Aplikace matematiky

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Aplikace matematiky, Vol. 27 (1982), No. 2, 87–95

Persistent URL: http://dml.cz/dmlcz/103950

ON THE TWO-SIDED QUALITY CONTROL

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(Received August 25, 1979)

1. INTRODUCTION

Statisticians often have to test whether $100(1-\Delta)\%$ of a population have values of an investigated quantity in a prescribed interval $\langle m-\delta, m+\delta\rangle$, where m, $\delta(\delta > 0)$ are fixed real numbers. This two-sided control is often performed by a graphical method, which can be found in [5], pp. 54–57 (cf. also [3]). The aim of this paper is to apply the maximum likelihood principle for the two-sided control. The second part of the paper contains an exact formula for the asymptotic distribution of the test statistic and the third part contains its critical values.

2. ASYMPTOTIC DISTRIBUTION OF THE MAXIMUM LIKELIHOOD STATISTIC

Let us denote

$$\Theta = \left\{\begin{pmatrix}\mu\\ \sigma\end{pmatrix}; \mu \in R, \sigma > 0\right\},$$

where R is the real line, and for $\theta = \begin{pmatrix}\mu\\ \sigma\end{pmatrix} \in \Theta$ put

$$F_{\mu,\sigma}(x) = \int_{-\infty}^{x} f_\theta(z)\,\mathrm{d}z\,, \quad f_\theta(z) = \frac{1}{\sqrt{(2\pi)}\,\sigma}\exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right).$$

Let $\Delta \in (0, 1)$, $\phi = F_{0,1}$ and $\phi(c_\Delta) = 1 - \Delta/2$. If we denote

$$(1) \qquad H_\Delta = \left\{\begin{pmatrix}\mu\\ \sigma\end{pmatrix} \in \Theta;\ \mu + c_\Delta\sigma \leqq m + \delta, \mu - c_\Delta\,\sigma \geqslant m - \delta\right\},$$

then for $(\mu, \sigma)' \in H_\Delta$ (where x' means the transpose of the vector x) we obtain

$$P_{\mu,\sigma}[x \in (m-\delta, m+\delta)] \geqq F_{\mu,\sigma}(\mu + c_\Delta\sigma) - F_{\mu,\sigma}(\mu - c_\Delta\sigma) =$$
$$= 2\phi(c_\Delta) - 1 = 1 - \Delta\,,$$

and the population has the mentioned property. Now we describe the maximum likelihood statistic for testing the hypothesis H_Δ against the alternative $\Theta - H_\Delta$.

Let us put for $\varphi \subset \Theta$

$$L(x^{(n)}, \varphi) = \sup_{\theta \in \varphi} \prod_{k=1}^{n} f_\theta(x_k)\,, \quad L(x^{(n)}) = \sup_{\theta \in \Theta} \prod_{k=1}^{n} f_\theta(x_k)\,, \tag{2}$$

where $x^{(n)} = (x_1, \ldots, x_n)$ consists of n independent realizations of the random variable X, and define a mapping $T_n : R^n \to H_\Delta$ as follows. Let us denote

$$\bar{x} = \frac{1}{n} \sum_{k=1}^{n} x_k\,, \quad s^2 = \frac{1}{n} \sum_{k=1}^{n} (x_k - \bar{x})^2$$

and put

$$T_n(x^{(n)}) = (M_n(x^{(n)}), D_n(x^{(n)}))\,, \tag{3}$$

where the mappings M_n, D_n are defined by the following formulas. If $\bar{x} \in (m - \delta, m + \delta)$ we put

(4)
$$M_n(x^{(n)}) = \bar{x}\,, \quad D_n(x^{(n)}) = \begin{cases} s & (\bar{x}, s) \in H_\Delta \\ -(\bar{x} - m + \delta)\, c_\Delta^{-1} & (\bar{x}, s) \notin H_\Delta, \bar{x} \in (m - \delta, m\rangle \\ (m + \delta - \bar{x})\, c_\Delta^{-1} & (\bar{x}, s) \notin H_\Delta, \bar{x} \in \langle m, m + \delta)\,. \end{cases}$$

