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ON THE SOLUTION OF A GENERALIZED SYSTEM OF VON KÁRMÁN EQUATIONS

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INTRODUCTION

A nonlinear system of equations generalizing von Kármán equations is studied. The system considered is derived in [1] under the assumption of a nonlinear relation between the intensity of stresses and deformations in the constitutive law $\sigma_i/e_i = E(1 - \omega)$ and stands as a model for large deformations of thin plates or shells. In the case $\omega \equiv 0$ this system reduces to the system von of Kármán equations. The function $\omega \equiv \omega(e)$ can also characterize the plasticity properties of the given material but the derived system is a model for large deformations of elastic-plastic plates for simple exterior stresses only (i.e. all exterior stresses arise from zero stresses in a monotonic way). From the numerical point of view the generalized system has been analysed also in [2]. The case $\omega \equiv \omega(x, y)$ has been considered in [8]. Our goal is to prove the existence of a solution and its properties for $\omega \to 0$. We use the technique developed in [3–6] and some results from [7].

1. NOTATION AND FORMULATION OF THE PROBLEM

Let $\Omega \subset R^2$ be a simply connected bounded domain describing the shape of a plate. We assume that the boundary $\partial\Omega$ is piecewise three times continuously differentiable (see [5]). Denote $w_x = \partial w/\partial x$, $w_y = \partial w/\partial y$, $w_{xy} = (w_x)_y$ etc.; $\Delta^2 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy}$; $[w, f] = w_{xx}f_{yy} + w_{yy}f_{xx} - 2w_{xy}f_{xy}$; w_ν stands for the outward normal derivative with respect to $\partial\Omega$. By means of (from the constitutive law) we define the functions a_i $(i = 1, 2, 3)$ in the following way:

$$Q_1 = \frac{2}{h}\int_{-h/2}^{h/2} \omega \, dz\,, \quad Q_2 = \frac{4}{h^2}\int_{-h/2}^{h/2} z\omega \, dz\,, \quad Q_3 = \frac{8}{h^3}\int_{-h/2}^{h/2} z^2\omega \, dz$$

$$a_1 = (1 - \tfrac{1}{2}Q_1)^{-1}\,, \quad a_2 = a_1 Q_2\,, \quad a_3 = \tfrac{3}{4}(2Q_3 + a_1 Q_2^2)\,,$$

where h is the thickness of the plate. Let w be the deflection and F Airy's stress function of the plate. Then a_i are the functions of w_{xx}, w_{xy}, w_{yy}, F_{xx}, F_{xy} and F_{yy}.

We assume a_i to be in the form $a_i \equiv a_i(x, y, w, w_x, w_y, w_{xx}, w_{xy}, w_{yy}; F, F_x, F_y, F_{xx}, F_{xy}, F_{yy}) \equiv a_i(Dw; DF)$. A corresponding system for unknown functions F, w, derived in [1] under the nonlinear constitutive law, is of the form

$$(E_1) \qquad \Delta^2 w - ((F_{xx} + \tfrac{1}{2}F_{yy})\, a_3(Dw; DF))_{xx} - ((F_{yy} + \tfrac{1}{2}F_{xx})\, a_3(Dv; DF))_{yy} -$$

$$- (w_{xy} a_3(Dw; DF))_{xy} + \frac{9}{4Eh}\Big\{((F_{yy}\, a_2(Dw; DF))_{xx} + ((F_{xx}\, a_2(Dw; DF))_{yy} -$$

$$- 2(F_{xy}\, a_2(Dw; DF))_{xy}\Big\} = \frac{9}{Eh^2}[F, w] + \frac{q}{P},$$

$$(E_2) \qquad ((F_{xx} - \tfrac{1}{2}F_{yy})\, a_1(Dw; DF))_{xx} + ((F_{yy} - \tfrac{1}{2}F_{xx})\, a_1(Dw; DF))_{yy} +$$

$$+ 3(F_{xy}\, a_1(Dw; DF))_{xy} - \frac{Eh}{4}\Big\{(w_{xx}\, a_2(Dw; DF))_{yy} + (w_{yy}\, a_2(Dw; DF))_{xx} -$$

$$- 2(w_{xy}\, a_2(Dw; DF))_{xy}\Big\} = -\frac{E}{2}[w, w]$$

for $(x, y) \in \Omega$, where E is the modulus of elasticity, $P = \frac{1}{9}Eh^3$ and q is the density of the perpendicular load.

Together with (E_1), (E_2) we consider the following boundary conditions

$$(B) \qquad w = w_\nu = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad F = F_0\,, \quad F_\nu = F_{0,\nu} \quad \text{on} \quad \partial\Omega\,,$$

where $F_0 \in C^2(\bar{\Omega})$ is a given function.

