# ESTIMATION OF POLYNOMIALS IN THE REGRESSION MODEL

JÚLIA VOLAUFOVÁ

Received October 6, 1980

## INTRODUCTION

The following problem often occurs in experimental practice. We have to verify if the investigated system characterized by the dependences $y = F_1(x, \boldsymbol{\theta})$ and $y = F_2(x, \boldsymbol{\beta})$ fulfils the following condition: there exists $x_0$ with the property $\mathbf{O} = F_1(x_0, \boldsymbol{\theta}) = F_2(x_0, \boldsymbol{\beta})$. If e.g. $F_1(x, \boldsymbol{\theta}) = \sum_{i=1}^{m} \theta_i x^i$ and $F_2(x, \boldsymbol{\beta}) = \sum_{i=1}^{n} \beta_i x^i$ then the condition is equivalent to the condition that the resultant of these polynomials is zero. The resultant is a polynomial in the variables $\theta_1, \ldots, \theta_m, \beta_1, \ldots, \beta_n$, hence an investigation of minimal variance estimates of polynomials is actual in this context.

The general case of estimating a polynomial $f(\boldsymbol{\theta}) = \sum_{i=1}^{m} a_i \theta_1^{i_1} \ldots \theta_k^{i_k}$, $\boldsymbol{\theta} \in \mathscr{R}^k$, from measured data is considered in this paper. The measured data are viewed as a realization of normally distributed random vector $\mathbf{Y}$ with a mean $\mathbf{A}\boldsymbol{\theta}$ and a covariance matrix $\mathbf{K}$. An arbitrary polynomial $f(\boldsymbol{\theta})$ can be expressed as a sum of homogeneous polynomials and these as sums of polynomials of the form given in a Note II. 1. Therefore the investigation in the paper is restricted to such polynomials. We can estimate the members of the sum as is shown in Theorem II.3. The criterion for the unbiased estimability of polynomials is in Theorem II.1. If the functional is not unbiasedly estimable we can use results of Theorem II.4 in some cases.

## I. PRELIMINARY ASSERTIONS

Let us consider a random vector $\mathbf{Y} \sim N_n(\mathbf{O}, \mathbf{K})$, where $R(\mathbf{K}) \leq n$. Following [7] we denote the linear subspace of $L^2(\Omega, \mathscr{A}, P_0)$ spanned by the real linear combinations $\sum_{i=1}^{n} a_i Y_i$ by $\mathscr{H} = L^2(\mathbf{Y}) \subset L^2(\Omega, \mathscr{A}, P_0)$. Let us consider the reproducing kernel

Hilbert space (RKHS) $\mathrm{H}(\boldsymbol{K}) = \mathcal{M}(\boldsymbol{K})$ (the space spanned by the columns of the matrix $\boldsymbol{K}$) with the kernel $\boldsymbol{K}$ and the inner product defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathrm{H}(\boldsymbol{K})} = \boldsymbol{x}'\boldsymbol{K}^-\boldsymbol{y}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{M}(\boldsymbol{K})$. (The symbol $\boldsymbol{K}^-$ means the g-inverse of the matrix $\boldsymbol{K}$ (cf. [6]).) We denote $\exp \odot \mathrm{H}(\boldsymbol{K}) = \bigoplus_{n=0}^{\infty} [\mathrm{H}(\boldsymbol{K})]^{n\odot}$, where $\bigoplus_{n=0}^{\infty} [\mathrm{H}(\boldsymbol{K})^{n\odot}$ is the direct sum of symmetric tensor products of $\mathrm{H}(\boldsymbol{K})$. Let $\mathcal{B}(\mathcal{H}) \subset \mathcal{A}$ be the σ-algebra generated by $\mathcal{H}$. We have $L^2(\mathcal{B}(\mathcal{H})) \subset L^2(\Omega, \mathcal{A}, \mathrm{P}_0)$. Then the assertions in [2] imply

Note I. 1. (i) $L^2(\boldsymbol{Y}) \simeq \mathrm{H}(\boldsymbol{K})$; $\exp \odot \mathrm{H}(\boldsymbol{K}) \simeq L^2(\mathcal{B}(\mathcal{H}))$, where the symbol "$\simeq$" means the isometric isomorphism between two Hilbert spaces such that $\exp \odot \boldsymbol{Ku} = \sum_{n=0}^{\infty} \frac{1}{n!} (\boldsymbol{Ku})^{n\odot} \in \exp \odot \mathrm{H}(\boldsymbol{K})$ is the image of the random variable $\exp(\boldsymbol{u}'\boldsymbol{Y} - \frac{1}{2}\boldsymbol{u}'\boldsymbol{Ku}) \in L^2(\mathcal{B}(\mathcal{H}))$ under this isomorphism. Let us denote by $L^2_n(\mathcal{B}(\mathcal{H}))$ the image of $\mathrm{H}(\boldsymbol{K})^{n\odot}$ under the above mentioned isomorphism. Then we can write $L^2(\mathcal{B}(\mathcal{H})) = \bigoplus_{n=0}^{\infty} L^2_n(\mathcal{B}(\mathcal{H}))$.

