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ON THE RESTRICTED RANGE IN THE SAMPLES FROM THE GAMMA POPULATION

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1. INTRODUCTION

The range plays an important role as is well known in the statistical theory. The multiple range test is a useful test in the analysis of variance. In industrial applications, control charts are based on this statistic. Duncan [1] and Levy [5] have made use of the multiple range tests and similarly Nelson [6] uses the range for testing homogeneity of variances. The case of the range from the non-normal populations is dealt with by Singh [8]. McDonald [7] considers the range in samples from the uniform populations. Lawless [2] deals with the problem of prediction in the exponential population using order statistics, which for special values reduces to the range consideration. Lingappaiah [3], [4], deals with the range in the exponential and gamma populations. Here we consider the samples from the gamma population which are censored both above and below. Distribution of the range in such censored samples called the restricted range is put in a closed form. For few small values of n, r and s. the actual form of the distribution is given. This form of the distribution of this restricted range can be compared with the distribution of the range in the complete samples which is given in Lingappaiah [4].

2(a). DISTRIBUTION OF THE RESTRICTED RANGE

Consider a sample of size n drawn from a gamma population

$$f(x) = e^{-\theta x}\,\theta(\theta x)^{\alpha-1}/\Gamma(\alpha)\,, \quad x > 0\,, \quad \theta > 0\,, \quad \alpha = 1, 2, \ldots \tag{1}$$

Let r observations below and s observations above among these n observations be censored. Then the joint density of the available observations $u_{r+1}, u_{r+2}, \ldots, u_{n-s}$, where $u_i = x_{(i)n}$, the i-th order statistic in n observations can be put in the form

$$(2) \qquad f(U) = c\Big(1 - \sum_{k=0}^{\alpha-1} e^{-\theta u_{r+1}}(\theta u_{r+1})^k/k!\Big)^r \cdot \Big(\prod_{i=r+1}^{n-s} [e^{-\theta u_i}\theta \cdot (\theta u_i)^{\alpha-1}/\Gamma(\alpha)]\Big) \cdot \Big(\sum_{k=0}^{\alpha-1} e^{-\theta u_{n-s}}(\theta u_{n-s})^k/k!\Big)^s ,$$

where $U = (u_{r+1}, \ldots, u_{n-s})$, $c = n!/r!s!$ and $F(x)$ from (1) is $1 - \sum_{k=0}^{\alpha-1} e^{-\theta x}(\theta x)^k/k!$. Now using Lingappaiah [4], we can write (2) in the following form.

(3)

$$f(U) = c \sum_{t=0}^{r} \binom{r}{t}(-1)^t e^{-t\theta u_{r+1}} \sum_{p=0}^{t(\alpha-1)} a_p(\alpha, t)\,(\theta u_{r+1})^p \cdot \prod_{i=r+1}^{n-s} [e^{-\theta u_i}\theta \,.\, (\theta u_i)^{\alpha-1}/\Gamma(\alpha)] \,. \sum_{q=0}^{s(\alpha-1)} b_q(\alpha, s)\,(\theta u_{n-s})^q e^{-\theta s u_{n-s}} ,$$

where $a_p(\alpha, t)$ is the coefficient of $(\theta u)^p$ in the expansion of $\left(\sum_{k=0}^{\alpha-1} \frac{(\theta u)^k}{k!}\right)^t$ and similarly $b_q(\alpha, s)$ is the coefficient of $(\theta w)^q$ in the expansion of $\left(\sum_{k=0}^{\alpha-1} \frac{(\theta w)^k}{k!}\right)^s$, where $u_{r+1} = u$ and $u_{n-s} = w$.

Now integrating out $u_{r+2}, \ldots, u_{n-s-1}$ (and noticing that $u \leqq u_{r+2} \leqq \ldots \ldots \leqq u_{n-s-1} \leqq w$) in (3), we get the joint density of u and w as

$$(4) \qquad f(u, w) = (B)\,\Big[\sum_{i=0}^{a} Aa_i\, e^{-(a-i)\theta u}\, e^{-i\theta w}\Big] ,$$

where $a = n - r - s - 2$, $u = {}_{r+1}$, $w = u_{n-s}$ and

$$(5) \qquad (B) = c \,.\, \Big[\sum_t \sum_p \sum_q \cdot \binom{r}{t} (-1)^t\, a_p(\alpha, t)\, b_q(\alpha, s) \,.\, \theta^2 \,.\, e^{-\theta u(t+1)}\, e^{-\theta w(s+1)}(\theta u)^{p+\alpha-1}\, (\theta w)^{q+\alpha-1}/\Gamma^2(\alpha)\Big] .$$