Let $\zeta : \bar{\Omega} \to \langle 0, 1 \rangle$ be an arbitrary function with the property

$$(P) \qquad \zeta \in C^2(\bar{\Omega}) \quad \text{and} \quad \zeta = 1\,, \quad \zeta_\nu = 0 \quad \text{on} \quad \partial\Omega\,.$$

We denote $f_0 = \zeta F_0$ and we consider F in the form $F = f + f_0$, where $f = f_\nu = 0$ on $\partial\bar{\Omega}$.

For the sake of simplicity we denote $(u, v)_W = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy})\,.$ $.\,\mathrm{d}x\,\mathrm{d}y$, $(u, v) = \int_\Omega uv\,\mathrm{d}x\,\mathrm{d}y$ and $B(u; v, w) = \int_\Omega (u_{xy}v_x w_y + u_{xy}v_y w_x - u_{yy}v_x w_x -$ $= u_{xx}v_y w_y)\,\mathrm{d}x\,\mathrm{d}y$ for $u, v, w \in W_2^2(\Omega)$ (Sobolev space).

Definition. *A couple $\{w, F\}$ is said to be a variational solution of $(E_1$, (E_2), (B), iff w, $F - f_0 \in \mathring{W}_2^2(\Omega)$ and the identities*

$$(1) \qquad ((L_1(w, F), \varphi)) \equiv (w, \varphi)_W - ((w_{xx} + \tfrac{1}{2}w_{yy})\, a_3(Dw; DF), \varphi_{xx}) -$$

$$- ((w_{yy} + \tfrac{1}{2}w_{xx})\, a_3(Dw; DF), \varphi_{yy}) - (w_{xy}\, a_3(Dw; DF), \varphi_{xy}) +$$

$$+ \frac{9}{4Eh}\{(F_{yy}\, a_2(Dw; DF), \varphi_{xx}) + (F_{xx}\, a_2(Dw; DF), \varphi_{yy}) -$$

$$- 2(F_{xy}\, a_2(Dw; FF), \varphi_{xy})\} - \frac{9}{Eh^2} B(w; F, \varphi) = \left(\frac{q}{P}, \varphi\right),$$

$$(2) \qquad ((L_2(w, F), \psi)) \equiv ((F_{xx} - \tfrac{1}{2}F_{yy})\, a_1(Dw; DF), \psi_{xx}) +$$

$$((F_{yy} - \tfrac{1}{2}F_{xx})\, a_1(Dw; DF), \psi_{yy}) + 3(F_{xy}\, a_1(Dw; DF), \psi_{xy}) -$$

$$- \tfrac{1}{4}Eh\{(w_{xx}\, a_2(Dw; DF), \psi_{yy}) + (w_{yy}\, a_2(Dw; DF), \psi_{xx}) -$$

$$- 2(w_{xy}\, a_2(Dw; DF), \psi_{xy}) + \tfrac{1}{2}E\, B(w; w, \psi) = 0$$

hold for all $\varphi, \psi \in \mathring{W}_2^2(\Omega)$.

Using Green's theorem in (1) and (2) we can easily find that a variational solution of (E_1), (E_2). (B) is also a classical solution under the regularity assumptions on w, F and a_i $(i = 1, 2, 3)$.

The expression $B(u; v, w)$ in (1) and (2) is well defined for $u, v, \in W_2^2(\Omega)$ since the inequality

$$(3) \qquad |B(u; v, w)| \leqq \|u\|_{W_2{}^2} \|v\|_{W_4{}^1} \|w\|_{W_4{}^1}$$

holds. Moreover, for $u, v \in W_2^2(\Omega)$ and $w \in \mathring{W}_2^2(\Omega)$ we have

$$(4) \qquad B(w; u, v) = B(v; u, w) = B(v; w, u)$$

(see, e.g., [3]).

2. EXISTENCE OF A SOLUTION

We prove the existence of a variational solution of the problem (E_1), (E_2), (B) using the abstract existence results for the corresponding operator equation $Au = G$. We deduce this equation in the following way: Let us denote $H = \mathring{W}_2^2(\Omega) \times \mathring{W}_2^2(\Omega)$ with the usual norm $\|\cdot\|_H$. Let $u \equiv \{w, f\}$, $v \equiv \{\varphi, \psi\} \in H$. We define the operator $A_\zeta : H \to H^*(H^* \equiv W_2^{-2} \times W_2^{-2})$ by means of the form $\langle A_\zeta u, v\rangle = ((L_1(w, f + f_0), \varphi)) + ((L_2(w, f + f_0), \psi))$ since $f_0 = \zeta F_0$ and ζ is a function with the property (P). In what follows we omit the index ζ in A_ζ. $G \in H^*$ is of the form $\{q/P, 0\}$. Clearly, the solvability of $Au = G$ in H is equivalent to the existence of a variational solution of (E_1), (E_2), (B).