(ii) The generator of the space $L^2_m(\mathcal{B}(\mathcal{H}))$ is a system of random variables $h_m(Y_{n_1}, Y_{n_2}, \ldots, Y_{n_m})$, where for any random variables $U_1, \ldots, U_m$ we define

$$h_m(U_1, \ldots, U_m) = \sum_{\boldsymbol{a} \in \{0,1\}^m} \sum_{\boldsymbol{A} \in \mathcal{M}_{\boldsymbol{a}}} (-\tfrac{1}{2})^{\sum_{i,j} a_{ij}} \prod_{i=1}^{m} U_i^{(1-a_i)} \prod_{j=1}^{m} \operatorname{cov}(U_i, U_j)^{a_{ij}} .$$

Here $\mathcal{M}_{\boldsymbol{a}}$ is the set of all $m \times m$ matrices with elements $a_{ij}$ equal to 0 or 1, $i, j =$ $i, j = 1, \ldots, m$, and such that $(\boldsymbol{A} + \boldsymbol{A}')\,\boldsymbol{i} = \boldsymbol{a}$, where $\boldsymbol{i} = (1, \ldots, 1)'$.

**Lemma I. 1.** [3] *Let $\boldsymbol{A}$ be a matrix of order $n \times k$, let $\mathcal{M}(\boldsymbol{A}) \subset \mathcal{M}(\boldsymbol{K})$. Then $\forall\{\boldsymbol{\theta} \in \mathcal{R}^k\}$ $\mathrm{P}_{\boldsymbol{A\theta}}$ is a probability measure equivalent to the measure $\mathrm{P}_0$ on $(\Omega, \mathcal{A})$, such that $\forall\{\boldsymbol{u} \in \mathcal{R}^n\}$ the random variable $\boldsymbol{u}'\boldsymbol{Y}$ is normally distributed with $E_{\boldsymbol{A\theta}}(\boldsymbol{u}'\boldsymbol{Y}) = \boldsymbol{u}'\boldsymbol{A\theta}$ and $D_{\boldsymbol{A\theta}}(\boldsymbol{u}'\boldsymbol{Y}) = \boldsymbol{u}'\boldsymbol{Ku}$; in particular $E_{\boldsymbol{A\theta}}(\boldsymbol{Y}) = \boldsymbol{A\theta}$ $D_{\boldsymbol{A\theta}}(\boldsymbol{Y}) = \boldsymbol{K}$ (the covariance matrix of $\boldsymbol{Y}$).*

**Theorem I. 1.** [7] *The Hilbert space $L^2(\mathcal{B}(\mathcal{H}), \mathrm{P}_0)$ is isomorphic to the following RKHS of functions defined on $\mathrm{H}(\boldsymbol{K})$:*

$$\mathrm{H}(\boldsymbol{G}) = \{f_U : f_U(\boldsymbol{Kv}) = \langle U, \exp(\boldsymbol{v}'\boldsymbol{Y} - \tfrac{1}{2}\boldsymbol{v}'\boldsymbol{Kv}) \rangle\}_{L^2(\mathcal{B}(\mathcal{H}))}, \quad \boldsymbol{v} \in \mathcal{R}^n \; U \in L^2(\mathcal{B}(\mathcal{H})) .$$

*The reproducing kernel of $\mathrm{H}(\boldsymbol{G})$ is $\boldsymbol{G}(\boldsymbol{u}, \boldsymbol{u}_1) = \exp(\boldsymbol{u}'\boldsymbol{K}^-\boldsymbol{u}_1)$, $\boldsymbol{u}, \boldsymbol{u}_1 \in \mathrm{K}(\boldsymbol{K})$. The space $\mathrm{H}(\boldsymbol{G})$ can be expressed as*

$$\mathrm{H}(\boldsymbol{G}) = \{f(\cdot) : f(\boldsymbol{u}) = \sum_{n=0}^{\infty} \langle \boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_n, \boldsymbol{u}^{n\otimes} \rangle_{[\mathrm{H}(\boldsymbol{K})]^{n\otimes}}, \quad \boldsymbol{u} \in \mathrm{H}(\boldsymbol{K}),$$

$$\boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_n \in [\mathrm{H}(\boldsymbol{K})]^{n\otimes},$$

$$\|f_U\|^2_{\mathrm{H}(\boldsymbol{G})} = \sum_{n=0}^{\infty} n!\, \|\mathrm{P}_{\mathrm{H}(\boldsymbol{K})^{n\odot}}(\boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_n)\|_{[\mathrm{H}(\boldsymbol{K})]^{n\otimes}} < \infty\} .$$

**Theorem I.2.** [7] *The Hilbert space* $L^2(\mathscr{B}(\mathscr{H}), P_k)$, $\boldsymbol{k} \in H(\boldsymbol{K})$ *is spanned by the system of random variables*

$$\left\{\frac{dP_u}{dP_k} = \exp\left[(\boldsymbol{u} - \boldsymbol{k})' \boldsymbol{K}^- \boldsymbol{Y} - \tfrac{1}{2}(\|\boldsymbol{u}\|^2_{H(K)} - \|\boldsymbol{k}\|^2_{H(K)})\right], \quad \boldsymbol{u} \in H(\boldsymbol{K})\right\}.$$

*Further* $L^2(\mathscr{B}(\mathscr{H}), P_k) \simeq H(\boldsymbol{G}_k)$, *where* $H(\boldsymbol{G}_k)$ *is a RKHS with the kernel* $\boldsymbol{G}_k(\boldsymbol{u}, \boldsymbol{u}_1) = \exp\langle \boldsymbol{u} - \boldsymbol{k}, \boldsymbol{u}_1 - \boldsymbol{k}\rangle_{H(K)}$, $\boldsymbol{u}, \boldsymbol{u}_1 \in H(\boldsymbol{K})$.

$$H(\boldsymbol{G}_k) = \{f(\cdot) : f(\boldsymbol{u}) = \sum_{n=0}^{\infty} \langle \boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_n, (\boldsymbol{u} - \boldsymbol{k})^{n\otimes}\rangle_{[H(K)]^{n\otimes}}, \quad \boldsymbol{u} \in H(\boldsymbol{K}),$$
$$\boldsymbol{u} \in H(\boldsymbol{K}),$$
$$\|f\|^2_{H(\boldsymbol{G}_k)} = \sum_{n=0}^{\infty} n! \, \|P_{H(K)^{n\odot}}(\boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_n)\|^2_{[H(K)]^{n\otimes}} < \infty\}.$$

(*The symbol* $P_{H(K)^{n\odot}}$ *means the projection to the space* $[H(\boldsymbol{K})]^{n\odot}$.