In (4), the terms A_{ai} satisfy the following identies:

$$(6a) \qquad A_{a0} = A_{a-1,0}d_a g_a + b_a g_a\Big[\sum_{i=1}^{a-1} A'_{a-1,i}c_a(i+1)\Big]$$

$$(6b) \qquad A_{a1} = A'_{a-1,0}d_a h_a = A'_{a1} h_a$$

$$(6c) \qquad A_{aj} = -A'_{a-1,j-1} b_a c_a(j)\, h_a = A'_{aj} h_a \,, \quad j = 2, 3, \ldots, a \,,$$

where $g_i = (\theta u)^{k_i}/k_i!$, $h_i = (\theta w)^{k_i}/k_i!$, $d_i = \sum_{k_i=0}^{\alpha-1}$,

$$b_{i+1} = \sum_{k_{i+1}=0}^{k_i+\alpha-1} \binom{k_i + \alpha - 1}{k_i}, \quad c_i(j) = j^{k_i}/j^{k_{i-1}+\alpha} \quad i, j = 2, 3, \ldots, a;$$

for example, $c_4(3) = 3^{k_4}/3^{k_3+\alpha}$, $A'_{i0} = A_{i0}$, $i = 0, 1, 2, \ldots, a$, $A_{00} = 1$. From (6c), by successive substitutions, we get

$$A'_{aj} = (-1)^j A'_{a-j,0}\, d_{a-j+1} \prod_{r=0}^{j-2} b_{a-r} c_{a-r}(j-r)\,. \tag{7}$$

(6) and (7) can be used to find all the coefficients A_{ai}, $i = 0, 1, 2, \ldots, a$, recursively from A_{10}, A_{20} and so on. We give below a few values of A_{ai} for $a = 1, 2, 3$.

$a = 1, (n - r - s = 3)$:

$$\begin{aligned} A_{10} &= A_{00} d_1 g_1 = d_1 g_1 \\ A_{11} &= -A'_{00} d_1 h_1 = A'_{11} h_1 \end{aligned} \tag{8}$$

$a = 2, (n - r - s = 4)$:

$$\begin{aligned} A_{20} &= A_{10} d_2 g_2 \times b_2 g_2 A'_{11} c_2(2) = d_1 d_2 g_1 g_2 - d_1 g_2 b_2 c_2(2) \\ A_{21} &= -A'_{10} d_2 h_2 = -d_1 d_2 g_1 h_2 = A'_{21} h_2 \\ A_{22} &= -A'_{11} b_2 c_2(2)\, h_2 = d_1 b_2 c_2(2)\, h_2 = A'_{22} h_2 \end{aligned} \tag{9}$$

$a = 3, (n - r - s = 5)$:

$$\begin{aligned} A_{30} &= A_{20} d_3 g_3 + b_3 g_3 [A'_{21} c_3(2) + A'_{22} c_3(3)] \\ A_{31} &= -A'_{20} d_3 h_3 = A'_{31} h_3 \\ A_{32} &= -A'_{21} b_3 c_3(2)\, h_3 = A'_{32} h_3 \\ A_{33} &= -A'_{22} b_3 c_3(3)\, h_3 = A'_{33} h_3 \end{aligned} \tag{10}$$

Now (4) can be written as

$$f(u, w) = (B)\left[A_{a0}\, e^{-a\theta u} + \sum_{i=1}^{a} e^{-(a-i)\theta u} A'_{ai}\, e^{-i\theta w} h_a\right]; \tag{11}$$

here A_{a0} is free of h_i's and contains only g_i's, and similarly A'_{ai}'s. Setting $R = w - u$ in (11), we have

$$\begin{aligned} f(u, R) = (Q)\, &[e^{-\theta(R+u)(s+1)} [\theta(R+u)]^{q+\alpha-1}]\,. \\ &.\left[A_{a0}\, e^{-a\theta u} + \sum_{i=1}^{a} A'_{ai}\, e^{-(a-1)\theta u} \,.\, [\theta(R+u)]^{k_a}/k_a!\right], \end{aligned} \tag{12}$$

where $B = (Q)\,(e^{-\theta w(s+1)}(\theta w)^{q+\alpha-1})$.

