Ján Lovíšek

Duality in the obstacle and unilateral problem for the biharmonic operator

Aplikace matematiky, Vol. 26 (1981), No. 4, 291–303

Persistent URL: http://dml.cz/dmlcz/103918

DUALITY IN THE OBSTACLE AND UNILATERAL PROBLEM FOR THE BIHARMONIC OPERATOR

JÁN LOVÍŠEK

(Received August 16, 1979)

INTRODUCTION

The problem of minimization and the problem of maximization-also called the dual problem have been examined in detail during the last years, together with their mutual relation in the problems of mechanics. In this work the problem of duality is formulated for the obstacle and unilateral biharmonic problem, which physicaly expresses the equilibrium of a thin plate with an obstacle inside the domain or on the boundary. The dual variational inequality is derived by introducing polar (or conjugate) functions (functions of Fenchel-Rockafellar), as well as by means of the saddle point of the Lagrangian.

1. FUNCTIONAL SPACES

The following functional spaces are essential for studying the problem given above:

$$H^1(\Omega) = \left\{ v \mid v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L_2(\Omega) \right\},$$

$$H_0^1(\Omega) = \{ v \mid v \in H^1(\Omega),\ v|_{\partial\Omega} = 0 \},$$

$$H^2(\Omega) = \left\{ v \mid v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x\, \partial y}, \frac{\partial^2 v}{\partial y^2} \in L_2(\Omega) \right\},$$

$$H_0^2(\Omega) = \overline{\mathscr{D}(\Omega)}^{H^2(\Omega)} = \left\{ v \mid v \in H^2(\Omega),\ v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

($\overline{\mathscr{D}(\Omega)}^{H^2(\Omega)}$ = the closure of the $\mathscr{D}(\Omega)$-functions infinitely times differentiable and with compact support in Ω in the norm of $H^2(\Omega)$).

$$H(\Omega, \nabla^2) = \{ v \mid v \in L_2(\Omega);\ \nabla^2 v \in L_2(\Omega) \}.$$

On the spaces $H^2(\Omega)$, $H^1(\Omega)$ and $H(\Omega, \nabla^2)$ we introduce scalar product by

$$(u, v)_{1,\Omega} = (u, v)_{0,\Omega} + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{0,\Omega} + \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)_{0,\Omega},$$

$$(u, v)_{2,\Omega} = (u, v)_{1,\Omega} + \left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial x^2}\right)_{0,\Omega} + \left(\frac{\partial^2 u}{\partial x\,\partial y}, \frac{\partial^2 v}{\partial x\,\partial y}\right)_{0,\Omega} + \left(\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 v}{\partial y^2}\right)_{0,\Omega},$$

$$(u, v)_{H(\Omega,\nabla^2)} = (u, v)_{0,\Omega} + (\nabla^2 u, \nabla^2 v)_{0,\Omega},$$

respectively, and the corresponding norms are

$$\|u\|_{1,\Omega} = [(u, u)_{1,\Omega}]^{1/2},$$
$$\|u\|_{2,\Omega} = [(u, u)_{2,\Omega}]^{1/2},$$
$$\|u\|_{H(\Omega,\nabla^2)} = [(u, u)_{H(\Omega,\nabla^2)}]^{1/2},$$

where

$$(u, v)_{0,\Omega} = \int_\Omega uv \, \mathrm{d}\Omega .$$

We assume that Ω is an open domain which is smooth C^∞ (or is convex polygon), with a boundary $\partial\Omega$.

Remark 1.1. The transformation $v \to \|\nabla^2 v\|_{0,\Omega}$ defines a norm on the space $H^2(\Omega) \cap H_0^1(\Omega)$ (or $H_0^2(\Omega)$) which is equivalent to the norm induced by the space $H^2(\Omega)$.

$$|v|_{H^2(\Omega)} = \left(\int_\Omega \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2\right] \partial\Omega\right)^{1/2}$$

is a seminorm on the space $H^2(\Omega)$.

The transformation

$$\left\{\gamma_0 v = v\big|_{\partial\Omega};\ \gamma_1 v = \frac{\partial v}{\partial u}\bigg|_{\partial\Omega}\right\} \text{ (operators of traces)}$$

is a linear continuous surjection from $H^2(\Omega)$ on to the product of the spaces $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$.