## II. ESTIMATION OF POLYNOMIALS

Let us consider a model $(\boldsymbol{Y}, \boldsymbol{A\theta}, \boldsymbol{K})$, assuming that we observe the random vector $\boldsymbol{Y}$, $N_n(\boldsymbol{A\theta}, \boldsymbol{K})$ distributed. The design matrix $\boldsymbol{A}$ is known, it is of order $n \times k$, $R(\boldsymbol{A}) \leqq k \leqq n$; the vector parameter $\boldsymbol{\theta} \in \Theta = \mathscr{R}^k$. Let $\mathscr{M}(\boldsymbol{A}) \subset \mathscr{M}(\boldsymbol{K})$.

**Definition II.1.** *A functional* $f : \Theta \to \mathscr{R}^1$ *is said to be unbiasedly estimable (u.e.) at* $\boldsymbol{\theta}_0$ *if there exists a statistic* $\tilde{f} = U(\boldsymbol{Y})$ ($\mathscr{B}(\mathscr{H})$*-measurable) such that* $\forall\{\boldsymbol{\theta} \in \Theta\}$ . $E_{\boldsymbol{A\theta}}(U(\boldsymbol{Y})) = f(\boldsymbol{\theta})$, $E_{\boldsymbol{A\theta}_0}\{(U(\boldsymbol{Y}) - f(\boldsymbol{\theta}_0))^2\} < \infty$. *We denote the class of unbiased estimators of the functional* $f$ *by* $\mathscr{U}^f_{\theta_0}$. *The minimum variance unbiased estimator (MVUE) at* $\boldsymbol{\theta}_0$ *is a statistic* $\hat{f} = U(\boldsymbol{Y})$ *such that*

$$\hat{f} \in \mathscr{U}^f_{\boldsymbol{\theta}_0} \;\&\; \forall'\{\tilde{f} \in \mathscr{U}^f_{\boldsymbol{\theta}_0}\}\; E_{\boldsymbol{A\theta}_0}\{(\hat{f} - f(\boldsymbol{\theta}_0))^2\} \leqq E_{\boldsymbol{A\theta}_0}\{(\tilde{f} - f(\boldsymbol{\theta}_0))^2\}.$$

**Lemma II.1.** [5] *Let* $\mathscr{U}_0$ *be the class of all estimators* $U_0$ *having the property:* $\forall\{\boldsymbol{\theta} \in \Theta\}\; E_{\boldsymbol{A\theta}}(U_0) = 0$. *Then the statistic* $U(\boldsymbol{Y})$ *is a MVUE of the functional* $f$ *at* $\boldsymbol{\theta}_0$ *iff* $\forall\{U_0 \in \mathscr{U}_0\}\; \mathrm{cov}_{\boldsymbol{A\theta}_0}(U(\boldsymbol{Y}), U_0) = 0$.

Note II. 1. It can be shown that every polynomial

$$f(\boldsymbol{\theta}) = \sum_{i=1}^{m} a_i \theta_1^{i_1} \cdot \theta_2^{i_2} \ldots \theta_k^{i_k}$$

can be expressed in the form of a sum of homogeneous polynomials and hence in the form

$$f(\boldsymbol{\theta}) = \sum_{i=1}^{m} \langle \boldsymbol{p}_1^{(i)} \otimes \ldots \otimes \boldsymbol{p}_i^{(i)}, \boldsymbol{\theta}^{i\otimes}\rangle, \quad \boldsymbol{p}_s^{(i)} \in \mathscr{R}^k \quad i = 1, \ldots, m, \quad s = 1, \ldots, i, \quad \boldsymbol{\theta} \in \mathscr{R}^k.$$

Due to Lemma II.1. it is obvious that we can consider functionals $f(\boldsymbol{\theta})$ of the form $f(\boldsymbol{\theta}) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle$ only, where $\boldsymbol{p}_i \in \mathscr{R}^k$, $i = 1, \ldots, m$, $\boldsymbol{\theta} \in \mathscr{R}^k$. The symbol $\boldsymbol{p}_1 \otimes \boldsymbol{p}_2 \otimes \ldots \otimes \boldsymbol{p}_m$ is the Kronecker product (cf. [6]) of vectors of order $k \times 1$, and $\langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle$ means the scalar product in the space $(\mathscr{R}^k)^{m\otimes}$;

$$\langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle = \prod_{i=1}^{m} \boldsymbol{p}_i' \boldsymbol{\theta} .$$

**Lemma II.2.** [7] *A functional $f(\boldsymbol{\theta})$ is u.e. at 0 iff $f \in \mathrm{H}(\boldsymbol{G})$.*

**Theorem II.1.** *A functional $f(\boldsymbol{\theta}) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle$ is u.e. at 0 iff $\forall\{i = 1, 2, \ldots, m\}$ $\boldsymbol{p}_i \in \mathscr{M}(\boldsymbol{A}')$, i.e. $\boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m \in [\mathscr{M}(\boldsymbol{A}')]^{m\otimes}$.*