Under certain assumptions on a_i $(i = 1, 2, 3)$ we prove that $A : H \to H^*$ is a continuous, bounded operator with the property S (i.e., $u_n \rightharpoonup u$ (weak convergence) and $\langle Au_n - Au, u_n - u\rangle \to 0$ implies $\|u_n - u\|_H \to 0$). Using the result from [5] (see [3], [6]), under a suitable choice of the function ζ we prove coercivity of the operator $A(A \equiv A_\zeta)$. Then from well known results (see, e.g., [7]) we obtain $A(H) = H^*$, which implies the existence of a variational solution of (E_1), (E_2), (B).

We assume that $a_i(x, y, \zeta, \tau)$ $(i = 1, 2, 3)$ are continuous functions in all variables defined on $\Omega \times R^6 \times R^6$, where the real vectors $\xi, \tau \in R^6$ stand instead of w, f and their derivatives up to the second order. We shall assume that there exist positive constants M_0 and M_i $(i = 1, 2, 3)$ such that

(5) $$a_1(x, y, \xi, \tau) \geqq M_0 ,$$

(6) $$|a_i(x, y, \xi, \tau)| \leqq M_i , \quad i = 1, 2, 3 ,$$

for all $(x, y) \in \Omega$ and $\xi, \tau \in R^6$.

Moreover, we shall assume that the partial derivatives $\partial a_i/\partial \xi_j$ and $\partial a_i/\partial \tau_j$ are continuous on $\Omega \times R^6 \times R^6$ for all $i = 1, 2, 3$ and $|j| \leqq 2$ where j is the multiindex $(j = (j_1, j_2), j_1, j_2 \geqq 0$ and $|j| = j_1 + j_2)$. To prove the property S of the operator A we shall assume that there exist $C_j \geqq 0$ $(|j| \leqq 2)$ and $s > 1$ such that the estimates

(7) $$\left|\frac{\partial a_i(x, y, \xi, \tau)}{\partial \xi_j}\right| + \left|\frac{\partial a_i(x, y, \xi, \tau)}{\partial \tau_j}\right| \leqq \frac{C_j}{1 + \sum_{|\alpha|=2} (|\xi_\alpha|^s + |\tau_\alpha|^s)}$$

hold for all $i = 1, 2, 3$, $|j| \leqq 2$, $(x, y) \in \Omega$ and $\xi, \tau \in R^6$.

Lemma 1. *Let* (6) *be satisfied. Then the operator A is continuous and bounded from H into H^*.*

Proof. Suppose $u_n \to u$ in H. It suffices to prove

(8) $$\sup_{\|v\|_H \leqq 1} |\langle Au_n - Au, v\rangle| \to 0 \quad \text{for} \quad n \to \infty$$

and $\sup_{\|v\|_H \leqq 1} |\langle Au, v\rangle| \leqq C_D < \infty$ for u from a bounded set D in H. Denote $u_n = \{w_n, f_n\}$, $u = \{w, f\}$ and $v = \{\varphi, \psi\}$. We have $w_n \to w$, $f_n \to f$ in $W \equiv W_2^2(\Omega)$. Let us estimate the members of the type

$$I_n^{(1)} = \sup_{\|\varphi\|_W \leqq 1} |B(w_n; f_n + f_0, \varphi) - B(w; f + f_0, \varphi)| \leqq$$

$$\sup_{\|\varphi\|_W \leqq 1} |B(w_n - w; f_n + f_0, \varphi)| + \sup_{\|\varphi\|_W \leqq 1} |B(w; f_n - f, \varphi)| .$$

Owing to (3) we obtain $I_n^{(1)} \to 0$ for $n \to \infty$. Now we estimate the members of the type

$$I_n^{(2)} = \sup_{\|\varphi\|_W \leqq 1} |((w_n)_{xx}\, a_3(Dw_n; D(f_n + f_0)) - w_{xx}\, a_3(Dw; D(f + f_0)), \varphi_{xx})| .$$

From the relations

$$(w_n)_{xx}\, a_3(Dw_n; D(f_n + f_0)) - w_{xx}\, a_3(Dw; D(f + f_0)) =$$

$$(w_n - w)_{xx}\, a_3(Dw_n; D(f_n + f_0)) + w_{xx}(a_3(Dw_n; D(f_n + f_0) -$$

$$a_3(Dw; D(f + f_0)) ,$$

$u_n \to u$ in H and (6) we easily deduce that $I_n^{(2)} \to 0$ for $n \to \infty$. From these facts we easily conclude (8). Boundedness of the operator can be proved analogously.

The coercivity of the operator A $(A \equiv A_\zeta)$ is proved by means of the result in [5] (see [3], [6]), which is based on the idea of Knightly [6], for a special choice of the function ζ.