Further, if $\bar{x} \notin (m - \delta, m + \delta)$, we put

(5) $M_n(x^{(n)}) = m + \delta - c_\Delta\, D_n(x^{(n)})\,,$

$$D_n(x^{(n)}) = \min \{\delta/c_\Delta, c_\Delta(\bar{x} - m - \delta)/2 + [s^2 + (\bar{x} - m - \delta)^2 (1 + c_\Delta^2/4)]^{1/2}\}$$

if $\bar{x} \geqq m + \delta$, and

(6) $M_n(x^{(n)}) = m - \delta + c_\Delta\, D_n(x^{(n)})\,,$

$$D_n(x^{(n)}) = \min \{\delta/c_\Delta, c_\Delta(m - \delta - \bar{x})/2 + [s^2 + (\bar{x} - m + \delta)^2 (1 + c_\Delta^2)/4]^{1/2}\}$$

if $\bar{x} \leqq m - \delta$.

Theorem 1. (i) *If we denote* $f_\theta^{(n)}(x^{(n)}) = \prod_{k=1}^{n} f_\theta(x_k)$, *then*

$$f_{T_n}^{(n)}(x^{(n)}) = L(x^{(n)}, H_\Delta)\,, \tag{7}$$

where $T_n = T_n(x^{(n)})$.

(ii) *If* $t > 0$, *then for every* $\theta \in H_\Delta$

$$\lim_{n \to \infty} P_\theta^{(n)} \left[-2 \ln \frac{L(x^{(n)}, H_\Delta)}{L(x^{(n)})} \geqq t \right] \leqq 1 - F_\Delta(t)\,, \tag{8}$$

where $F_\Delta(t) = 0$ *if* $t < 0$. If $t \geqq 0$, *then*

$$(9) \qquad F_\Delta(t) = [2^{-1} - (1/\pi)\arctan(2^{1/2}/c_\Delta)] + 2^{-1} F_1(t) + \\ + [2^{-1} - (1/\pi)\arctan(c_\Delta/2^{1/2})] F_2(t),$$

where F_j *is the chi-square distribution function on* j *degrees of freedom and the function* arctan *takes its values in the interval* $(-\pi/2, \pi/2)$. *If* $\theta' = (m, \delta/c_\Delta)$, *then* (8) *holds with the equality* sign.

Proof. First we prove the first part of the assertion. Since ns^2 is chi-square distributed, we may assume that $s > 0$.

If $(\bar{x}, s) \in H_\Delta$, then (7) holds (cf. [4], p. 504). Let $\bar{x} \in (m - \delta, m + \delta)$ and $(\bar{x}, s) \notin \notin H_\Delta$. If we put

$$\lambda_{\mu,\sigma}(x^{(n)}) = \ln f^{(n)}_{(\mu,\sigma)'}(x^{(n)}),$$

then

$$(10) \qquad \frac{\partial \lambda_{\mu,\sigma}}{\partial \mu} = n\sigma^{-2}(\bar{x} - \mu), \quad \frac{\partial \lambda_{\mu,\sigma}}{\partial \sigma} = -n\sigma^{-1} + \sum_{k=1}^{n}(x_k - \mu)^2 \sigma^{-3},$$

which means that

$$(11) \qquad \lambda_{\mu,\sigma} \leqq \lambda_{\bar{x},\sigma}$$

for every $\mu \in R$, $\sigma > 0$. Further, if we denote $g(\alpha) = \lambda_{\bar{x},\alpha s}$, we see that the function g is increasing on $(0, 1\rangle$. Hence if $\bar{x} \in (m - \delta, m\rangle$, the relations

$$s > c_\Delta^{-1}(\bar{x} - (m - \delta)) \geqq \sigma \quad \text{if} \quad (\bar{x}, \sigma)' \in H_\Delta, \\ (\bar{x}, c_\Delta^{-1}(\bar{x} - (m - \delta)) \in H_\Delta$$

together with (11) and (4) imply (7). The case $\bar{x} \in \langle m, m + \delta)$ can be treated similarly.

