integral on the right-hand side of (14).)

If $z \in R \cap \Omega$ tends to $z_0 \in \partial A$, then the length of μ_z tends to 0; as F is continuous on $\overline{R \cap \Omega}$, it is bounded. This implies $\int_{\mu_z} F \to 0$ for $z \in R \cap \Omega$, $z \to z_0$. As $\operatorname{Im} \int_{\lambda | \langle 0, s(z) \rangle} F = \int_0^{s(z)} \operatorname{Im} [(F \circ \lambda) \lambda'] = 0$ by (10), we have $v(z) = \operatorname{Im} \int_{\omega_z} F \to 0$ for $z \in R \cap \Omega$, $z \to z_0$, Q.E.D.

Thus, f flows around A; as $A \in \mathscr{P}(\Omega)$ was arbitrary, f flows around $\partial \Omega$.

4. Supposing the field f admits a (finite) continuous extension onto $\Omega \cup \bigcup_{A \in \mathscr{P}(\Omega)} \partial A$, we can prove the reverse of the implication in Theorem 4 as well:

Theorem 5. *Suppose the region Ω as in (5) and let (8′) and (8″) hold. Suppose the field f admits a (finite) continuous extension onto $\Omega \cap \bigcup_{A \in \mathscr{P}(\Omega)} \partial A$.*

Then f flows around $\partial \Omega$, if and only if its normal component is zero almost everywhere on $\partial \Omega$.

Proof. Choose $A \in \mathscr{P}(\Omega)$ arbitrarily and use the hitherto (abbreviated) notations. Let $s_0 \in (0, V)$ be an arbitrary point; as may be shown, there is a Jordan region U containing the point $\lambda(s_0)$ and points s', s'' with $0 < s' < s_0 < s'' < V$ such that

$$(15) \qquad U \cap \partial A = \lambda((s', s'')), \quad U - \Omega = U \cap A .$$ [9])

By the theorem on θ-curves (see [3] or [5]), this implies that

$$(16) \qquad U - \partial A = U_1 \cap U_2 ,$$

where U_1, U_2 are disjoint Jordan regions (components of the set $U - \partial A$); by the Jordan Theorem (see [3] or [5]) one of these regions – say U_1 – is part of the exterior of A, hence of Ω (the other region U_2 being then part of the interior of A).

[9]) By a well known theorem of the plane topology (see e.g. [3] or [5]), there is a homeomorphism H of $\mathbf{S}$ onto $\mathbf{S}$ such that $H(\partial A)$ is the unit circle. For the unit circle the existence of a Jordan region U with properties analogous to the mentioned ones is evident. Hence, such a region U exist in the general case as well.

Choose $s_1 \in (s', s'')$ so that $\lambda(s_1)$ is linearly accessible from U_1, and to each $z \in U_1$ assign a piecewise linear curve $\omega_z : \langle 0, 1\rangle \to \overline{U}_1$ such that $\omega_z(0) = \lambda(s_1)$, $\omega_z(1) = z$, $\omega_z((0, 1\rangle) \subset U_1$. Then define

$$(17) \qquad \Phi(z) = \int_{\omega_z} F \quad \text{for all} \quad z \in U_1 \,.$$

The function Φ is primitive of F in U_1, hence a holomorphic branch in U_1 of some complex potential $\mathscr{F}$ of the field $\boldsymbol{f}$; $v = \operatorname{Im} \Phi$ is a stream function of $\boldsymbol{f} \mid U_1$. By our assumptions, v may be extended onto $U_1 \cup \lambda((s', s''))$ continuously by setting it equal to an appropriate constant c on $\lambda((s', s''))$. Choose an arbitrary $z \in U_1$ and set $\omega^t = \omega_z \mid \langle 0, t\rangle$ for each $t \in (0, 1\rangle$; then obviously $\Phi(\omega_z(t)) = \int_{\omega^t} F \to 0$ for $t \to 0+$. Hence $c = 0$.

If for some $s \in (s', s'')$ the point $\lambda(s)$ is linearly accessible from U_1, there is a linear curve $u : \langle 0, 1\rangle \to \overline{U}_1$ with $u(0) = \lambda(s)$, $u((0, 1\rangle \subset U_1$. Setting (for each positive integer n) $u_n = u \mid \langle 0, 1/n\rangle$, $w_n = \omega_{u(1/n)}$ we show easily with aid of the Cauchy-Goursat Theorem that

$$(18) \qquad \int_{s_1}^{s} (F \circ \lambda)\, \lambda' = \int_{u_n} F - \int_{w_n} F \,.$$

Passing to the imaginary parts on both sides we obtain $\operatorname{Im} \int_{u_n} F = \operatorname{Im} \Phi(u(1/n)) = v(u(1/n)) \to v(\lambda(s)) = 0$ and $\operatorname{Im} \int_{w_n} F \to 0$ (as F is bounded on U_1 and the length of the curve w_n tends to zero). Thus, $\int_{s_1}^{s} \operatorname{Im} [(F \circ \lambda)\, \lambda'] = 0$ for each $s \in (s', s'')$ with $\lambda(s)$ linearly accessible from U_1. Taking in account the continuity of the integral (as a function of $s \in (s', s'')$) and the density of linearly accessible points, we see that $\int_{s_1}^{s} \operatorname{Im} [(F \circ \lambda)\, \lambda'] = 0$ for each $s \in (s', s'')$. Thus, $\operatorname{Im} [(F \circ \lambda)\, \lambda'] = 0$ almost everywhere in (s', s'').

For each $s_0 \in (0, V)$ we have $\operatorname{Im} [(F \circ \lambda)\, \lambda'] = 0$ almost everywhere on some interval (s', s'') containing s_0; thus, $\operatorname{Im} [(F \circ \lambda)\, \lambda'] = 0$ almost everywhere in $\langle 0, V\rangle$. As $A \in \mathscr{P}(\Omega)$ was arbitrary, the normal component of $\boldsymbol{f}$ is zero almost everywhere on $\partial\Omega$, Q.E.D.

Remark. Let the field $\boldsymbol{f}$ (continuous on $\Omega \cup \bigcup_{A \in \mathscr{P}(\Omega)} \partial A$) flow around $\partial\Omega$. It may be shown (by a method slightly simpler than the above one) that the identity $\operatorname{Im} [(F(\lambda(s)) \,.\, \lambda'(s)] = 0$ holds e.g. for any $s \in (0, V)$ at which λ' is continuous. An analogous assertion holds for one-sided derivatives. Hence, provided the profile ∂A is piecewise smooth, the normal component of the field f is equal to 0 at each point $s \in \langle 0, V \rangle$ with the exception of a finite set where only one-sided identities hold.

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Souhrn

NĚKOLIK METODICKÝCH POZNÁMEK O OBTÉKÁNÍ LIBOVOLNÝCH PROFILŮ

ILJA ČERNÝ

Srovnávají se dvě dobře známé definice obtékání hranice oblasti Ω rovinným vektorovým polem. Ukazuje se, že (vhodným způsobem upravená) definice založená na konstantnosti proudové funkce na každém profilu je nejen invariantní vůči konformním zobrazením, ale že je i obecnější než definice založená na nulovosti normálové složky pole v $\partial\Omega$.

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