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NOTE ON STEADY FLOW OF HEAT IN A SEMI-INFINITE STRIP

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1. Let us consider a semi-infinite strip of homogeneous metal and let its two edges $x = 0$, $x = \pi$ be kept at zero temperature.

Let the temperature U be given in the region $0 < x < c$, $y = 0$ and $\partial U/\partial y$ be prescribed on $c < x < \pi$, $y = 0$.

To determine the temperature function U we have to solve the differential equation (cf. [6])

$$\nabla^2 U = 0 \quad \text{in} \quad 0 < x < \pi\,, \quad y > 0$$

subject to the boundary conditions:

(i) $U = f_2(x)\,, \quad y = 0\,, \quad 0 < x < c\,,$

(ii) $\partial U/\partial y = -f_1(x)\,, \quad y = 0\,, \quad c < x < \pi\,,$

(iii) $\partial U/\partial x = 0 \quad \text{on} \quad x = 0, \pi\,,$

(iv) $|U| < \infty \quad \text{as} \quad y \to \infty\,.$

The solution of this problem depends on the solution of a dual trigonometric series of the type

$$\sum_{n=1}^{\infty} a_n \sin nx = f_2(x)\,, \quad 0 < x < c\,, \tag{1.1}$$

$$\sum_{n=1}^{\infty} n a_n \sin nx = f_1(x)\,, \quad c < x < \pi\,,$$

where $f_1(x), f_2(x)$ belong to the space $L(0, \pi)$.

The formal solution of the dual trigonometric series of the type

$$\sum_{n=1}^{\infty} n a_n \sin nx = f_2(x)\,, \quad 0 < x < c\,, \tag{1.2}$$

$$\sum_{n=1}^{\infty} a_n \sin nx = f_1(x)\,, \quad c < x < \pi$$

have been discussed by Tranter [1], Srivastav [2], Sneddson [3]. Noble and Whiteman [4] have proved the existence of (1.2), where the known functions are in $L^2(0, \pi)$. In this note we will discuss the existence of a solution of the dual sine series of the type (1.1).

2. It is well known that a solution of the Laplace's equation is the real or imaginary part of the function of a complex variable which is analytic in the region under consideration. Thus our problem is to determine an analytic function $U(x, y) + iv(x, y)$ of the complex variable $z = x + iy$ in the region $0 < x < \pi$, $y > 0$, whose real part satisfy the conditions (i) to (iv).

Consider a pair of dual series

$$(2.1) \qquad \sum_{n=1}^{\infty} b_n \sin nx = 0 , \quad 0 < x < c ,$$

$$\sum_{n=1}^{\infty} n b_n \sin nx = f_1(x) , \quad c < x < \pi$$

and

$$(2.2) \qquad \sum_{n=1}^{\infty} c_n \sin nx = f_2(x) , \quad 0 < x < c ,$$

$$\sum_{n=1}^{\infty} n c_n \sin nx = 0 , \quad c < x < \pi ,$$

where $a_n = b_n + c_n$ will be the solutions of (1.1) provided b_n and c_n satisfy (2.1) and (2.2), respectively.

To determine b_n, consider a bounded analytic function

$$(2.3) \qquad \varphi(z) = \int_c^{\pi} \frac{\cos (z/2) f(t)\, dt}{(\cos t - \cos z)} + ik .$$

From (2.1) and (2.3) we have

$$(2.4) \qquad \frac{F(x)}{\cos (x/2)} = \int_x^{\pi} \frac{f(t)\, dt}{\sqrt{(\cos x - \cos t)}} ,$$

where $F(x) = \int_p^{\pi} f_1(t)\, dt$.