Proof. Let us suppose $\forall\{i = 1, \ldots, m\}$ $\boldsymbol{p}_i \in \mathscr{M}(\boldsymbol{A}')$. Then $\boldsymbol{A}'(\boldsymbol{A}')^- \boldsymbol{p}_i = \boldsymbol{p}_i$ and for $f(\boldsymbol{\theta})$ we have

$$f(\boldsymbol{\theta}) = \langle \boldsymbol{A}'(\boldsymbol{A}')^- \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{A}'(\boldsymbol{A}')^- \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle =$$
$$= \langle [\boldsymbol{K}(\boldsymbol{A}')^-]^{m\otimes} (\boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m), \quad (\boldsymbol{A\theta})^{m\otimes} \rangle_{[\mathrm{H}(\boldsymbol{K})]^{m\otimes}} .$$

Consequently, due to Lemma II.2 and Theorem I.1. $f$ is u.e.

Conversely, let $f$ be an u.e. functional. Then according to Lemma II.2 and Theorem I.1, $\exists\{\boldsymbol{u}_1 \otimes \ldots \otimes \boldsymbol{u}_m \in (\mathscr{R}^k)^{m\otimes}\}$ such that

$$\forall\{\boldsymbol{\theta} \in \mathscr{R}^k\} \quad f(\boldsymbol{\theta}) = \langle \boldsymbol{K}^{m\otimes}(\boldsymbol{u}_1 \otimes \ldots \otimes \boldsymbol{u}_m), (\boldsymbol{A\theta})^{m\otimes} \rangle_{[\mathrm{H}(\boldsymbol{K})]^{m\otimes}} =$$
$$= \langle \boldsymbol{A}'\boldsymbol{u}_1 \otimes \ldots \otimes \boldsymbol{A}'\boldsymbol{u}_m, \boldsymbol{\theta}^{m\otimes} \rangle = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle$$

and finally $\boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m \in [\mathscr{M}(\boldsymbol{A}')]^{m\otimes}$. The proof is complete.

An analogous result is given for uncorrelated observations in [4].

**Lemma II.3.** [7] *A functional $f$ is u.e. at $\boldsymbol{\theta}_0$ iff $f \in \mathrm{H}(\boldsymbol{G}_{\boldsymbol{A}\boldsymbol{\theta}_0})$. If the functional $f$ is u.e. at 0 then it is u.e. at every $\boldsymbol{\theta} \in \mathscr{R}^k$.*

Note II.2. It follows from Lemma II.3. that there exists a class $\mathscr{F} \in \mathrm{H}(\boldsymbol{G}_{\boldsymbol{A\theta}})$ for $f(\boldsymbol{\theta}) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle$ an u.e. functional, such that

$$\mathscr{F} = \{g(\cdot) : g(\boldsymbol{u}) = \langle \boldsymbol{K}^{m\otimes}(\boldsymbol{A}')_1^- \boldsymbol{p}_1 \otimes \ldots \otimes (\boldsymbol{A}')_m^- \boldsymbol{p}_m, \boldsymbol{u}^{m\otimes} \rangle_{[\mathrm{H}(\boldsymbol{K})]^{m\otimes}} , \quad \boldsymbol{u} \in \mathrm{H}(\boldsymbol{K}) ,$$
$$(\boldsymbol{A}')_i^- \in (\mathscr{A}')^- \; i = 1, \ldots, m, \; g(\boldsymbol{A\theta}) = f(\boldsymbol{A\theta}) \; \forall\{\boldsymbol{\theta} \in \mathscr{R}^k\}\} .$$

$(\mathscr{A}')^-$ is the class of $g$-inverses of the matrix $\boldsymbol{A}'$.

**Theorem II.2.** *Let $f(\boldsymbol{\theta}) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle$ be u.e. under the model $(\boldsymbol{Y}, \boldsymbol{A\theta}, \boldsymbol{K})$. Then $\tilde{f}_g(\boldsymbol{\theta}) = \mathrm{h}_m(\boldsymbol{p}_1'[(\boldsymbol{A}')_1^-]' \boldsymbol{Y}, \boldsymbol{p}_2'[(\boldsymbol{A}')_2^-]' \boldsymbol{Y}, \ldots, \boldsymbol{p}_m'[(\boldsymbol{A}')_m^-]' \boldsymbol{Y}$ is an unabiased estimator of $f(\boldsymbol{\theta})$, where $\mathrm{h}_m$ is a random variable from Note I.1.*

Proof. We have

$$f(\theta) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \theta^{m\otimes} \rangle = \langle (\boldsymbol{A}')_1^- \boldsymbol{p}_1 \otimes \ldots \otimes (\boldsymbol{A}')_m^- \boldsymbol{p}_m, (\boldsymbol{A}\theta)^{m\otimes} \rangle =$$

$$= \prod_{i=1}^{m} \langle \boldsymbol{K}(\boldsymbol{A}')_i^- \, p_i, \boldsymbol{A}\theta \rangle_{\mathrm{H}(\boldsymbol{K})} \,.$$