The transformation γ_1 is a surjection from V on to $H^{1/2}(\partial\Omega)$. Let

$$\langle \cdot; \cdot \rangle_{H^{1/2}(\partial\Omega)} \text{ (resp. } \langle \cdot; \cdot \rangle_{H^{3/2}(\partial\Omega)})$$

denote the bilinear form of duality between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$; (resp. $H^{3/2}(\partial\Omega)$ and $H^{-3/2}(\partial\Omega)$), which extends $(\cdot; \cdot)$ $L_2(\Omega)$, i.e.,

$$\langle u, v \rangle_{H^{1/2}(\partial\Omega)} = \int_{\partial\Omega} uv \, \mathrm{d}s \quad \text{for any} \quad u \in H^{1/2}(\partial\Omega),\ v \in L_2(\partial\Omega),$$

resp.

$$\langle u, v\rangle_{H^{3/2}(\partial\Omega)} = \int_{\partial\Omega} uv \, \mathrm{d}s \quad \text{for any} \quad u \in H^{3/2}(\partial\Omega)\,, \quad v \in L_2(\partial\Omega)\,.$$

We can write Green's formula in the form

$$\int_\Omega \nabla^4 uv \, \mathrm{d}\Omega - \int_\Omega u\nabla^4 v \, \mathrm{d}\Omega = \langle \gamma_1 u, \gamma_0 v\rangle_{H^{1/2}(\partial\Omega)} - \langle \gamma_0 u, \gamma_1 v\rangle_{H^{3/2}(\partial\Omega)} \quad \text{for any} \quad u \in H^2(\Omega)\,, \quad v \in H(\Omega, \nabla^2)\,.$$

Remarks on duality and formulation of the problem in the sense of Fenchel-Rockafellar.

In the following we shall need some results on duality as presented in Ekeland-Temam [3].

If V; (Y) is a Banach space (its dual space will be V^*; (Y^*) with topology generated by norm. For every $u \in V$ and $u^* \in V^*$ (resp. $y \in Y$; $y^* \in Y^*$) we denote the by symbol $\langle u, u^*\rangle_V$ (resp. $\langle y^*, y\rangle_Y$) the value of the functional u^* at the point u (resp. of y^* at the point y). We shall assume the existence of a linear continuous operator $\Lambda \in \mathscr{L}(V, Y)$ and the adjoint operator $\Lambda^* \in \mathscr{L}(Y^*, V^*)$. Further, we introduce functionals F and G defined on V, and Y, respectively with values in $(-\infty, +\infty\rangle$, convex and lower semi-continuous on V (resp. Y) and proper (i.e. not identically equal to $+\infty$). Then $F \in \Gamma_0(V) \subset \Gamma(V)$, $G \in \Gamma_0(Y) \subset \Gamma(Y)$), where $\Gamma(V)$ is the set of functions $F : V \to R$, which are pointwise suprema of family us of continuous affine functions, so that

$$F(v) = \sup_{i\in I} (l_i(v) - \alpha_i) = \sup_{i\in I} (\langle v, v_i^*\rangle_V - \alpha_i)\,,$$

if
$v_i^* \in V^*$, $\alpha_i \in R$ and I is a set of indices ($\Gamma_0(V)$ denotes the subset of functions from $\Gamma(v)$, which are not equal to $+\infty$ nor to $-\infty$). The polar (conjugate) functionals to F and G defined on the spaces V^* and Y^*, respectively, are introduced in the form

$$F^*(v^*) = \sup_{v\in V} \{\langle v, v^*\rangle_V - F(v)\}\,.$$

Note that $F^* \in \Gamma_0(V^*)$ ($G^* \in \Gamma_0(Y^*)$). They are convex, lower semi-continuous and not identically equal to $+\infty$. Given the problem of minimization (or primary problem)

$$(\mathscr{P}) \qquad \inf_{u\in V} \{F(u) + G(\Lambda u)\}\,,$$

the dual problem in the sense of Fenchel-Rockafellar is defined by

$$(\mathscr{P}^*) \qquad \sup_{p^*\in Y^*} \{-F^*(\Lambda^* p^*) - G^*(-p^*)\}\,.$$

The following relations ([3]) between the problems $(\mathscr{P})$ and $(\mathscr{P}^*)$ hold

(a) 1° $\sup(\mathscr{P}^*) \leqq \inf(\mathscr{P})$;