Equation (2.4) is an integral equation of Abel type, the solution of which gives

$$(2.5) \qquad f(x) = \frac{2}{\pi} \int_x^{\pi} \frac{\cos (t/2) \tan (x/2)}{\sqrt{(\cos x - \cos t)}} f_1(t)\, dt .$$

Hence by (2.3), (2·5) we have

$$(2.6) \quad b_n = \frac{\sqrt{2}}{\pi} \int_c^{\pi} [P_n(\cos \theta) + P_{n-1}(\cos \theta)] \tan (\theta/2)\, d\theta \int_\theta^{\pi} \frac{f_1(t) \cos (t/2)\, dt}{\sqrt{(\cos \theta - \cos t)}} ,$$

where we have used the result

$$\sum_{n=1}^{\infty} [P_{n-1}(\cos\theta) + P_n(\cos\theta)] \sin nx = \frac{\sqrt{2}\cos(x/2)}{\sqrt{(\cos\theta - \cos x)}},$$

which is obtained from the Mehler integral formula (cf. [5]).

To determine c_n, consider the analytic function

$$\psi(z) = \frac{\cos(z/2)\, g(t)\, dt}{\sqrt{(\cos t - \cos z)}} + ik. \tag{2.7}$$

From (2.2) and (2.7) we have an Abel type integral equation, the solution of which gives

$$g(x) = -\frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{\sin(t/2) f_2(t)\, dt}{\sqrt{(\cos t - \cos x)}}. \tag{2.8}$$

Therefore by (2.7) and (2.8) we have

$$c_n = -\frac{\sqrt{2}}{\pi}\int_0^c [P_n(\cos\theta) + P_{n-1}(\cos\theta)]\, d\theta \frac{d}{d\theta}\int_0^\theta \frac{\sin(t/2) f_2(t)\, dt}{\sqrt{(\cos t - \cos\theta)}}. \tag{2.9}$$

Hence by (2.6) and (2.9) we have

$$a_n = \frac{\sqrt{2}}{\pi}\int_c^\pi [P_n(\cos\theta) + P_{n-1}(\cos\theta)] \tan(\theta/2)\, d\theta \int_\theta^\pi \frac{f_1(t)\cos(t/2)\, dt}{\sqrt{(\cos\theta - \cos t)}} - \tag{2.10}$$

$$- \frac{\sqrt{2}}{\pi}\int_0^c [P_n(\cos\theta) + P_{n-1}(\cos\theta)]\, d\theta \frac{d}{d\theta}\int_0^\theta \frac{\sin(t/2) f_2(t)\, dt}{(\cos t - \cos\theta)}.$$

3. The existence of a solution (1.1) depends on the following two theorems.

Theorem 1. *Let $f_1(x)$ be a function defined on (c, π) and satisfy Dirichlet's conditions in the same interval, then*

$$\sum_{n=1}^{\infty} n b_n \sin nx = \tfrac{1}{2}[f_1(x+0) + f_1(x-0)], \quad x \in (c, \pi), \tag{3.1}$$

$$\sum_{n=1}^{\infty} b_n \sin nx = 0, \quad x \in (0, c) \tag{3.2}$$

where

$$b_n = \frac{\sqrt{2}}{\pi}\int_c^\pi [P_n(\cos\theta) + P_{n-1}(\cos\theta)] \tan(\theta/2)\, d\theta \int_\theta^\pi \frac{f_1(t)\cos(t/2)}{\sqrt{(\cos\theta - \cos t)}}\, dt, \tag{3.3}$$

$n = 1, 2, 3, \ldots$.

Theorem 2. *Let $f_2(x)$ be a function of x defined on $(0, c)$ and let $f_2'(x)$ exists almost everywhere in $(0, c)$. Moreover, let $f_2(x)$ satisfy Dirichlet's conditions in the same interval; then*

$$\text{(3.4)} \qquad \sum_{n=1}^{\infty} nc_n \sin nx = 0\,, \qquad x \in (c, \pi)\,,$$

$$\text{(3.5)} \qquad \sum_{n=1}^{\infty} c_n \sin nx = \tfrac{1}{2}[f_2(x+0) + f_2(x-0)]\,, \qquad x \in (0, c)\,,$$

where

$$\text{(3.6)} \quad c_n = -\frac{\sqrt{2}}{\pi}\int_0^c [P_n(\cos\theta) + P_{n-1}(\cos\theta)] \tan(\theta/2)\, d\theta \int_0^\theta \frac{f_2'(t)\cos(t/2)}{\sqrt{(\cos t - \cos\theta)}}\, dt\,,$$

$n = 1, 2, 3, \ldots$.