Lemma 2. *Suppose* (5), (6). *If the inequality*

$$\frac{3}{2}M_3 + 81 M_0^{-1} M_2^2 < 1 \tag{9}$$

is satisfied then there exists a $\zeta \in C^2(\bar{\Omega})$ *with the property* (P) *and constants* C_1, C_2 $(C_1 \equiv C_1(\zeta) > 0,\ C_2 \equiv C_2(\zeta) > 0)$ *such that the estimate*

$$\langle Au, u\rangle \geqq C_1 \|u\|_H^2 - C_2 \tag{10}$$

holds for all $u \in H$.

Proof. Let us put $u = \{w, f\}$ into (1), (2). Using (4) and eliminating $B(w; w, f)$ from (1), (2) we successively obtain the estimate

$$\langle Au, u\rangle \geqq \|w\|_W^2 \left(1 - \frac{3}{2} M_3 - \frac{9M_2 L^2}{2Eh} - \frac{9M_2\varepsilon^2}{4Eh}\right) + \tag{11}$$

$$\frac{17}{E^2h^2}\, \|f + f_0\|_W^2 \left(\frac{M_0}{2} - \frac{9EhM_2}{2L^2} - \frac{E^2h^2M_1\varepsilon^2}{12}\right) - \frac{9}{Eh^2}\, B(w; f_0, w) - C(\varepsilon)\,.\, \|f_0\|_{W_2{}^2}\,,$$

where $L > 0$ is an arbitrary number $C(\varepsilon) \to \infty$ for $\varepsilon \to 0$, $f_0 = \zeta F_0$ (see (B)) and $\|v\|_W^2 = \|v_{xx}\|_{L_2}^2 + \|v_{yy}\|_{L_2}^2 + 2\|v_{xy}\|_{L_2}^2$. In (11) Young's inequality $(ab \leqq 2^{-1}\varepsilon^2 a^2 + 2^{-1}\varepsilon^{-2} b^2)$ has been used. Let us take $L^2 = (M_0 - \gamma)^{-1}\, 9EhM_2$ where $(0 < \gamma < M_0/2)$ is sufficiently small. Then owing to (9) we have

$$C_0 = 1 - \frac{3}{2} M_3 - \frac{9M_3L^2}{2Eh} > 0 \quad \text{and} \quad \frac{M_0}{2} - \frac{9EhM_2}{2L^2} > 0\,.$$

Using the result from [5] (see also [3], [6]) we can choose such a ξ with the property (P) that the estimate

$$|B(w; \xi F_0, w)| < \frac{C_0}{4}\, \|w\|_W^2 \tag{12}$$

holds. From (11), (12) and for sufficiently small ε we obtain the estimate (10) and Lemma 2 is proved.

Henceforth let $\zeta \in C^2(\bar{\Omega})$ be a fixed function for which Lemma 2 holds true. In order to prove the property S for A we use the following lemma.

Lemma 3. *Let* $a = (a_i)$, $b = (b_i)$, $A = (A_i)$, $B = (B_i)$ *be real vectors in* E^n. *If* $s > 1$ *then there exists a constant* $K > 0$ *(independent of* a, b, A, B*) such that the* *estimates*

$$I_i = \int_0^1 \frac{|a_i| + |b_i|}{1 + |a + t(A - a)|^s + |b + t(B - b)|^s} \,\mathrm{d}t \leqq K$$

hold for all $i = 1, 2, \ldots, n$.

Proof. Denote $x = a_i$, $y = A_i$. We assume $x, y \geqq 0$.

For $0 \leqq x \leqq y$ we have

$$I_i \leqq I \equiv \int_0^1 \frac{x}{1 + |x + t(y - x)|^s} \,\mathrm{d}t = \frac{x}{y - x} \int_x^y \frac{\mathrm{d}z}{1 + z^s} \leqq \frac{x}{1 + x^s} \leqq 1 .$$

If $x \leqq 1$ then $I \leqq 1$. Thus we assume $x \geqq 1$. For $0 \leqq y \leqq x$ we consider the cases 1) $0 \leqq y \leqq \frac{1}{2}x$ abd 2) $x \geqq y \geqq \frac{1}{2}x$. In the case 1) we have

$$I \leqq 2 \int_0^\infty \frac{\mathrm{d}z}{1 + z^s} = 2K_s = \frac{2\pi}{s} \left(\sin \frac{\pi}{s} \right)^{-1} .$$

In the case 2) we have

$$I \leqq \frac{x}{1 + y^s} \leqq \frac{x}{1 + 2^{-s} x^s} \leqq 2 .$$

Analogously, for $y < 0$, $x \geqq 0$ we obtain $I \leqq 2K_s$. Hence Lemma 3 is proved with $K = 4 \max (K_s, 1)$.

Denote

$$C = K \left(14 + \frac{21}{Eh} + 3Eh \right), \quad \delta = \max_{|i| = 2} \{C_i\} , \tag{13}$$

where K is from Lemma 3 and C_j are from (7). Our main lemma is

Lemma 4. *Let* (5)–(7) *be satisfied. If the inequalities*

$$M_3 < \tfrac{2}{3} , \quad 1 - \tfrac{3}{2} M_3 + \frac{M_0}{2} - \left(\left(1 - \tfrac{3}{2} M_3 - \frac{M_0}{2} \right)^2 + \right. \tag{14}$$
$$\left. + \, 4M_2^2 \left(\frac{9}{8Eh} + \frac{Eh}{8} \right)^2 \right)^{1/2} > 2C\delta$$

hold then the operator A *possesses the property* S.