Now we assume that $\bar{x} \geqq m + \delta$. Making use of (10) we obtain that

$$(12) \qquad \ln L(x^{(n)}, H_\Delta) = \sup\{\lambda_{m+\delta-c_\Delta\sigma,\sigma}(x^{(n)});\ \sigma \in (0, \delta c_\Delta^{-1}\rangle\}.$$

Denoting $\tilde{\lambda}_\sigma = \lambda_{m+\delta-c_\Delta\sigma,\sigma}(x^{(n)})$ we see that

$$\frac{d\tilde{\lambda}_\sigma}{d\sigma} = \frac{n}{\sigma}[-1 + \sigma^{-2}(s^2 + (\bar{x} - (m + \delta))^2) + \sigma^{-1} c_\Delta(\bar{x} - m + \delta))]$$

and the equation $d\tilde{\lambda}\sigma/d\sigma = 0$ has a unique positive solution

$$\sigma_1 = c_\Delta(\bar{x} - (m + \delta))/2 + \varepsilon_{s,\bar{x}},$$

where

$$\varepsilon_{s,\bar{x}} = [s^2 + (\bar{x} - (m + \delta))^2 (1 + c_\Delta^2/4)]^{1/2}.$$

Since the function $\tilde{\lambda}_\sigma$ is increasing on $(0, \sigma_1\rangle$ and reaches its maximum in the right end-point of this interval, taking into account both (12) and (5) we see that (7) holds. The case $\bar{x} \leqq m - \delta$ can be treated similarly.

We begin the second part of the proof with the definition of the approximability (cf. also [1], [6]). A set $\varphi \subset R^m$ is said to be approximable at a point $\theta \in \bar{\varphi}$ by a cone $C \subset R^m$, if

$$\sup \{\varrho(x, C + \theta);\ x \in \varphi,\ \|x - \theta\| \leqq a_n\} = o(a_n)\,,$$
$$\sup \{\varrho(y + \theta, \varphi);\ y \in C,\ \|y\| \leqq a_n\} = o(a_n)$$

for every sequence $\{a_n\}$ of positive numbers which tend to zero. By a cone we understand any closed convex set $C \subset R^m$ satisfying the relation $y \in C, \alpha > 0 \Rightarrow \alpha y \in C$, and

$$\varrho(z, D) = \inf \{\|z - d\|;\ d \in D\} \tag{13}$$

is the usual distance of a point z from a set D. To prove the second part of the theorem we shall need a version of the Chernoff theorem. Before stating it we introduce regularity conditions of the Rao-Cramer type (cf. also [1], [4] and [7]). We assume that a class of probabilities $\mathscr{P} = \{P_\theta;\ \theta \in \Theta\}$, where $\Theta \subset R^m$ is an open set, is defined on $(X, \mathscr{S})$ by densities $f_\theta(x) = \mathrm{d}P_\theta(x)/\mathrm{d}\mu$ which for every $\theta \in \Theta$ satisfy

(C1) $f_\theta(x)$ is positive on $X \times \Theta$ and has all partial derivatives of the third order in θ.

(C2) There are a P_θ-integrable non-negative function G and a neighbourhood $U \subset \Theta$ of the point θ such that

$$\sup_{i,j,k} \sup_{\theta \in U} \left| \frac{\partial^3}{\partial\theta_i\, \partial\theta_j\, \partial\theta_k} \ln f_\theta(x) \right| \leqq G(x)$$

for every $x \in X$.

(C3) The coordinates of the vector $(\partial \ln f_\theta(x)/\partial\theta_i)_{i=1,\ldots,m}$ belong to $L_2(P_\theta)$ and its covariance matrix $J(\theta)$ is strictly positive definite.

(C4) The identities

$$\int \frac{\partial}{\partial\theta_i} f_\theta(x)\, \mathrm{d}\mu(x) = 0\,, \quad \int \frac{\partial^2}{\partial\theta_i\, \partial\theta_j} f_\theta(x)\, \mathrm{d}\mu(x) = 0$$

hold for $i, j = 1, \ldots, m$.