2° if there exists $u_0 \in V$ such that $F(u_0) \langle +\infty$, $G(\Lambda u_0) \langle +\infty$ and if G^* is finit and continuous in Λu_0, then $\inf(\mathscr{P}) = \sup(\mathscr{P}^*)$
and the problem $(\mathscr{P}^*)$ has at least one solution $\bar{p}^*$;

3° $\bar{p}^* \in Y^*$ is a solution of the problem $(\mathscr{P}^*)$ and $\bar{u} \in V$ is a solution of the problem $(\mathscr{P})$ and $\inf(\mathscr{P}) = \sup(\mathscr{P}^*)$ if and only if the following relations of extremality hold:

$$F(\bar{u}) \quad + F^*(\Lambda^* \bar{p}^*) = \langle \Lambda^* \bar{p}^*, \bar{u} \rangle_V ,$$

$$G(\Lambda \bar{u}) + G^*(-\bar{p}^*) = -\langle \bar{p}^*, \Lambda \bar{u} \rangle_Y ,$$

i.e.,
$\Lambda^* \bar{p}^* \in \partial F(\bar{u})$ (subdifferential F at the point $\bar{u}$),
$-\ \bar{p}^* \in \partial G(\Lambda \bar{u})$ (subdifferential G at the point $\Lambda \bar{u}$).
Further, $(\bar{u}, \bar{p}^*)$ is a saddle point of the following Lagrangian:

$$\mathscr{L}(u, p^*) = F(u) + \langle \Lambda u, p^* \rangle_Y - G^*(p^*) .$$

On the other hand, if $(\bar{u}, \bar{p}^*)$ is a saddle point of $\mathscr{L}(u, p^*)$ on $X \times Y^*$ then $\bar{u}$ is a solution of $(\mathscr{P})$ and $\bar{p}^*$ is a solution of $(\mathscr{P}^*)$.

The primary and the dual problem allow the following interpretation by the Lagrangian:

$$(\mathscr{P}) \Leftrightarrow \inf_{u \in V} \sup_{p^* \in Y^*} \mathscr{L}(u, p^*) ,$$

$$(\mathscr{P}^*) = \sup_{p^* \in Y^*} \inf_{v \in V} \mathscr{L}(u, p^*) .$$

Definition 2.1. *The polarity generates a bijection between $\Gamma(V)$ and $\Gamma(V^*)$. $F \in \Gamma(V)$ and $G \in \Gamma(V^*)$ are in duality if they coincide in the bijection $F = G^*$ and $G = F^*$.*

In terms of an even function $\varphi \in \Gamma_0(R)$ and its conjugate convex function φ^*, which is from $\Gamma_0(R)$, we define

$$F\colon V \to R \text{ and } G\colon V^* \to R, \text{ so that}$$

$$F(u) = \varphi(\|u\|_V) \tag{2.2}$$

$$G(u^*) = \varphi^*(\|u^*\|_{V^*})$$

Lemma 1. *Under the above assumptions, F and G are in dualtiy ([3]). Further, we shall examine the case, when the function $\varphi(t)$ is of the form $\varphi(t) = \frac{1}{2}|t|^2$, the conjugate function is $\varphi^*(t) = \frac{1}{2}|t|^2$,*
where

$$\varphi(t), \varphi^*(t) \in \Gamma_0(R) .$$

Then $F(u) = \frac{1}{2}\|u\|_V^2$; $G(u^*) = \frac{1}{2}\|u^*\|_V^2$ *is a conjugate function.*

On a convex closed set $K \subset V(K \neq \emptyset)$, the indicator function $\chi_K(v)$ of K is defined by

(2.3)
$$\chi_K(v) = \begin{cases} 0 & \text{if} \quad v \in K\,, \\ +\infty & \text{if} \quad v \notin K\,. \end{cases}$$

The polar function $\chi_K^*(v^*)$ will have the form $\chi_K^*(v^*) = \sup_{v \in K} \langle v^*, v \rangle_V$, hence

(2.4)
$$\chi_K^*(v^*) = \begin{cases} 0 & \text{if} \quad v^* \leqq 0\,, \\ +\infty & \text{if} \quad v^* > 0 \end{cases}$$

provided K is a convex cone with its vertex at the origin.

If we introduce $K^* = \{v^* \mid v^* \in V^*,\ v^* \leqq 0\}$, it will be $\chi_K^*(v^*) = \chi_{K^*}(v^*)$.