It can be shown that the random variable $\mathrm{h}_m(\boldsymbol{p}_1'[(\boldsymbol{A}')_1^-]' \boldsymbol{Y}, \ldots, \boldsymbol{p}_m'[(\boldsymbol{A}')_m^-]' \boldsymbol{Y}$ corresponds to $g = \boldsymbol{K}(\boldsymbol{A}')_1^- \boldsymbol{p}_1 \odot \boldsymbol{K}(\boldsymbol{A}')_2^- \boldsymbol{p}_2 \odot \ldots \odot \boldsymbol{K}(\boldsymbol{A}')_m^- \boldsymbol{p}_m$ under the isometric isomorphism between $\exp \odot \mathrm{H}(\boldsymbol{K})$ and $L^2(\mathscr{B}(\mathscr{H}))$ (cf. Note I.1); and

$$\forall\{\theta \in \Theta\} \; E_{\boldsymbol{A}\theta}(\mathrm{h}_m(\boldsymbol{p}_1'[(\boldsymbol{A}')_1^-]' \boldsymbol{Y}, \ldots, \boldsymbol{p}_m'[(\boldsymbol{A}')_m^-]' \boldsymbol{Y}) = \left\langle h_m, \frac{\mathrm{dP}_{\boldsymbol{A}\theta}}{\mathrm{dP}_0} \right\rangle_{L^2(\mathscr{B}(\mathscr{H}))} = f(\theta) \,.$$

The last identity is obvious if we take into account the above mentioned isomorphism. The proof is complete.

Note II.3. It is shown in [7] that the variance of an unbiased estimator $\tilde{f}_g(\theta)$ of $f(\cdot)$ at $\theta_0$ is

$$D_{\boldsymbol{A}\theta_0}(\tilde{f}_g) = \|g\|^2_{\mathrm{H}(\boldsymbol{G}_{\boldsymbol{A}\theta_0})} - f^2(\theta_0) \,,$$

where $g \in \mathscr{F}$ (cf. Note II.2) and

$$\|g\|^2_{\mathrm{H}(\boldsymbol{G}_{\boldsymbol{A}\theta_0})} = \sum_{i=0}^{m} \binom{m}{i}^2 i! \| \langle \mathrm{P}_{[\mathrm{H}(\boldsymbol{K})]^{m\odot}}\{\boldsymbol{K}^{m\otimes}(\boldsymbol{A}')_1^- \boldsymbol{p}_1 \otimes \ldots \otimes (\boldsymbol{A}')_m^- \boldsymbol{p}_m\} \,,$$

$$(\boldsymbol{A}\theta_0)^{(m-i)\otimes} \rangle_{\mathrm{H}(\boldsymbol{K})^{(m-i)\otimes}} \|^2_{\mathrm{H}(\boldsymbol{K})^{i\otimes}} \,.$$

The symbol $\mathrm{P}_{\mathrm{H}(\boldsymbol{K})^{m\odot}}$ means a projection to the space $[\mathrm{H}(\boldsymbol{K})]^{m\odot}$. It can be shown that if $\boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_m \in [\mathrm{H}(\boldsymbol{K})]^{m\otimes}$ then

$$\mathrm{P}_{\mathrm{H}(\boldsymbol{K})^{m\odot}}(\boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_m) = \frac{1}{\mathrm{m}!} \sum_{\sigma} \boldsymbol{g}_{\sigma_1} \otimes \ldots \otimes \boldsymbol{g}_{\sigma_m} \,,$$

where the summation runs over all permutations $\sigma$ of the set $\{1, \ldots, \mathrm{m}\}$. The symbol

$$\langle \mathrm{P}_{\mathrm{H}(\boldsymbol{K})^{m\odot}}\{\boldsymbol{K}^{m\otimes}(\boldsymbol{A}')_1^- \boldsymbol{p}_1 \otimes \ldots \otimes (\boldsymbol{A}')_m^- \boldsymbol{p}_m\}, (\boldsymbol{A}\theta)^{(m-i)\otimes} \rangle_{\mathrm{H}(\boldsymbol{K})^{(m-i)\otimes}}$$

means the vector

$$\frac{1}{\mathrm{m}!} \sum_{\sigma} \langle \boldsymbol{p}_{\sigma_1} \otimes \ldots \otimes \boldsymbol{p}_{\sigma_{m-i}}, \theta^{(m-i)\otimes} \rangle \, (\boldsymbol{K}(\boldsymbol{A}')^-_{\sigma_{m-i+1}} \boldsymbol{p}_{\sigma_{m-i+1}}) \otimes \ldots \otimes (\boldsymbol{K}(\boldsymbol{A}')^-_{\sigma_m} \boldsymbol{p}_{\sigma_m}) \,.$$

**Theorem II.3.** *Let* $f(\theta) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \theta^{m\otimes} \rangle$ *be u.e. under the model* $(\boldsymbol{Y}, \boldsymbol{A}\theta, \boldsymbol{K})$. *Then* $\hat{f}(\theta) = h_m(\boldsymbol{p}_1'[(\boldsymbol{A}')^-_{\mathrm{m}(\boldsymbol{K})}]' \boldsymbol{Y}, \ldots, \boldsymbol{p}_m'[(\boldsymbol{A}')^-_{\mathrm{m}(\boldsymbol{K})}]' \boldsymbol{Y})$ *is a MVUE of* $f(\theta)$ *in the class of all unbiased estimators corresponding to the class* $\mathscr{F}$ *(cf. Note* II.2, *Theorem* II.2), *where* $\mathrm{h}_m$ *is the random variable from Note* I.1 *and* $(\boldsymbol{A}')^-_{\mathrm{m}(\boldsymbol{K})}$ *denotes the minimum* $\boldsymbol{K}$*-norm (seminorm) g-inverse of the matrix* $\boldsymbol{A}'$ *(cf.* [6]).