Proof of Theorem 1. Let $S_m(x) = \sum_{n=1}^{m} nb_n \sin nx$, $x \in (c, \pi)$. Then by (2.3) and

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\varphi}{\sqrt{(\cos\varphi - \cos\theta)}}\, d\varphi$$

we have

$$S_m(x) = \frac{4}{\pi^2}\sum_{n=1}^{m} n \sin nx \int_c^\pi \tan(\theta/2)\, d\theta \int_0^\theta \frac{\cos n\varphi \cos\varphi/2}{\sqrt{(\cos\varphi - \cos\theta)}}\, d\varphi \times$$

$$\times \int_\theta^\pi \frac{f_1(t)\cos(t/2)\, dt}{\sqrt{(\cos\theta - \cos t)}} = \frac{2}{\pi}\sum_{n=1}^{m} n \sin nx \int_c^\pi \frac{\sin nt}{n} f_1(t)\, dt -$$

$$-\frac{4}{\pi^2}\sum_{n=1}^{m} n \sin nx \int_0^c \frac{\sin n\varphi}{n}\, d\varphi \int_c^\pi f_1(t) \frac{d}{d\varphi}\left\{\arc\tan\left(\frac{\cos\varphi/2}{\cos t/2}\sqrt{\frac{\cos c - \cos t}{\cos\varphi - \cos c}}\right)\right\} dt\,.$$

Interchanging the order of summation and integration, we get after some elementary mathematical operations

$$S_m(x) = \frac{1}{\pi}\int_c^\pi \left[\frac{\sin(m+\frac{1}{2})(t-x)}{2\sin\left(\frac{t-x}{2}\right)} - \frac{\sin(m+\frac{1}{2})(t+x)}{2\sin\left(\frac{t+x}{2}\right)}\right] f_1(t)\, dt -$$

$$-\frac{2}{\pi^2}\int_0^c \left[\frac{\sin(m+\frac{1}{2})(\varphi-x)}{2\sin\left(\frac{\varphi-x}{2}\right)} - \frac{\sin(m+\frac{1}{2})(\varphi+x)}{2\sin\left(\frac{\varphi+x}{2}\right)}\right] d\varphi \times$$

$$\times \int_c^\pi f_1(t)\frac{d}{d\varphi}\left\{\arc\tan\left(\frac{\cos\varphi/2}{\cos t/2}\sqrt{\frac{\cos c - \cos t}{\cos\varphi - \cos c}}\right)\right\} dt\,,$$

Now using the well known results

$$\text{(3.7)} \qquad \lim_{m\to\infty}\int_0^a \frac{\sin mx}{\sin x} f(x)\, dx = \frac{\pi}{2} f(0+)$$

and

$$\text{(3.8)} \qquad \lim_{m\to\infty}\int_a^b \frac{\sin mx}{\sin x} f(x)\, dx = 0\,, \quad (b > a > 0)$$

we have

$$\lim_{m\to\infty} S_m(x) = \tfrac{1}{2}[f_1(x+0) + f_1(x-0)] \quad \text{in} \quad c < x < \pi ,$$

i.e., we get (3.1).

To prove (3.2), consider

$$s_m(x) = \sum_{n=1}^{m} b_n \sin nx ,$$

where $x \in (0, c)$. On the right hand side of the above expression, substitute the value of b_n given by (3.3) and

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\varphi}{\sqrt{(\cos\theta - \cos\varphi)}}\,d\varphi .$$