Proof. Let $u_n = \{w_n, f_n\}$, $u = \{w, f\} \in H$ and $u_n \rightharpoonup u$, $P_n \equiv \langle Au_n - Au, u_n - u \rangle \to 0$ for $n \to \infty$. For simplicity we denote $F_n = f_n + f_0$, $F = f + f_0$. $a_i(n) \equiv \equiv a_i(Dw_n; DF_n)$ and $a_i(0) \equiv a_i(Dw; DF)$ $(i = 1, 2, 3)$. Using Young's inequality we successively estimate

$$(15)\quad P_n \geqq \|w_n - w\|_W^2 - \tfrac{3}{2}M_3\|w_n - w\|_W^2 - ((a_3(n) - a_3(0))(w_{xx} + \tfrac{1}{2}w_{yy}),$$

$$(w_n - w)_{xx}) - ((a_3(n) - a_3(0))(w_{yy} + \tfrac{1}{2}w_{xx}), (w_n - w)_{yy}) -$$

$$- ((a_3(n) - a_3(0))\, w_{xy}, (w_n - w)_{xy}) - \frac{L_1^2\, 9M_2}{8Eh}\|w_n - w\|_W^2 - \frac{9M_2}{8EhL_1^2}\|f_n - f\|_W^2 +$$

$$+ \frac{9}{4Eh}\{(F_{yy}(a_2(n) - a_2(0)), (w_n - w)_{xx}) + (F_{xx}(a_2(n) - a_2(0)), (w_n - w)_{yy}) -$$

$$- 2(F_{xy}(a_2(n) - a_2(0)), (w_n - w)_{xy})\} + \frac{M_0}{2}\|f_n - f\|_W^2 -$$

$$- ((a_1(n) - a_1(0))(F_{xx} - \tfrac{1}{2}F_{yy}), (f_n - f)_{xx}) - ((a_1(n) - a_1(0))(F_{yy} - \tfrac{1}{2}F_{xx}), (f_n - f)_{yy}) -$$

$$- 3(F_{xy}(a_1(n) - a_1(0)), (f_n - f)_{xy}) - \frac{EhM_2}{8}L_2^2\|w_n - w\|_W^2 -$$

$$- \frac{EhM_2}{8L_2^2}\|f_n - f\|_W^2 - \frac{Eh}{4}\{(w_{xx}(a_2(n) - a_2(0)), (f_n - f)_{yy}) - (w_{yy}(a_2(n) -$$

$$- a_2(0)), (f_n - f)_{xx}) + 2(w_{xy}(a_2(n) - a_2(0)), (f_n - f)_{xy})\} - Z_n\,,$$

where $L_1, L_2 > 0$ are arbitrary numbers and

$$Z_n = \frac{9}{Eh^2}\{B(w_n; f_n, w_n - w) - B(w; f, w_n - w) + B(w_n; f_0, w_n - w) -$$

$$- B(w; f_0, w_n - w)\} + \frac{E}{2}\{B(w_n; w_n, f_n - f) - B(w; w, f_n - f)\}\,.$$

From the compactness of the imbedding $W_2^2(\Omega) \to W_4^1(\Omega)$ $(n = 2)$ and from (3) we obtain $\lim_{n\to\infty} Z_n = 0$. All the members containing the expression $a_i(n) - a_i(0)$ are estimated in the same way. Let us consider, e.g., the integral

$$J = (w_{xx}(a_3(n) - a_3(0)), (w_n - w)_{xx})\,.$$

We have

$$(16)\quad J = \left(w_{xx}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\, a_3(D(w + t(w_n - w)); D(F + t(F_n - F)))\,\mathrm{d}t, (w_n - w)_{xx}\right) =$$

$$= \left(\sum_{|i|\leqq 2} D^i(w_n - w)\int_0^1 \frac{\partial a_3}{\partial \xi_i}\, w_{xx}\,\mathrm{d}t, (w_n - w)_{xx}\right) +$$

$$+ \left(\sum_{|i|\leqq 2} D^i(f_n - f)\int_0^1 \frac{\partial a_3}{\partial \tau_i}\, w_{xx}\,\mathrm{d}t, (w_n - w)_{xx}\right),$$

where $i = (i_1, i_2)$ is a multiindex and $D^i v = \partial^{|i|} v / (\partial x^{i_1} \, \partial y^{i_2})$. Owing to Lemma 3 we conclude from (7) that