Now from (12), we can integrate out u and get the distribution of R. However, integrating out u in (12) depends on A'_{ai}'s, $i = 0, 1, 2, \ldots,$ a, which include g_i's. For example, if $a = 1$ $(n - r - s = 3)$, $n = 5$, $r = s = 1$, $\alpha = 2$, then using (3) and (8), we have (12) in the form

$$\begin{aligned} f(u, R) = (5!) &\sum_{t=0}^{1} \sum_{p=0}^{1} \sum_{q=0}^{1} (-1)^t \binom{1}{t} a_p(2, t)\, b_q(2, s)\,. \\ &.\, e^{-\theta u(t+1)}\, e^{-2\theta(R+u)} \theta^2 \,.\, (\theta u)^{p+1} [\theta(R+u)]^{q+1}\,. \\ &.\left[\sum_{k_1=0}^{1} \left(\frac{e^{-\theta u}(\theta u)^{k_1}}{k_1!} - \frac{e^{-\theta w}(\theta w)^{k_1}}{k_1!}\right)\right]. \end{aligned} \tag{13}$$

For $r = s = 0$, that is, for the complete sample case, the distribution of R is given in Lingappaiah [4].

2(b). ALTERNATE FORM

Now, in the place of (12), we give another form which can also be used easily. Using (7), we can write (4) or (11) as

$$(14) \qquad f(u, w) = (B)\left[A_{a0}\, e^{-a\theta u} + \sum_{i=1}^{a} A_{a-i,0}(-1)^i D_{ai}\, e^{-(a-i)\theta u}\, e^{-i\theta w}\right],$$

where $D_{ai} = d_{a-i+1} \prod_{r=0}^{i-2} b_{a-r} c_{a-r}(i-r)$. But now, A_{r0} has its own representation. That is,

$$(15) \quad A_{r0} = g_r \phi_{r-1}(g_1, \ldots, g_{r-1}) = g_r \sum_{i=0}^{r-1} \sum z_i \prod_{j=1}^{i} (g_{l_j}) (-1)^{(r-1)-i},$$

where $\sum$ is taken over all permutations of g_{l_j}'s, where $l_j = 1, 2, \ldots, r-1$. z_i's depend upon the permutations $\sum$ and these include the operators d_i, b_i and the quantities $c_i(j)$. For example,

$$(15\text{a}) \qquad \begin{aligned} A_{10} &= g_1 d_1\,, \\ A_{20} &= d_1 d_2 g_1 g_2 - d_1 g_2 b_2 c_2(2) = g_2\, \phi_1(g_1)\,, \\ A_{30} &= A_{20} d_3 g_3 + g_3 b_3[-A'_{10} d_2 c_3(2) + A'_{00} d_1 c_3(3) \,.\, b_2 c_2(2)] = \\ &= g_3[d_1 d_2 d_3 g_1 g_2 - d_1 d_3 b_2 c_2(2)\, g_2 - d_1 d_2 b_3 c_3(2)\, g_1 + \\ &\quad + d_1 b_2 b_3 c_2(2)\, c_3(3)] = g_3\, \phi_2(g_1, g_2). \end{aligned}$$

Similarly, using (7) in (6a), we have

$$(16\text{a}) \qquad \begin{aligned} A_{40} = A_{30} d_4 g_4 &+ b_4 g_4[-A'_{20} d_3 c_4(2) + A_{10} d_2 c_3(3)\, b_3 c_3(2) - \\ &- A_{00} d_1 c_4(4) \,.\, b_2 c_2(2)\, b_3 c_3(3)]\,. \end{aligned}$$