3. FORMULATION OF THE BIHARMONIC PROBLEM WITH AN OBSTACLE IN THE DOMAIN

Let Ω be an open bounded domain in E_2 with a sufficiently smooth boundary $\partial\Omega$. The density of the loading of the plate is defined by a function $f(x, y) \in L_2(\Omega)$. We look for a solution w – (deflection of plate) of the variational inequality

(3.1)
$$\begin{cases} w(x, y) \in K\,, \\ D \int_\Omega \nabla^2 w\, \nabla^2(v - w)\, \mathrm{d}\Omega \geqq \int_\Omega f(v - w)\, \mathrm{d}\Omega \quad \text{for any} \quad v \in K\,, \end{cases}$$

where K is a closed convex set in $V = H_0^2(\Omega)$, which is defined by

(3.2)
$$K = \{v \in H_0^2(\Omega) \mid v \geqq \psi \text{ a.e. on } \Omega\}\,,$$
$$\psi \in H_0^2(\Omega)\,,$$

D is the cylindrical stiffness of the plate. The solution of the variational inequality (3.1) is equivalent to the solution of the following problem:

($\mathscr{P}$) find
$$\begin{cases} w(x, y) \in K & \text{such that} \\ J(w) \leqq J(v) & \text{for any} \quad v \in K \end{cases}$$

where

$$J(v) = D/2\|\nabla^2 v\|_{0,\Omega}^2 - (f, v)_{0,\Omega}\,.$$

The problem ($\mathscr{P}$) corresponds to the formulation in the weak sense of the so-called unilateral Dirichlet problem (or the problem with an obstacle) for the bending of a thin plate.

Theorem 1. *The problem* ($\mathscr{P}$) *has a unique solution* w *for every* $f \in L_2(\Omega)$.

Proof. K is a closed convex subset of $H_0^2(\Omega)$. $J(v)$ is a convex, quadratic and coercive functional on $H_0^2(\Omega)$. Then the rest of the proof follows directly from [[2] – Th. 4.04, s. 126].

Furthermore it can be shown that, if the solution is sufficiently smooth i.e., $w \in H^4(\Omega)$, it satisfies the following set of relations:

$$\text{(3.3)} \qquad \begin{cases} D\nabla^4 w - f \geqq 0 \\ w \geqq \psi \\ (w - \psi)(D\nabla^4 w - f) = 0 \quad \text{in} \quad \Omega \end{cases}$$

$$w\big|_{\partial\Omega} = \frac{\partial w}{\partial n}\bigg|_{\partial\Omega} = 0 \, .$$

These relations suggest that the solution w of the variational inequality (3.1) is a function which in a certain set $\Omega_0 \subset \Omega$ satisfies $D\nabla^4 w - f = 0$, while in another set $\Omega_1 \subset \Omega$ it is equal to ψ. We shall call the boundary $\partial\Omega_0$ of the free boundary. The above set of relations (3.3) will be referred to as the set of complementary differential inequalities.

Indeed we can write

$$D \int_\Omega \nabla^2 w \nabla^2 (v - w) \, d\Omega \geqq \int_\Omega f(v - w) \, d\Omega \, ,$$

but

$$D \int_\Omega \nabla^2 w \nabla^2 (v - w) \, d\Omega = D \int_\Omega \nabla^4 w (v - w) \, d\Omega \, .$$

Then we have

$$D \int_\Omega \nabla^4 w (v - w) \, d\Omega \geqq \int_\Omega f(v - w) \, d\Omega$$

whence

$$\int_\Omega (D\nabla^4 w - f)(v - w) \, d\Omega \geqq 0 \, .$$

Let $\varphi \in \mathscr{D}(\Omega)$ be such that $\varphi \geqq 0$; then $v = w + \varphi \in K$, and hence

$$\int_\Omega (D\nabla^4 w - f) \varphi \, d\Omega \geqq 0 \, , \quad \text{which implies that} \quad D\nabla^4 w - f \geqq 0 \, .$$

Next we take

$$\text{(3.4)} \qquad \begin{aligned} v &= \psi \in K \, , \\ v &= 2w - \psi = w - (\psi - w) \in K \, . \end{aligned}$$