If we denote by $x^{(n)} = (x_1, \ldots, x_n) \in X^n$ independent realizations of the random variable X, then under the preceding regularity conditions the following assertion holds.

Theorem 2. *Let ω, τ be subsets of Θ such that*

(i) $\theta_0 \in \omega \cap \tau$.

(ii) *If* $\varphi \in \{\omega, \tau\}$, *then there is a sequence* $\{\hat{\theta}_n^{(\varphi)}\}_{n=1}^{\infty}$ *of measurable mappings* $\hat{\theta}_n^{(\varphi)} : X^n \to \varphi$ *such that* $\hat{\theta}_n^{(\varphi)} \to \theta_0$ *in the probability* P_{θ_0} *and (cf. (2))*

$$\lim_{n\to\infty} P_{\theta_0}[L(x^{(n)}, \varphi) = L(x^{(n)}, \hat{\theta}_n^{(\varphi)})] = 1 .$$

If ω, τ *are approximable at* θ_0 *by cones* C_ω, C_τ, *then*

$$\mathscr{L}\left[-2 \ln \frac{L(\cdot, \omega)}{L(\cdot, \tau)} \,\Big|\, P_{\theta_0}\right] \to \mathscr{L}[g \mid N(0, J^{-1}(\Theta_0))] , \tag{14}$$

$$g(z) = \inf_{\theta \in C_\omega} (\theta - z)' J(\theta_0)(\theta - z) - \inf_{\theta \in C_\tau} (\theta - z)' J(\theta_0)(\theta - z) ,$$

where the symbol $\mathscr{L}(Z \mid P)$ *denotes the distribution function of the of the random variable* Z *under the probability* P, $\to$ *denotes the usual weak convergence of probability distributions and* $N(0, J^{-1})$ *is the normal distribution with zero mean and covariance matrix* J^{-1}.

We remark that in contradistinction to [1] and [2], p. 20 we have omitted the condition of the disjointness of the cones C_ω, C_τ. The proof of the preceding theorem can be performed similarly as proofs in [1] or [6].

Now we can return to our hypothesis H_Δ (cf. (1)). Since $T_n \to \Theta$ in the probability P_θ for each $\theta \in H_\Delta$ and the regularity conditions (C1)–(C4) are fulfilled, we may use the preceding theorem.

If θ_0 is an inner point of H_Δ, then according to (4)

$$L(x^{(n)}, H_\Delta)/L(x^{(n)}) \to 1 \quad \text{in the probability} \quad P_{\theta_0}$$

and (8) holds.

Let θ_0 be a boundary point of H_Δ. If $\theta_0' = (\mu_0, \sigma_0)$, where $\mu_0 \in (m - \delta, m)$, then the set H_Δ can be approximated at θ_0 by the cone

$$K = \left\{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} ; \quad y_1 - c_\Delta y_2 \geqq 0\right\}$$

and according to Theorem 2,

$$\mathscr{L}[-2 \ln L(x^{(n)}, H_\Delta)/L(x^{(n)}) \mid P_{\theta_0}] \to \mathscr{L}[\inf_{\theta \in J^{1/2}K} \|\theta - z\|^2 \,|N(0, I)] . \tag{15}$$

where $J = J(\theta_0)$ and I is the unit matrix. Since $J^{1/2} K = \{z \in R^2; c'z \leqq 0\}$ where c is a non-zero vector, making use of the mapping $x \to -x$ and the notation $D = J^{1/2}K$, $g_1(z) = \varrho^2(z, D)$ (cf. (13)), we obtain for every $t > 0$

$$P[g_1(z) \leqq t \mid N(0, I)] = 1/2 + (1/2)\, P[g_1(z) \leqq t, z \notin D \mid N(0, I)] . \tag{16}$$