Using A_{10}, A_{20} and A_{30}, we have

$$(16\text{b}) \qquad \begin{aligned} A_{40} &= g_4[(d_1 d_2 d_3 d_4)(g_1 g_2 g_3) - (d_1 d_3 d_4)[b_2 c_2(2)]\, g_2 g_3 - \\ &\quad - (d_1 d_2 d_4[b_3 c_3(2)]\, g_1 g_3 - (d_1 d_2 d_3[b_4 c_4(2)]\, g_1 g_2 + \\ &\quad + (d_1 d_4)[(b_2 b_3)\, c_2(2)\, c_3(3)](g_3) + (d_1 d_3)[(b_2 b_4)\, c_2(2)\, c_4(2)](g_2) + \\ &\quad + (d_1 d_2)[(b_3 b_4)\, c_3(2)\, c_4(3)](g_1) - d_1(b_2 b_3 b_4)[c_2(2)\, c_3(3)\, c_4(4)]] = \\ &= g_4\, \phi_3(g_1, g_2, g_3). \end{aligned}$$

From (15) and (16a) it is seen that

$$(17) \qquad A_{r0} = g_r \sum_{i=0}^{r-1} z'_i\, \phi_i(g_1, g_2, \ldots, g_i) \quad \text{with} \quad \phi_0 = 1\,,$$

where again z'_i's include the operators d_i, b_i and the quantities $c_i(j)$'s.

Using (17), we can write (4) using (14) as

$$(18) \qquad f(u, R) = (Q)\left[e^{-a\theta u} g_a\, \phi_{a-1}(g_1, g_2, \ldots, g_{a-1}) + \right.$$
$$+ \sum_{i=1}^{a} (-1)^i\, \phi_{a-i-1}(g_1, g_2, \ldots, g_{a-i-1})\, g_{a-i} D_{ai} \,.$$
$$. \, e^{-(a-i)\theta u}\, e^{-i\theta(R+u)} \,.\, \left.[\theta(R+u)]^{k_a}/k_a!\right] .$$
$$. \left[[\theta(R+u)]^{q+\alpha-1} \,.\, e^{-\theta(R+u)(s+1)}\right] .$$

From (18), we can get the dustribution of R by integrating out u depending on φ_i's.

3. SPECIAL CASE ($\alpha = 1$)

Since $k_i = 0$, $i = 1, 2, \ldots, a$, we now have from (7) with $g_i = 1$, $i = 1, 2, \ldots, a$

$$(19) \qquad A'_{aj} = (-1)^j A'_{a-j,0}/j! \,.$$

Now using (19) in (6a), we get

(20)

$$A_{a0} = A_{a-1,0} - \frac{A_{a-2,0}}{2!} + \frac{A_{a-3,0}}{3!} + \ldots + \frac{(-1)^{j-1} A_{a-j,0}}{j!} + \ldots + (-1)^{a-1}/a! \,.$$

From (20), one recursively gets

$$(21) \qquad A'_{aj} = 1/(a-j)!\, j! \,, \quad j = 0, 1, 2, \ldots, a \,.$$

Hence (4) reduces to

$$(22) \qquad f(u, w) = c \sum_{i=0}^{a} (-1)^i\, \theta^2 e^{-\theta u(a-i+1)} (1 - e^{-\theta u})^r\, e^{-b\theta w}/(a-i)!\, i! \,,$$

where $b = s + i + 1$.

Hence we have the *pdf* of R as

$$(23) \qquad f(R) = \sum_{i=0}^{a} (-1)^i\, \theta\, e^{-b\theta R}\, \Gamma(n-r)/s!\, i!\, (a-i)! \,.$$

If $s = r = 0$, (23) reduces to the distribution of R in the complete sample case, which is

$$(24) \qquad f(R) = \sum_{i=0}^{n-2} (-1)^i\, \theta\, e^{-\theta R(i+1)} \binom{n-2}{i} (n-1) \,.$$

In (24), it is to be noted that

$$(25) \qquad \sum_{i=0}^{n-2} (-1)^i \binom{n-2}{i} \frac{1}{(i+1)} = \frac{(n-2)!}{\prod_{i=0}^{n-2} (i+1)} \,,$$

which provides a *pdf* for (24).

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Souhrn

O ROZPĚTÍ V CENSOROVANÝCH VÝBĚRECH Z GAMMA ROZLOŽENÍ

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Mějme dány výběry z gamma rozložení, které jsou censorovány zdola i shora, tj. mezi n pozorováními chybí r pozorování zdola a s pozorování shora. V článku se studuje rozpětí v takto censorovaných výběrech a odvozuje se jeho rozložení; to lze porovnat s předchozími autorovými výsledky pro rozpětí v úplných výběrech.

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