Proof. Due to Theorem II.2, $\tilde{f}_g(\theta) = h_m(p'_1[(A')_1^-]' Y, \ldots, p'_m[(A')_m^-]' Y)$ is an unbiased estimator of $f(\theta)$. The variance of $\tilde{f}_g(\theta)$ at $\theta_0$ is

$$D_{A\theta_0}(\tilde{f}_g(\theta)) = \|g\|^2_{H(G_{A\theta_0})} - f^2(\theta_0), \quad \text{where} \quad \|g\|^2_{H(G_{A\theta_0})}$$

is the norm of the functional $g \in \mathscr{F}$ from Note II.2 and Note II.3. Further we have

$$D_{A\theta_0}(\tilde{f}_g(\theta)) = \sum_{i=0}^{m} \binom{m}{i}^2 i! \left\| \frac{1}{m!} \langle K^{m\otimes} \sum_\sigma (A')_{\sigma_1}^- p_{\sigma_1} \otimes \ldots \otimes (A')_{\sigma_m}^- p_{\sigma_m}, (A\theta_0)^{(m-i)\otimes} \rangle_{H(K)^{(m-i)\otimes}} \right\|^2_{H(K)^{i\otimes}} - f^2(\theta_0) =$$

$$= \sum_{i=1}^{m} \binom{m}{i}^2 i! \left\| \sum_\sigma \frac{1}{m!} \langle p_{\sigma_1} \otimes \ldots \otimes p_{\sigma_{m-i}}, \theta_0^{(m-i)\otimes} \rangle K(A')_{\sigma_{m-i+1}}^- p_{\sigma_{m-i+1}} \otimes \ldots \otimes K(A')_{\sigma_m}^- p_{\sigma_m} \right\|^2_{[H(K)]^{i\otimes}} =$$

$$= \sum_{i=1}^{m} \binom{m}{i}^2 i! \left\| K^{i\otimes} \sum_\sigma \frac{1}{m!} (A')_{\sigma_{m-i+1}}^- \otimes (A')_{\sigma_{m-i+2}}^- \otimes \ldots \otimes (A')_{\sigma_m}^- \langle p_{\sigma_1} \otimes \ldots \otimes p_{\sigma_{m-i}}, \theta_0^{(m-i)\otimes} \rangle . (p_{\sigma_{m-i+1}} \otimes \ldots \otimes p_{\sigma_m}) \right\|^2_{[H(K)]^{i\otimes}} =$$

$$= \sum_{i=1}^{m} \binom{m}{i}^2 i! \left\| K^{i\otimes} \frac{1}{m!} [(A')_{\sigma_{m-i+1}}^- \otimes \ldots \otimes (A')_{\sigma_m}^-; \ldots; (A')_{\varkappa_{m-i+1}}^- \otimes \ldots \otimes (A')_{\varkappa_m}^-] \times \right.$$
$$\left. \times \begin{bmatrix} \langle p_{\sigma_1} \otimes \ldots \otimes p_{\sigma_{m-i}}, \theta_0^{(m-i)\otimes} \rangle p_{\sigma_{m-i+1}} \otimes \ldots \otimes p_{\sigma_m} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \langle p_{\varkappa_1} \otimes \ldots \otimes p_{\varkappa_{m-i}}, \theta_0^{(m-i)\otimes} \rangle p_{\varkappa_{m-i+1}} \otimes \ldots \otimes p_{\varkappa_m} \end{bmatrix} \right\|^2_{[H(K)]^{i\otimes}} .$$

To minimize the variance it is sufficient to minimize every member of this sum. Let us consider the $j$-th member. We need to choose the $g$-inverse from $(\mathscr{A}')^-$ so that the $K^{j\otimes}$-norm (seminorm) of this $j$-th member be minimal. It can be shown that the matrix

$$\frac{1}{m!} [(A')_{\sigma_{m-j+1}}^- \otimes \ldots \otimes (A')_{\sigma_m}^-, \ldots, (A')_{\varkappa_{m-j+1}}^- \otimes \ldots \otimes (A')_{\varkappa_m}^-]$$

is a $g$-inverse of the matrix $((A')^{j\otimes\prime}, \ldots, (A')^{j\otimes\prime})'$. To get the minimal $K^{j\otimes}$-norm (seminorm) we have to choose

$$\{[((A')^{j\otimes})', \ldots, ((A')^{j\otimes})']'\}^-_{m(K^{j\otimes})}$$

$g$-inverse. The minimum norm (seminorm) solution is independent of various choices of minimum norm (seminorm) $g$-inverses. One choice of such a $g$-inverse is $1/m!$ . . $\{[(A')^-_{m(K)}]^{j\otimes}, \ldots, [(A')^-_{m(K)}]^{j\otimes}\}$. If we utilize the last result in the expression for $h_m$, we obtain the MVUE at $\boldsymbol{\theta}_0$ of the form $h_m(\boldsymbol{p}_1'[(\boldsymbol{A}')^-_{m(\boldsymbol{K})}]' \boldsymbol{Y}, \ldots, \boldsymbol{p}_m'[(\boldsymbol{A}')^-_{m(\boldsymbol{K})}]' \boldsymbol{Y})$. This is valid $\forall\{\boldsymbol{\theta}_0 \in \Theta\}$, so this unbiased estimator has the uniformly minimum variance. The proof is complete.

Let us now consider a functional $f(\boldsymbol{\theta}) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle$ such that $\exists\{i_0\}$ . . $\boldsymbol{p}_{i_0} \notin \mathscr{M}(\boldsymbol{A}')$, so that the functional is not u.e. For this kind of functionals we can prove the following theorem.