We obtain

$$\left.\begin{aligned} \int_\Omega (D\nabla^4 w - f)(\psi - w) \, d\Omega \geqq 0 \\ \int_\Omega (D\nabla^4 w - f)(w - \psi) \, d\Omega \geqq 0 \end{aligned}\right\} \Rightarrow \int_\Omega (D\nabla^4 w - f)(w - \psi) \, d\Omega = 0 \, .$$

This implies

$$(D\nabla^4 w - f)(w - \psi) = 0 \quad \text{over } \Omega$$

and we deduce that

$$D\nabla^4 w = f \quad \text{over} \quad \Omega_0 \subset \Omega\,, \tag{3.5}$$
$$D\nabla^4 w > f \quad \text{over} \quad \Omega_1 \subset \Omega\,,$$

where

$$\Omega_0 = \{(x, y) \in \Omega \mid w(x, y) > \psi(x, y)\}\,,$$
$$\Omega_1 = \{(x, y) \in \Omega \mid w(x, y) = \psi(x, y)\}\,.$$

If $\psi \in H_0^2(\Omega)$ we can write $K = \psi + U$, where

$$U = \{u \in H_0^2(\Omega) \mid u \geqq 0 \text{ a.e. over } \Omega\}$$

is a positive cone with its vertex at the origin. Further putting

$$w = \psi + u^*\,, \quad u^* \in U \quad \text{we have}\,. \tag{3.6}$$

Lemma 2. *We can write*

$$\langle D\nabla^4 w - f, u^* \rangle_{H_0^2(\Omega)} = 0\,. \tag{3.7}$$

Proof. Substitute $v = \psi$ in to (3.1) and then put $v = \psi + 2u^*$. If we apply (3.6) and Green's formula we get (3.7).

Now we derive the dual formulation of the problem $(\mathscr{P})$, which we write in the form

$$\inf_{v \in H_0^2(\Omega)} \{F(v) + G(\Lambda v)\} \tag{$\mathscr{P}$}$$

where

$$F(v) = -(f, v)_{0,\Omega} + \chi_K(v)\,, \tag{3.8}$$

$\chi_K(v)$ is the indicator function of K and

$$\Lambda = \nabla^2 (\Lambda^* = \nabla^2 \in \mathscr{L}(L_2(\Omega), H^{-2}(\Omega))\,, \tag{3.9}$$
$$\Lambda \in \mathscr{L}(H_0^2(\Omega), L_2(\Omega))\,,$$

$$G(p) = D/2\, \|p\|_{0,\Omega}^2\,, \quad p \in Y = L_2(\Omega)\,. \tag{3.10}$$

We shall now consider a function (3.11) of $H_0^2(\Omega) \times L_2(\Omega)$ into $\bar{R}$ such that

$$\Phi(v, 0) = F(v) + G(\Lambda v)$$

and for every $p \in L_2(\Omega)$ we shall consider the minimization problem:

$$\inf_{v \in H_0^2(\Omega)} \Phi(v, p) \tag{$\mathscr{P}_p$}$$

Clearly, for $p = 0$, $\mathscr{P}_0$ is nothing else than the problem $\mathscr{P}$. The problem $\mathscr{P}_p$ will be called the pertrubed problem of $\mathscr{P}$ (with respect to the given perturbation).

By introducing the perturbed function

$$\Phi(v, p) = F(v) + G(\Lambda v - p) \tag{3.11}$$

we shall formulate the dual problem $(\mathscr{P}^*)$ by

$$(\mathscr{P}^*) \qquad \sup_{p^* \in L_2(\Omega)} \{-F^*(\Lambda^* p^*) - G^*(-p^*)\} .$$

Lemma 3. *For $p \in L_2(\Omega)$, we have*

$$F^*(\Lambda^* p^*) = \begin{cases} (\Lambda\psi, p)_{0,\Omega} + (\psi, f)_{0,\Omega} & \text{if} \quad \Lambda^* p^* + f \leqq 0 , \\ +\infty & \textit{otherwise} . \end{cases} \tag{3.12}$$

Proof.