If we denote

$$\pi_D(z) = z - c'z\|c\|^{-2}\, c , \tag{17}$$

then $\pi_D(z)$ is the projection of z into D and making use of the relation

$$\mathscr{L}[\|\pi_D(z) - z\|^2 \mid N(0, I)] = \mathscr{L}[x^2 \mid N(0, (\|c\|^{-1} c)' I \|c\|^{-1} c)]$$

we see that the right hand side in (16) is of the form

$$1/2 + (1/2) F_1(t) .$$

But $\arctan \gamma + \arctan \gamma^{-1} = \pi/2$ implies

$$F_\Delta(t) - (1/2 + (1/2) F_1(t)) \leqq 0 ,$$

which means that (8) holds. Since the case $\mu_0 \in (m, m + \delta)$ can be treated similarly, we assume that $\theta_0 = (m, \delta/c_\Delta)$. It is easy to see that H_Δ can be approximated at θ_0 by the cone

$$K = \{y \in R^2;\ y_1 \in \langle c_\Delta y_2, -c_\Delta y_2 \rangle\} ,$$

and Theorem 2 implies that

$$\mathscr{L}[-2 \ln L(x^{(n)}, H_\Delta)/L(x^{(n)}) \mid P_{\theta_0}] \to \mathscr{L}[\varrho^2(z, J^{1/2}K) \mid N(0, I)] ,$$

where

$$J^{1/2}K = \{x \in R^2;\ x_2 \leqq \gamma x_1,\ x_2 \leqq -\gamma x_1\} , \quad \gamma = \sqrt{(2)}/c_\Delta .$$

Hence to complete the proof of Theorem 1, we have to prove

Lemma 1. *If* $D = \{y \in R^2;\ y_2 \leqq \gamma y_1,\ y_2 \leqq -\gamma y_1\}$ *with* $\gamma > 0$, *then for every* $t \in R$

(18) $$P[\varrho^2(z, D) \leqq t \mid N(0, I)] = F(t) ,$$

where the function F *is defined by* (9) *with* $\sqrt{(2)}/c_\Delta$ *replaced by* γ.

Proof. Since $N(0, I)$ is a symmetric distribution, we have

(19) $$P[\varrho^2(z, D) \leqq t \mid N(0, I)] = 2P[z_1 < 0,\ \|z - \pi_D(z)\|^2 \leqq t \mid N(0, I)] ,$$

where $\pi_D(z)$ is the projection of z on the convex set D.

Let $z_1 < 0$, $z_2 \in (\gamma z_1, -\gamma^{-1} z_1)$ (cf. Fig. 1). Then $\pi_D(z)$ is the projection on the cone

$$C = \{x \in R^2;\ c'x = 0\} ,$$

where $c' = (\gamma, -1)$. Making use of the transformation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\gamma^{-1} z_1 - z_2 \\ \gamma z_1 - z_2 \end{pmatrix}$$

we obtain

(20) $$P[z_1 < 0, \gamma z_1 < z_2 < -\gamma^{-1} z_1,\ \|z - \pi_D(z)\|^2 \leqq t \mid N(0, I)] = \\ = (1/2) P[y_2 < 0,\ y_2^2 \leqq t(1 + \gamma^2) \mid N(0, \gamma^2 + 1)] = (1/4) F_1(t) .$$

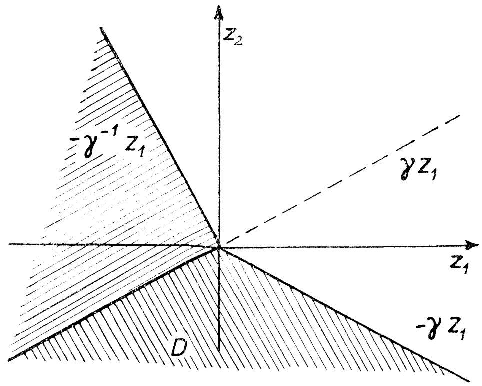

Fig. 1.