$$\left| \int_0^1 \frac{\partial a_3}{\partial \xi_i} \, w_{xx} \, dt \right| + \left| \int_0^1 \frac{\partial a_3}{\partial \tau_i} \, w_{xx} \, dt \right| \leqq K C_i \quad \text{for a.e.} \quad (x, y) \in \Omega$$

and $|i| \leqq 2$. For $|i| = 2$ we estimate

$$\left| \left(D^i(w_n - w) \int_0^1 \frac{\partial a_3}{\partial \xi_i} \, w_{xx} \, dt, \, (w_n - w)_{xx} \right) \right| \leqq \delta K \left(\tfrac{1}{2} \| D^i(w_n - w) \|^2 + \tfrac{1}{2} \| (w_n - w)_{xx} \|^2 \right)$$

and

$$\left| \left(D^i f_n - f \right) \int_0^1 \frac{\partial a_3}{\partial \xi_i} \, w_{xx} \, dt, \, (w_n - w)_{xx} \right) \right| \leqq \delta K \left(\tfrac{1}{2} \| D^i(f_n - f) \|^2 + \tfrac{1}{2} \| (w_n - w)_{xx} \|^2 \right).$$

For $|i| < 2$ we estimate

$$J_n(1, i) = \left| \left(D^i(w_n - w) \int_0^1 \frac{\partial a_3}{\partial \xi_i} \, w_{xx} \, dt, \, (w_n - w)_{xx} \right) \right| \leqq C_i K \| D^i(w_n - w) \| \, \| (w_n - w)_{xx} \|$$

and

$$J_n(2, i) = \left| \left(D^i(f_n - f) \int_0^1 \frac{\partial a_3}{\partial \xi_i} \, w_{xx} \, dt, \, (w_n - w)_{xx} \right) \right| \leqq C_i K \| D^i(f_n - f) \| \, \| (w_n - w)_{xx} \| .$$

Hence and from (16) we obtain

$$|J| \leqq \delta K \left(\| w_n - w \|_W^2 + \| f_n - f \|_W^2 + 3 \| (w_n - w)_{xx} \|^2 \right) + G_n(J) ,$$

where $G_n(J) = \sum_{|i| < 2} (J_n(1, i) + J_n(2, i))$ and $\lim_{n \to \infty} G_n(J) = 0$.

Analogously we estimate, e.g., the integral

$$I = \left| \left(F_{yy}(a_1(n) - a_1(0)), \, (f_n - f)_{yy} \right) \right| \leqq \delta K \left(\| w_n - w \|_W^2 + \right.$$
$$\left. + \| f_n - f \|_W^2 + 3 \| (f_n - f)_{yy} \|^2 \right) + G_n(I) ,$$

where $\lim_{m \to \infty} G_n(I) = 0$. Let $G_n = \sum_J G_n(J)$, where the sum is taken over all integrals J corresponding to (15). Summarizing the previous estimates from (14) we conclude that

$$(17) \quad P_n + Z_n + G_n \geqq \| w_n - w \|_W^2 \left(1 - \tfrac{3}{2} M_3 - \frac{9 L_1^2 M_2}{8 E h} - \frac{E h M_2 L_2^2}{8} - C \delta \right) +$$
$$+ \| f_n - f \|_W^2 \left(\frac{M_0}{2} - \frac{9 M_2}{8 E h L_1^2} - \frac{E h M_2}{8 L_2^2} - C \delta \right),$$

where C and δ are from (13) and $\lim_{n\to\infty} G_n = 0$. Let us choose

$$L_1^2 = L_2^2 = \tfrac{1}{2}(a + (a^2 + 4M_2^2 b^2)^{1/2})/b \quad \text{where} \quad a = 1 - \tfrac{3}{2}M_3 - \tfrac{1}{2}M_0\,, \quad b = 9/(8Eh) + \tfrac{1}{8}Eh\,.$$

If (14) is satisfied then

$$1 - \tfrac{3}{2}M_3 - \frac{9L_1^2 M_2}{8Eh} - \frac{EhM_2L_2^2}{8} - C\delta > 0$$

and

$$\frac{M_0}{2} - \frac{9M_2}{8EhL_1^2} - \frac{EhM_2}{8L_2^2} - C\delta > 0\,.$$

Hence and from (17) we conclude that $u_n \to u$ in H because $\lim_{n\to\infty} G_n = 0$. Thus, Lemma 3 is proved.

Applying known results (see, e.g., [7]) as a consequence of Lemmas 1–4 we have $A(H) = H^*$, i.e., we can formulate the following theorem.