**Theorem II.4.** *Let $f(\theta) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \theta^{m\otimes} \rangle$ be not u.e. under the model $(\boldsymbol{Y}, \boldsymbol{K}\theta, \boldsymbol{K})$. Then $\tilde{f}(\theta) = h_m(\boldsymbol{p}_1'[(\boldsymbol{A}')]^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}]' \boldsymbol{Y}, \ldots, \boldsymbol{p}_m'[(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m\boldsymbol{K})}]' \boldsymbol{Y})$ is the statistic which minimizes both the maximal bias, $\max\{|E_{\boldsymbol{A}\theta}(\tilde{f}(\theta) - f(\theta))|; \|\theta\| = 1\}$ and the variance. $(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}$ is a minimum $\boldsymbol{K}$-norm (seminorm), $\boldsymbol{I}$-least squares $g$-inverse of the matrix $\boldsymbol{A}'$ (cf. [6]). The minimization of the variance is in the class of statistics of the form $h_m$ (cf. Note I.1) corresponding to functionals from the class*

$$\mathscr{G}_f = \{g : g(\boldsymbol{u}) = \langle \boldsymbol{K}^{m\otimes}(\boldsymbol{A}')_1^- \, P_{\mathscr{M}(\boldsymbol{A}')} \boldsymbol{p}_1 \otimes \ldots \otimes (\boldsymbol{A}')_m^- \, P_{\mathscr{M}(\boldsymbol{A}')} \boldsymbol{p}_m, \boldsymbol{u}^{m\otimes} \rangle_{[H(\boldsymbol{K})]^{m\otimes}},$$
$$\boldsymbol{u} \in H(\boldsymbol{K}), (\boldsymbol{A}')_i^- \in (\mathscr{A}')^-, i = 1, \ldots, m\} .$$

Proof. In virtue of the isomorphism between $L^2(\mathscr{B}(\mathscr{H}))$ and $\exp \odot H(\boldsymbol{K})$ the following identity holds:

$$E_{\boldsymbol{A}\theta}(h_m(\boldsymbol{p}_1'[(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}]' \boldsymbol{Y}, \ldots, \boldsymbol{p}_m'[(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}]' \boldsymbol{Y}) = \left\langle h_m, \frac{dP_{\boldsymbol{A}\theta}}{dP_0} \right\rangle_{L^2(\mathscr{B}(\mathscr{H}))} =$$
$$= \langle \boldsymbol{K}^{m\otimes}[(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}]^{m\otimes} \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, (\boldsymbol{A}\theta)^{m\otimes} \rangle_{H(\boldsymbol{K})^{m\otimes}} =$$
$$= \langle [\boldsymbol{A}'(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}]^{m\otimes} \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \theta^{m\otimes} \rangle.$$

It is easy to verify that $[\boldsymbol{A}'(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}]^{m\otimes}$ is a projection operator on $[\mathscr{M}(\boldsymbol{A}')]^{m\otimes}$. It follows that

$$\sup_{\|\theta\|=1} |E_{\boldsymbol{A}\theta}(h_m(\boldsymbol{p}_1'[(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}]' \boldsymbol{Y}, \ldots, \boldsymbol{p}_m'[(\boldsymbol{A}')^+_{L(\boldsymbol{I}),m(\boldsymbol{K})}]' \boldsymbol{Y}) - f(\theta)| =$$
$$= \sup_{\|\theta\|=1} |\langle P_{[\mathscr{M}(\boldsymbol{A}')]^{m\otimes}}(\boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m) - \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \theta^{m\otimes} \rangle| =$$
$$= \min_{\boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_m \in \mathscr{M}(\boldsymbol{A}')^{m\otimes}} |\langle \boldsymbol{g}_1 \otimes \ldots \otimes \boldsymbol{g}_m - \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \theta^{m\otimes} \}| .$$

Due to the last equalities it is clear that our choice of the statistic fulfils the condition concerning the bias. The proof of the minimality of the variance is similar to that in Theorem II.3. The proof is complete.

Example. Let us consider the following experiment. Let $y_1, y_2, y_3$ be the measured data of the dependence $y = \theta_1 x + \theta_2$ at the points $x_1, x_2, x_3$ and let $y_4, y_5, y_6$ be the measured data of the dependence $y = \theta_3 x^2 + \theta_4 x + \theta_5$ at the same points $x_1, x_2, x_3$. We assume the covariance matrix of the vector $\boldsymbol{Y} = (Y_1, \ldots, Y_6)'$, $D(\boldsymbol{Y}) = = \sigma^2 \boldsymbol{I}$. The aim of the experiment is to verify the hypothesis that there exists a point

$x_0$ such that $\theta_1 x_0 + \theta_2 = \theta_3 x_0^2 + \theta_4 x_0 + \theta_5 = 0$. The measurement was made in the interval $\langle a, b \rangle \not\ni x_0$. If the hypothesis is valid then the resultant $f(\boldsymbol{\theta})$ of these polynomials is zero. The procedure for testing the hypothesis can be established on the statistic $\hat{f}(\boldsymbol{\theta})$ from Theorem II.3. In our case the design matrix is

$$\boldsymbol{A} = \begin{bmatrix} x_1 & 1 & 0 & 0 & 0 \\ x_2 & 1 & 0 & 0 & 0 \\ x_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & x_1^2 & x_1 & 1 \\ 0 & 0 & x_2^2 & x_2 & 1 \\ 0 & 0 & x_3^2 & x_3 & 1 \end{bmatrix}; \quad \begin{aligned} &\boldsymbol{p}_1' = \boldsymbol{p}_2' = \boldsymbol{p}_4' = (1, 0, \ldots, 0) \\ &\boldsymbol{p}_5' = \boldsymbol{p}_7' = \boldsymbol{p}_8' = (0, 1, 0, \ldots, 0) \\ &\boldsymbol{p}_3' = (0, \ldots, 0, 1) \quad \boldsymbol{p}_6' = (0, 0, 0, 1, 0) \\ &\boldsymbol{p}_9' = (0, 0, 1, 0, 0); \end{aligned}$$

the resultant is

$$f(\boldsymbol{\theta}) = \begin{vmatrix} \theta_1 & \theta_2 & 0 \\ 0 & \theta_1 & \theta_2 \\ \theta_3 & \theta_4 & \theta_5 \end{vmatrix} = \theta_1^2\theta_5 + \theta_2^2\theta_3 - \theta_1\theta_2\theta_4 \, ;$$