$$v \in K \Leftrightarrow v = \psi + u , \quad \text{where} \quad u \in U .$$

Now we have from the definition

$$F^*(\Lambda^* p^*) = \sup_{v \in H_0^2(\Omega)} \{\langle v, \Lambda^* p^* \rangle_{H_0^2(\Omega)} + \langle f, v \rangle_{H_0^2(\Omega)} - \chi_K(v)\} ,$$

where

$$F(v) = -\langle f, v \rangle_{H_0^2(\Omega)} + \chi_K(v) .$$

Hence we further get

$$F^*(\Lambda^* p^*) = \langle \psi, \Lambda^* p^* \rangle_{H_0^2(\Omega)} + \langle \psi, f \rangle_{H_0^2(\Omega)} + \sup_{u \in H_0^2(\Omega)} \{\langle u, \Lambda^* p^* \rangle_{H_0^2(\Omega)} + \langle f, u \rangle_{H_0^2(\Omega)} - \chi_K(\psi + u)\} = \langle \psi, \Lambda^* p^* \rangle_{H_0^2(\Omega)} + \langle \psi, f \rangle_{H_0^2(\Omega)} + \sup_{u \in U} \langle u, \Lambda^* p^* + f \rangle_{H_0^2(\Omega)} .$$

However

$$\sup_{u \in U} \langle u, \Lambda^* p^* + f \rangle_{H_0^2(\Omega)} = 0$$

for $\Lambda^* p^* + f \leqq 0$, because U is a positive cone. It is readily seen that

$$(\Lambda\psi, p^*)_{0,\Omega} = \langle \psi, \Lambda p^* \rangle_{H_0^2(\Omega)} \quad \text{for any} \quad p^* \in Y^* , \quad \psi \in H_0^2(\Omega) .$$

Summarizing we have

$$F^*(\Lambda^* p^*) = (\Lambda\psi, p^*)_{0,\Omega} + \langle \psi, f \rangle_{H_0^2(\Omega)} + \chi_{U^*}(\Lambda^* p^* + f) ,$$

where

$$\chi_{U^*}(\Lambda^* p^* + f)$$

is the indicator function of the polar cone U^*,

$$U^* = \{q^* \in H^{-2}(\Omega) \mid \langle q^*, u \rangle_{H_0^2(\Omega)} \leqq 0 \text{ for any } u \in U\} .$$

Lemma 1 implies that the conjugate function $G^*(p^*) \in \Gamma(L_2(\Omega))$ has the form

$$G^*(p^*) = \tfrac{1}{2} D \|p^*\|_{0,\Omega}^2 .$$

This the dual problem $(\mathscr{P}^*)$ is expressed by

$$(\mathscr{P}^*) \qquad \sup_{p^* \in L_2(\Omega)} \{-(\Lambda\psi, p^*)_{0,\Omega} - \langle \psi, f \rangle_{H_0{}^2(\Omega)} - \chi_{U^*}(\Lambda^* p^* + f) - \tfrac{1}{2} D \|p^*\|^2_{0,\Omega}\} .$$

After eliminating those elements p^* for which $F^*(\Lambda^* p^*) = +\infty$, i.e. taking supremum only on the elements p^* satisfying

$$\chi_{U^*}(\Lambda^* p^* + f) = 0 \Leftrightarrow \Lambda^* p^* + f \leqq 0 , \quad \text{we get}$$

$$(P^*) \qquad \sup_{p^* \in L_2(\Omega)} \{-(\Lambda\psi, p^*)_{0,\Omega} - \langle \psi, f \rangle_{H_0{}^2(\Omega)} - \tfrac{1}{2} D \|p^*\|^2_{0,\Omega}\}$$

$$\Lambda^* p^* + f \leqq 0 .$$

Lemma 4. *The problem $(\mathscr{P}^*)$ has a unique solution.*

Proof. The functional $G^*(p^*)$ is strictly convex on the space $L_2(\Omega)$ and the set of permissible p^*, i.e. $\Lambda^* p^* + f \leqq 0$, is closed and convex in $L_2(\Omega)$.

Lemma 5. *Let w be a solution of the problem $(\mathscr{P})$, $\bar{p}^*$ a solution of the problem $(\mathscr{P}^*)$. Then we have*

$$\inf (\mathscr{P}) = \sup (\mathscr{P}^*) \tag{3.13}$$

and w and $\bar{p}^*$ are satisfying the extremality relations

(3.14)
$$D\nabla^2 w + \bar{p}^* = 0 ,$$
$$\nabla^2 \bar{p}^* + f \leqq 0 , \quad \text{if} \quad w = \psi \quad (\text{i.e. in the domain } \Omega_1)$$
$$w \in K , \quad \bar{p}^* \in L_2(\Omega) .$$