Let $z_1 < 0$, $z_2 < \gamma z_1$. Then $\pi_D(z) = z$ and the substitution

(21) $$z_1 = r \cos \psi\,, \quad z_2 = r \sin \psi$$

yields

(22) $$P[z_1 < 0, z_2 < \gamma z_1, \|z - \pi_D(z)\|^2 \leqq t \mid N(0, I)] = 1/4 - (1/2\pi) \arctan \gamma\,.$$

Finally, let $z_1 < 0$, $z_2 > -\gamma^{-1} z_1$. Then $\pi_D(z) = 0$ and the substitution (21) yields

(23) $$P[z_1 < 0, z_2 > -\gamma^{-1} z_1, \|z - \pi_D(z)\|^2 \leqq t \mid N(0, I)] =$$

$$= (2\pi)^{-1} v_L[\psi \in (\pi/2, 3\pi/2);\ \tan \psi < -\gamma^{-1}] \int_0^{\sqrt{t}} \exp(-r^2/2)\, r\, \mathrm{d}r =$$

$$= 2^{-1} \left[[1/2 - \frac{1}{\pi} \arctan \gamma^{-1} \right] F_2(t)\,,$$

where v_L is the Lebesgue measure on the line. Combining relations (19)–(23) we see that (18) holds.

3. REMARKS AND TABLES.

If we denote for $\Delta \in (0,1)$

$$t_n^{(\Delta)}(x_1, \ldots, x_n) = n \left(2 \ln \frac{D_n}{s} - 1 \right) + \sum_{k=1}^{n} \frac{(x_k - M_n)^2}{D_n^2}\,,$$

where the quantities $D_n = D_n(x_1, \ldots, x_n)$, $M_n = M_n(x_1, \ldots, x_n)$ are defined by the formulas (4)–(6), then the inequality (8) implies

$$\sup_{\theta \in H_\Delta} \lim_{n \to \infty} P^{(n)}[t_n^{(\Delta)} \geqq t] = 1 - F_\Delta(t)\,,$$

whenever $t > 0$. Obviously, if we find a suitable constant $t(\Delta, \alpha)$, then the tests

$$\Psi_n(x_1, \ldots, x_n) = \begin{cases} \text{reject } H_\Delta & \text{if } \; t_n^{(\Delta)}(x_1, \ldots, x_n) \geqq t(\Delta, \alpha) \\ \text{accept } H_\Delta & \text{if } \; t_n^{(\Delta)}(x_1, \ldots, x_n) < t(\Delta, \alpha) \end{cases}$$

will have the asymptotic size α. The values of $t(\Delta, \alpha)$ for various Δ, α are given in the following Table 1.

Table 1.

Δ	α	$t(\Delta, \alpha)$
	0·05	4·11833
0·1	0·02	5·84051
	0·01	7·16359
	0·05	3·98800
0·05	0·02	5·69907
	0·01	7·01569
	0·05	3·91063
0·03	0·02	5·61418
	0·01	6·92601
	0·05	3·85830
0·02	0·02	5·55679
	0·01	6·86568
	0·05	3·78258
0·01	0·02	5·47337
	0·01	6·77779

We remark that for every $t > 0$

$$\inf_{\theta \in \Theta - H_\Delta} \lim_{n \to \infty} P^{(n)}[t_n^{(\Delta)} \geqq t] = 1 ,$$

which means that the test Ψ_n not only have the asynptotic size α but are consistent as well.

Acknowledgement. I would like to thank Dr. Ivica Mišíková for computing the values of $t(\Delta, \alpha)$ in Table 1.

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Súhrn

O DVOJSTRANNEJ KONTROLE KVALITY

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Nech náhodná premenná X má normálne rozdelenie $N(\mu, \sigma^2)$. V článku sú odvodené explicitné formuly pre odhad maximálnej vierohodnosti pre parametre μ,σ za predpokladu platnosti hypotézy $\mu + c\sigma \leqq m + \delta$, $\mu - c\sigma \geqq m - \delta$, kde c, m, δ sú hocijaké pevne zvolené čísla. Táto hypotéza je testovaná pomocou pomeru vierohodností, uvádzame jeho asymptotické rozdelenie a niektoré jeho kvantily.

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