We have

$$s_m(x) = \frac{4}{\pi^2}\sum_{n=1}^{m} \sin nx \int_c^\pi \tan\theta/2\, d\theta \int_\theta^\pi \frac{\sin n\varphi\cos\varphi/2}{\sqrt{(\cos\theta - \cos\varphi)}}\,d\varphi \int_\theta^\pi \frac{f_1(t)\cos(t/2)}{\sqrt{(\cos\theta - \cos t)}}\,dt =$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{m} \sin nx \int_c^\pi \sin n\varphi\, F(\varphi)\,d\varphi ,$$

where

$$F(\varphi) = \int_c^\phi f_1(t)\left\{\text{arc}\tanh\left(\frac{\cos\varphi/2}{\cos t/2}\sqrt{\frac{\cos c - \cos t}{\cos\varphi - \cos c}}\right)\right\} dt +$$

$$+ \int_\phi^\pi f_1(t)\left\{\text{arc}\tanh\left(\frac{\cos t/2}{\cos\varphi/2}\sqrt{\frac{\cos\varphi - \cos c}{\cos c - \cos t}}\right)\right\} dt .$$

Changing the order of integration and summation we have

$$s_m(x) = \frac{2}{\pi^2}\int_c^\pi \sum_{n=1}^{m} [\cos(\varphi - x)n - \cos(\varphi + x)n]\, F(\varphi)\,d\varphi =$$

$$= \frac{2}{\pi^2}\int_c^\pi \left[\frac{\sin(m+\frac{1}{2})(\varphi - x)}{2\sin\left(\frac{\varphi - x}{2}\right)} - \frac{\sin(m+\frac{1}{2})(\varphi + x)}{2\sin\left(\frac{\varphi + x}{2}\right)}\right] F(\varphi)\,d\varphi .$$

Since $x \notin (c, \pi)$, hence (3.8) implies

$$\lim_{m\to\infty} s_m(x) = 0 \quad \text{for all} \quad x \in (0, c) ,$$

i.e., we get (3.2).

Proof of Theorem 2. From (3.6) and

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\varphi}{\sqrt{(\cos\varphi - \cos\theta)}}\,d\varphi$$

one can easily prove that

$$c_n = -\frac{4}{\pi^2}\int_0^c \frac{\sin n\varphi}{n}\,[G_1(\varphi) + G_2(\varphi)]\,\mathrm{d}\varphi,$$

where

$$G_1(\varphi) = \int_\phi^c f_2'(t)\,\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[\operatorname{arc\,tan}h\left(\frac{\cos\varphi/2}{\cos t/2}\sqrt{\frac{\cos t - \cos c}{\cos\varphi - \cos c}}\right)\right]\mathrm{d}t,$$

$$G_2(\varphi) = \int_0^\phi f_2'(t)\,\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[\operatorname{arc\,tan}h\left(\frac{\cos t/2}{\cos\varphi/2}\sqrt{\frac{\cos\varphi - \cos c}{\cos t - \cos c}}\right)\right]\mathrm{d}t.$$

Therefore,

$$S_m'(x) = \sum_{m=1}^{m} nc_n \sin nx =$$

$$= -\frac{2}{\pi^2}\int_0^c \left[\frac{\sin(m+\frac{1}{2})(\varphi - x)}{2\sin\left(\frac{\varphi - x}{2}\right)} - \frac{\sin(m+\frac{1}{2})(\varphi + x)}{2\sin\left(\frac{\varphi + x}{2}\right)}\right][G_1(\varphi) + G_2(\varphi)]\,\mathrm{d}\varphi.$$

Hence from (3.8) it is obvious that

$$\forall x \in (c, \pi), \quad \lim_{m\to\infty} S_m'(x) = 0,$$

i.e., we get (3.4).

To get (3.5) let us assume

$$s_m'(x) = \sum_{n=1}^{m} c_n \sin nx.$$

Using (3.6) and

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\varphi}{\sqrt{(\cos\theta - \cos\varphi)}}\,\mathrm{d}\varphi,$$

$$s_m'(x) = \frac{2}{\pi}\sum_{n=1}^{m}\sin nx\int_0^c \sin n\varphi\,\mathrm{d}\varphi\int_0^\phi f_2'(t)\,\mathrm{d}t +$$

$$+\frac{4}{\pi^2}\sum_{n=1}^{m}\sin nx\int_c^\pi \sin n\varphi\,\mathrm{d}\varphi\int_0^c f_2'(t)\left[\operatorname{arc\,tan}\left(\frac{\cos\varphi/2}{\cos t/2}\sqrt{\frac{\cos t - \cos c}{\cos c - \cos\varphi}}\right)\right]\mathrm{d}t.$$