$\mathrm{h}_3(u_1, u_2, u_3) = u_1 u_2 u_3 - u_1 \operatorname{cov}(u_2, u_3) - u_2 \operatorname{cov}(u_1, u_3) - u_3 \operatorname{cov}(u_1, u_2)$ (cf. Note I. 1 (ii)) and finally, the estimator is

$$\begin{aligned} \widehat{f(\boldsymbol{\theta})} = {} & \boldsymbol{p}_1'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_2'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_3'\boldsymbol{CA'Y} - \sigma^2(\boldsymbol{p}_1'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_2'\boldsymbol{C}\boldsymbol{p}_3 + \boldsymbol{p}_2'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_1'\boldsymbol{C}\boldsymbol{p}_3 + \\ & + \boldsymbol{p}_3'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_1'\boldsymbol{C}\boldsymbol{p}_2) + \boldsymbol{p}_4'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_5'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_6'\boldsymbol{CA'Y} - \sigma^2(\boldsymbol{p}_4'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_5'\boldsymbol{C}\boldsymbol{p}_6 + \\ & + \boldsymbol{p}_5'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_4'\boldsymbol{C}\boldsymbol{p}_6 + \boldsymbol{p}_6'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_4'\boldsymbol{C}\boldsymbol{p}_5) - \boldsymbol{p}_7'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_8'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_9'\boldsymbol{CA'Y} + \\ & + \sigma^2(\boldsymbol{p}_7'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_8'\boldsymbol{C}\boldsymbol{p}_9 + \boldsymbol{p}_8'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_7'\boldsymbol{C}\boldsymbol{p}_9 + \boldsymbol{p}_9'\boldsymbol{CA'Y} \,.\, \boldsymbol{p}_7'\boldsymbol{C}\boldsymbol{p}_8) \,, \end{aligned}$$

where

$$\boldsymbol{C} = \begin{bmatrix} \sum_{i=1}^{3} x_i^2 & \sum_{i=1}^{3} x_i & 0 & 0 & 0 \\ \sum_{i=1}^{3} x_i & 3 & 0 & 0 & 0 \\ 0 & 0 & \sum_{i=1}^{3} x_i^4 & \sum_{i=1}^{3} x_i^3 & \sum_{i=1}^{3} x_i^2 \\ 0 & 0 & \sum_{i=1}^{3} x_i^3 & \sum_{i=1}^{3} x_i^2 & \sum_{i=1}^{3} x_i \\ 0 & 0 & \sum_{i=1}^{3} x_i^2 & \sum_{i=1}^{3} x_i & 3 \end{bmatrix}^{-1} .$$

The value of $D(\hat{f}(\boldsymbol{\theta}))$ can help us to get a criterion for rejecting or accepting the null hypothesis. (The distribution function of the statistic $\hat{f}(\boldsymbol{\theta})$ is rather complicated and is the matter of further study.)

*References*


[1] *N. Aronszajn:* Theory of Reproducing Kernels. Trans. Amer. Math. Soc. 68 (1950), 337—404.

[2] *G. Kallianpur:* The Role of RKHS in the Study of Gaussian Processes. In Advances in Probability, vol. 2, M. Dekker INC., New York 1970, 59—83.

[3] *I. A. Ibragimov, J. A. Rozanov:* Gaussovskie slučajnye procesy. Nauka, Moskva 1970.

[4] *A. Pázman:* Optimal Designs for the Estimation of Polynomial Functionals. Kybernetika 17 (1981) (in print.)

[5] *R. C. Rao:* Lineární metody statistické indukce a jejich aplikace. Academia, Praha 1978.

[6] *R. C. Rao, S. K. Mitra:* Generalized Inverse of Matrices and Its Applications. John Willey, New York 1971.

[7] *F. Štulajter:* Nonlinear Estimators of Polynomials in Mean Values of a Gaussian Stochastic Process. Kybernetika 14 (1978), 3, 206—220.


## Souhrn

## ODHADY POLYNÓMOV V REGRESNOM MODELI


JULIA VOLAUFOVÁ



Nech $(\boldsymbol{Y}_{n,1}, \boldsymbol{A}_{n,k}\boldsymbol{\theta}_{k,1}, \boldsymbol{K}_{n,n})$ je všeobecný regresný model, kde $\mathrm{R}(\boldsymbol{A}_{n,k}) \leqq k \leqq n$ a $\mathrm{R}(\boldsymbol{K}_{n,n}) \leqq n$. Uvažujme polynóm $f(\boldsymbol{\theta})$ neznámého vektorového parametra $\boldsymbol{\theta} \in \mathscr{R}^k$ tvaru $f(\boldsymbol{\theta}) = \langle \boldsymbol{p}_1 \otimes \ldots \otimes \boldsymbol{p}_m, \boldsymbol{\theta}^{m\otimes} \rangle$. V práci je pre nevychýlene odhadnuteľný funkcionál $f(\boldsymbol{\theta})$ ukázaný najlepší nevychýlený odhad.



*Author's address:* RNDr. *Júlia Volaufová*, Ústav merania a meracej techniky SAV, Dúbravská cesta, 842 19 Bratislava.