Theorem 1. *Suppose (5)–(7). If (9) and (14) are fulfilled then there exists a variational solution of (E_1), (E_2), (B) for all $g \in W_2^{-2}$ and $F_0 \in C^2(\bar{\Omega})$.*

3. ASYMPTOTICAL BEHAVIOUR OF THE SOLUTION FOR $\omega \to 0$

The system (E_1), (E_2) for $a_1 \equiv 1$, $a_i \equiv 0$, $i = 1, 2$ (this is the case we obtain for $\omega \equiv 0$ in the constitutive law) can be identified with the system of von Kármán equations. In this section we shall be concerned with the behaviour of the solutions u_ω of the operator equations $A_\omega\, u = G$ for $\omega \to 0$, where $A_\omega \equiv A$ is the operator corresponding to the system (E_1), (E_2). Denote by $A_0 \equiv A_\omega|_{\omega=0}$ the operator corresponding to the system of von Kármán (i.e. $a_1 = 1$, $a_2 = a_3 = 0$). Evidently, the operator $A_0 : H \to H^*$ is a bounded, continuous and coercive operator with the property S. The functions a_i $(i = 1, 2, 3)$ in (E_1), (E_2) need not necessarily be derived from a function ω. Convergence $\omega_n \to 0$ is to be understood in the following sense: $a_{1,n} \rightrightarrows 1$, $a_{i,n} \rightrightarrows 0$ $(i = 1, 2)$ on $\bar{\Omega} \times R^6 \times R^6$.

Theorem 2. *We assume that the sequences of the functions $\{a_{i,n}(x, y, \xi, \tau)\}_{n=1}^{\infty}$ $(i = 1, 2, 3)$ satisfy (5)–(7) uniformly with respect to n (i.e., the constants M_i $(i = 0, 1, 2, 3)$ and C_j $(|j| \leqq 2)$ are independent of n). Suppose (9), (14) and*

$$(18) \qquad a_{1,n} \rightrightarrows 1\,, \quad a_{2,n} \rightrightarrows 0\,, \quad a_{3,n} \rightrightarrows 0 \quad \text{for} \quad n \to \infty$$

uniformly on the set $\bar{\Omega} \times R^6 \times R^6$. Then from each sequence $\{u_n\}_{n=1}^{\infty}$ $(u_n \equiv u_{\omega_n}$ is a solution of $A_{\omega_n} u = G)$ it is possible to choose a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that $u_{n_k} \to u$ in H, where u is a solution of $A_0 u = G$.

Proof. Existence of the solutions u_n, $n = 1, 2, \ldots$ is guaranteed by Theorem 1. Owing to the assumptions for $\{a_{i,n}\}$ $(i = 1, 2, 3)$ we easily find out that there exists a $\zeta \in C^2(\bar{\Omega})$ with the property (P) and C_1, C_2 (all independent of n) such that the estimate

$$\langle A_{\omega_n} u, u \rangle \geqq C_1 \|u\|_H^2 - C_2 \quad (C_1 > 0)$$

holds for all $u \in H$ (see the proof of Lemma 2). Hence and from $A_{\omega_n} u_n = G$ we obtain $\|u_n\|_H \leqq C$. Thus there exists a subsequence $v_k = u_{n_k}$ and $u \in H$ such that $v_k \rightharpoonup u$ in H. First we prove $v_k \to u$ in H and then $A_0 u = G$.

For $D_k = \langle A_k v_k, v_k - u \rangle$ we have $\lim_{k \to \infty} D_k = 0$ since $A_k v_k = G$ $(A_k \equiv A_{\omega_{nk}})$. By the same method as in Lemma 4 we obtain

$$\text{(19)} \qquad D_k = \langle A_k v_k - A_k u, v_k - u \rangle + \langle A_k u, v_k - u \rangle \geqq C\|v_k - u\|_H^2 - \\ - |\langle A_k u - A_0 u, v_k - u \rangle| - |\langle A_0 u, v_k - u \rangle| ,$$

where $C > 0$ is independent of k and $\lim_{k \to \infty} |\langle A_0 u, v_k - u \rangle| = 0$. Now we estimate

$$\text{(20)} \qquad |\langle A_k u - A_0 u, v_k - u \rangle| \leqq C_1 \|A_k u - A_0 u\|_{H^*} \leqq C_1(\|u\|_H + \\ + \|F_0\|_W) \cdot \tfrac{3}{2} \sup |a_{1,k}(x, y, \xi, \tau) - 1| + \left(\frac{3}{2} + \frac{3Eh}{4} + \frac{9}{4Eh}\right) \sup | a_{2,k}(x, y, \xi, \tau)| + \\ + \tfrac{3}{2} \sup |a_{3,k}(x, y, \xi, \tau)| \equiv C_1 T_k(\|u\|) ,$$

where the supremum is taken over the set $\Omega \times R^6 \times R^6$ and $T_k(\|u\|) \to 0$ for $k \to \infty$ because of (18). The last inequality follows easily from (1), (2) and from the definition

$$\|A_k u - A_0 u\|_{H^*} = \sup_{\|v\|_H \leqq 1} |\langle A_k u - A_0 u, v \rangle| ,$$

where $v = \{\varphi, \psi\} \in H$. The estimates (20) and (19) imply $v_k \to u$ in H. Analogously as in (20) we obtain $\|A_k v_k - A_0 v_k\|_{H^*} \leqq T_k(\|v_k\|)$ with $T_k(\|v_k\|) \to 0$ for $k \to \infty$ since $\|v_k\|_H \leqq C$. Hence and from the continuity of A_0 we conclude

$$G = \lim_{k \to \infty} A_k v_k = \lim_{k \to \infty} A_0 v_k = A_0 u$$

since $v_k \to u$ and Theorem 2 is proved.