Changing the order of integration and summation we have

$$s_m'(x) = \frac{1}{\pi}\int_0^c\left[\frac{\sin(m+\frac{1}{2})(\varphi - x)}{2\sin\left(\frac{\varphi - x}{2}\right)} - \frac{\sin(m+\frac{1}{2})(\varphi + x)}{2\sin\left(\frac{\varphi + x}{2}\right)}\right]\mathrm{d}\varphi\int_0^\phi f_2'(t)\,\mathrm{d}t +$$

$$+\frac{2}{\pi^2}\int_c^\pi\left[\frac{\sin(m+\frac{1}{2})(\varphi - x)}{2\sin\left(\frac{\varphi - x}{2}\right)} - \frac{\sin(m+\frac{1}{2})(\varphi + x)}{2\sin\left(\frac{\varphi + x}{2}\right)}\right]\mathrm{d}\varphi\int_0^c f_2'(t)\,\times$$

$$\times \left[\arc\tan \left(\frac{\cos \varphi/2}{\cos t/2} \sqrt{\frac{\cos t - \cos c}{\cos c - \cos \varphi}} \right) \right] \mathrm{d}t.$$

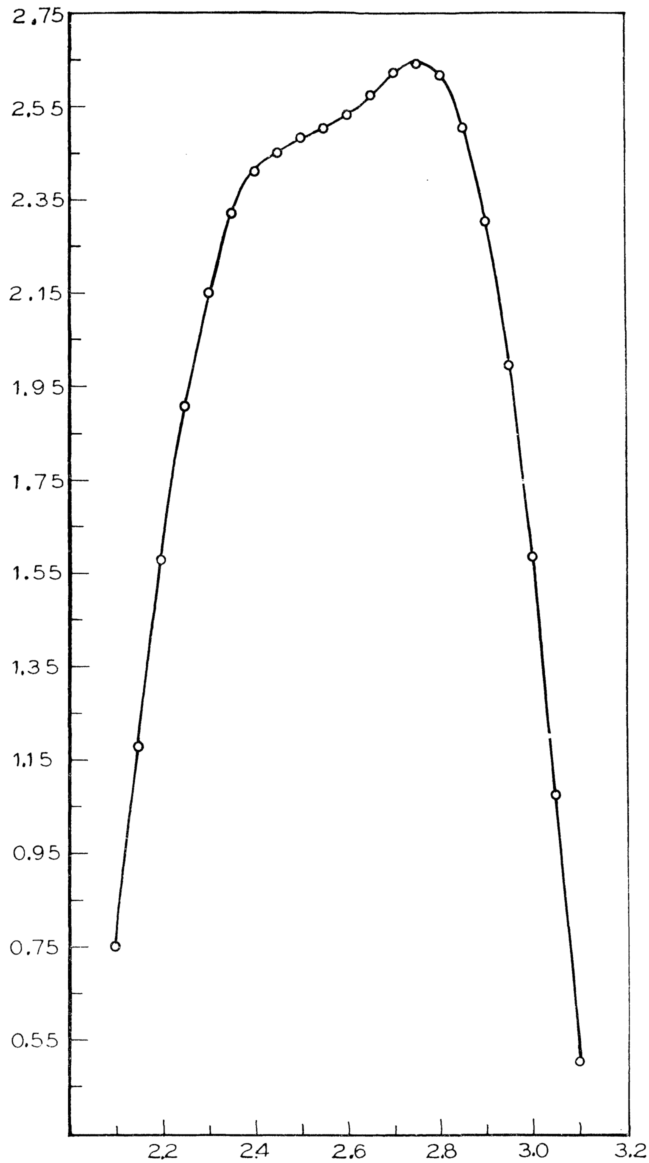

Fig. 1. Curve for temperature distribution.