Proof. By virtue of Theorem III.4.1 ([3]), since $F(\psi) < +\infty$ and G is finite and continuous at the point $\Lambda\psi$ we have

$$\inf (\mathscr{P}) = \sup (\mathscr{P}^*) \tag{3.15}$$

and the problem $(\mathscr{P}^*)$ possesses at least one solution $\bar{p}^*$. Then due to (a, 3°) we can write

(3.16)
$$F(w) + F^*(\Lambda^* \bar{p}^*) = -\langle f, w \rangle_{H_0{}^2(\Omega)} + \chi_K(w) +$$
$$+ \langle \psi, \Lambda^* \bar{p}^* \rangle_{H_0{}^2(\Omega)} + \langle \psi, f \rangle_{H_0{}^2(\Omega)} + \chi_{U^*}(\Lambda^* \bar{p}^* + f) = \langle \Lambda^* \bar{p}^*, w \rangle_{H_0{}^2(\Omega)}$$

only if $\Lambda^* \bar{p}^* + f \leqq 0$.

On the other hand, the second extremality relation

$$G(\Lambda w) + G^*(-\bar{p}^*) = \langle -\bar{p}^*, \Lambda w \rangle_{H_0{}^2(\Omega)} \tag{3.17}$$

easily yields that

$$D/2 \|\nabla^2 w\|^2_{0,\Omega} + \tfrac{1}{2} D \|\bar{p}^*\|^2_{0,\Omega} = \langle -\bar{p}^*, \nabla^2 w \rangle_{H_0{}^2(\Omega)} .$$

This equality can be rewriten in the form

$$\left\| \sqrt{(D)}\, \nabla^2 w + \frac{1}{\sqrt{D}}\, \bar{p}^* \right\|_{0,\Omega}^2 = 0 .$$

Therefore

$$D\nabla^2 w = -\bar{p}^* .$$

4. Formulation of the unilateral problem (obstacles on the boundary).

For the unilateral problem we replace the convex set in (3.2) by

(4.1) $$K = \{v \in H^2(\Omega) \cap H_0^1(\Omega) = V \mid \gamma_1 v \geqq 0 \text{ a.e. in } \Omega\}$$

(the relation $\gamma_1 v \geqq 0$ is to be understood in the sense of $H^{1/2}(\partial\Omega)$).

Let us consider the variational inequality:

Find

(4.2) $w \in K$ such that

$$a(w, v - w) \geqq (\tilde{f}, v - w)_{0,\Omega} \quad \text{for any} \quad v \in K ,$$

where

$$a(u, v) = \int_\Omega \left\{ \nabla^2 u \nabla^2 v + (1 - \mu)\left(2\frac{\partial^2 u}{\partial x\, \partial y}\frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} \right) \right\} \mathrm{d}\Omega =$$

$$= \int_\Omega \left\{ \mu \nabla^2 u \nabla^2 v + (1 - \mu)\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\, \partial y}\frac{\partial^2 v}{\partial x\, \partial y} \right) \right\} \partial\Omega$$

$$\tilde{f} = \frac{12(1 - \mu^2)}{Eh^3} f = D^{-1} f ;$$

$\frac{1}{2} > \mu > 0$ is Poisson's number.

The bilinear form $a(u, v)$ is continuous on $H^2(\Omega) \times H^2(\Omega)$ and V – elliptic.

We can write

(4.4) $$a(v, v) = \mu \|\nabla^2 v\|_{0,\Omega}^2 + (1 - \mu)\, |v|_{2,\Omega}^2 .$$

Clearly, the problem (4.2) is equivalent to the problem of minimization

(4.5) $$\inf_{v \in V} \{\tfrac{1}{2}\, a(v, v) - (\tilde{f}, v)_{0,\Omega} + \chi_K(v)\} .$$

Theorem 2. *The minimization problem* (4.5) *has one and only one solution.*

Proof. See [2]. Observe that the problem (4.2) coincides with the weak formulation of the unilateral Dirichlet problem with a homogeneous obstacle on the boundary for the transverse slope of the plate. Define a linear bounded operator $A : V \to V^*$

satisfying $\langle Au, v\rangle_V = a(u, v)$ for all $v \in V$. Then we can write (4.1) in the form

$$(4.6) \qquad w \in K \cap D(A), \quad \langle Aw, v - w\rangle_V \geqq \langle \tilde{f}, v - w\rangle_V \quad \text{for any} \quad v \in K$$

or in an equivalent form

$$(4.7) \qquad w \in D(A), \quad \langle Aw, v - w\rangle_V + \chi_K(v) - \chi_K(w) \geqq \langle \tilde{f}, v - w\rangle_V \quad \text{for any} \quad v \in V.$$