Consequence of Theorem 2. If there exists a unique solution u of the system of von Kármán $A_0 u = G$, then $u_{\omega_n} \to u$ in H where u_{ω_n} is a solution of $A_{\omega_n} u = G$.

Now, we prove that the topological degree of A for small ω (i.e., $|a_1 - 1|$, $|a_2|$, $|a_3|$ are sufficiently small) equals that of A_0. The topological degree for the operators with the property S was introduced in [7] and is a generalization of the topological degree for continuous mappings in E_n with analogous properties (see [7]).

We denote $G_R(v) \equiv \{w \in H;\ \|w - v\|_H \leqq R ,\ S_R(v) \equiv \{w \in H;\ \|v - w\|_H = R\}$, $A_g u = Au - g$ and $A_{0,g} u = A_0 u - g$ (for all $u \in H$), where $g \in H^*$ and $A \equiv A_\omega$.

Theorem 3. *Let* (5)–(7), (9) *and* (14) *be satisfied. Suppose* $g \in H^*$, $\sup_{\Omega \times R^6 \times R^6} |a_1 . (x, y, \xi, \tau) - 1| < L$, $M_2 < L$, $M_3 < L$. If *L is sufficiently small then the topological degree of A_g equals that of $A_{0,g}$ with respect to $R_R(0)$ for sufficiently large R, ($R = R(g, L)$).*

Proof. From the properties of the operators A and A_0 (see Lemmas 1–4) we deduce that the operator

$$A(t, u) = tA_{0,g}u + (1 - t) A_g u$$

defined on $(t, u) \in \langle 0, 1\rangle \times H$ is continuous (in all the variables) and differs from zero on the set $\langle 0, 1\rangle \times S_R$ for sufficiently large $R = R(g)$. From the S-property of A and A_0 (see Lemma 4) we easily find out that $t_n \to t \in \langle 0, 1\rangle$, $u_n \rightharpoonup u$ in H and $\lim_{n\to\infty} \langle A(t_n, u_n), \; u_n - u\rangle \leqq 0$ implies $u_n \to u$ in H. Thus, the operators $A_{0,g}$ and A_g are homotopic (see [7]). To prove Theorem 3 it suffices (see [7]) to prove the estimate

$$\|A_g u - A_{0,g}u\|_{H^*} < \|A_{0,g}u\|_{H^*} \tag{21}$$

for all $u \in S_R(0)$. We have

$$\|A_g u - A_{0,g}u\|_{H^*} = \sup_{\|v\|_H \leqq 1} |\langle Au - A_0 u, v\rangle| \leqq \tfrac{3}{2}M_2\|w\|_W + \tag{22}$$

$$+ \frac{9}{4Eh} M_2\|F\|_W + \sup_{\Omega \times R^6 \times R^6} |1 - a_1| \tfrac{3}{2}\|F\|_W + \frac{Eh}{4} 3M_2\|w\|_W ,$$

where $u = \{w, f\rangle$, $F = f + f_0$. On the other hand, the coercivity of A_0 yields

$$\|A_0 u - g\|_{H^*} \geqq \|u\|_H^{-1} (C_1\|u\|_H^2 - C_2) .$$

Hence and from (22) we obtain (21) and Theorem 3 is proved.

Remark. If u_0 is an isolated solution of $A_g u = 0$ then the topological degree of A_g with respect to $G_{u_0}(r)$ (which is independent of r for sufficiently small r) is called the index of u_0. Theorem 3 implies the following assertion: If there exist only isolated solutions of the equations

i) $Au - g = 0$, ii) $A_0 u - g = 0$

in $G_R(0)$, then the sum of indices of the solutions of i) is equal to the sum of indices of the solutions of ii).

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Súhrn

O RIEŠENÍ ISTÉHO ZOVŠEOBECNENÉHO SYSTÉMU VON KÁRMÁNOVÝCH ROVNÍC

JOZEF KAČUR

V práci sa dokazuje existencia riešenia istého nelineárneho systému rovníc, ktorý je zovšeobecnením známeho systému von Kármánových rovníc. Ďalej sa zkúma vzťah riešení tohoto systému k riešeniam von Kármánových rovníc. Zkúmaný systém je modelom pre veľké deformácie tenkých dosák a škrupín a bol odvodený v [1] za predpokladu nelineárneho vzťahu medzi napätiami a deformáciami v konštitutívnych rovniciach.

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