Therefore, using (3.7) and making $m \to \infty$ we have

$$\forall x \in (0, c), \quad \lim_{m \to \infty} s'_m(x) = \tfrac{1}{2}[f_2(x + 0) + f_2(x - 0)],$$

i.e., we get (3.5).

4. For numerical evaluation let us take $f_1(x) = x + 0{\cdot}5x^2$ in $2 < x < \pi$ and

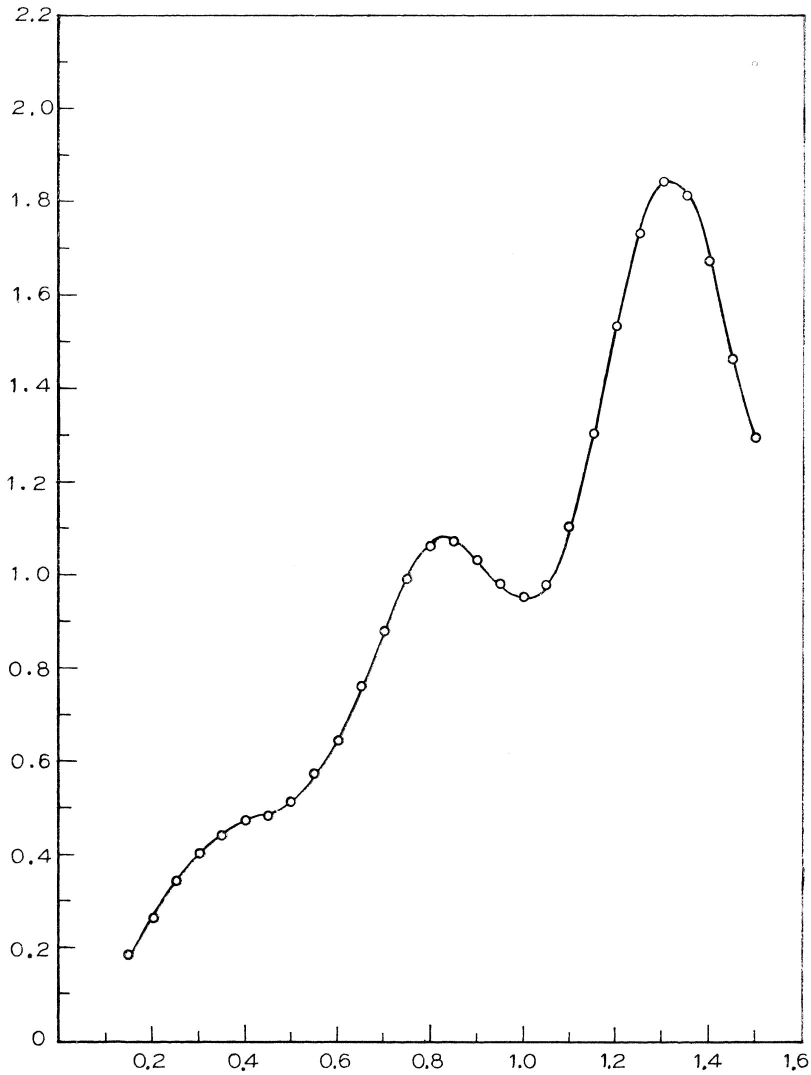

Fig. 2. Curve for normal flux.

$f_2(x) = x - x^3/3!$ in $0 < x < 2$. Knowing the nature of the function in one interval here, we calculate its nature in the other intervals.

After performing the numerical integration of equation (2.10) we have a set of values of a_n. Taking the first 20 terms of a_n we calculate the nature of the function in the other intervals

i) $\sum a_n \sin nx$ in $2 < x < \pi$,

ii) $\sum n a_n \sin nx$ in $0 < x < 2$,

and the corresponding shape of the curve for the temperature distribution and the normal flux given in Fig. 1 and Fig. 2.

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Souhrn

POZNÁMKA O USTÁLENÉM TEPELNÉM TOKU V POLONEKONEČNÉM PÁSU

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Autor vyšetřuje existenci řešení dvoudimenzionální rovnice pro vedení tepla se smíšenými okrajovými podmínkami na polonekonečném pásu.

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