For the dual formulation of variational inequality (4.7) we introduce the mapping

$$(4.8) \qquad A^{-1}\colon V^* \to V.$$

Then the dual variational inequality conjugate with (4.7) can be written as follows ([5]):

$$(4.9) \qquad w^* \in D(A^{-1}) \quad \langle A^{-1}w^*, v^* - w^*\rangle_V + \sigma_K(v^*) - \sigma_K(w^*) \geqq \langle h, v^* - w^*\rangle_V \quad \text{for any} \quad v^* \in V^*,$$

where $h = A^{-1}\tilde{f}$,
$\sigma_K(v^*)$ = support function of K ($\equiv$ conjugate function of $\chi_K(v)$).

Theorem 3 ([5]).
An element $w \in V$ is a solution of the variational inequality (4.7) if and only if the element $w^* = -Aw + \tilde{f}$ of V^* is a solution of (4.9), where $w = A^{-1}w^* + h$ and

$$(4.10) \qquad w \in D(A) \cap K \quad \sigma_K(w^*) = \langle w^*, w\rangle_V$$

Since $\tilde{f} \in L_2(\Omega)$, we have $w \in H^3(\Omega) \cap H_0^1(\Omega)$ (see [4]).

$$(\text{i}) \qquad \gamma_0(\nabla^2 w) \in H^{1/2}(\partial\Omega) \subset L_2(\partial\Omega).$$

Under the regularity hypothesis (i) it is possible to prove, in the case of the problem (3.3), that the solution w of (4.2) or equivalently the minimizer w of (4.5) can be characterized by

$$(4.11) \qquad \nabla^4 w = \tilde{f} \text{ over } \Omega, \quad w \in H^2(\Omega), \quad w = 0 \text{ on } \partial\Omega, \quad M_n(w) \geqq 0; \quad \gamma_1 w \geqq 0; \quad M_n(w)\,\gamma_1(w) = 0 \quad \text{on} \quad \partial\Omega.$$

Then we define the Lagrangian $\mathscr{L}\colon V \times L_2^+(\partial\Omega) \to R$ by

$$(4.12) \qquad \mathscr{L}(v, \nu) = J(v) - \int_{\partial\Omega} \nu\gamma_1 v \, ds,$$

where

$$J(v) = \tfrac{1}{2}\, a(v, v) - (\tilde{f}, v)_{0,\Omega}\,,$$

$$\text{(4.13)} \qquad L_2^+(\partial\Omega)\ (= \text{positive cone}) = \{v \mid v \in L_2(\partial\Omega),\ v = 0 \text{ a.e. in } \Omega\}\,.$$

Theorem 4. *Let w be a solution of the problem (4.6) with $\tilde{f} \in L_2(\Omega)$, then the Lagrangian defined in (4.12) possesses a saddle unique point $\{w, M_n(w)\}$ on the Cartesian product $V \times L_2^+(\partial\Omega)$.*

Proof.

1° By (i) we have $\lambda = M_n(w) \in H^{1/2}(\partial\Omega) \subset L_2^+(\partial\Omega)$ and $\gamma_1 w \geqq 0$ a.e. by (4.11). However,

$$\gamma_1 w \geqq 0 \Rightarrow \int_{\partial\Omega} v\gamma_1 w \,\mathrm{d}s \geqq 0 \quad \text{for any} \quad v \in L_2^+(\partial\Omega)\,.$$

Thus we may write

$$\text{(4.14)} \qquad \mathscr{L}(w, v) \leqq \mathscr{L}(w, \lambda)\ (= J(w)) \quad \text{for any} \quad v \in L_2^+(\partial\Omega)\,.$$

2° Let $\lambda \in L_2^+(\partial\Omega)$ be fixed; solve the problem of minimization

$$\text{(4.15)} \qquad \inf_{v \in V} \mathscr{L}(v, \lambda) = \inf_{v \in V} \left[J(v) - \int_{\partial\Omega} \lambda \gamma_1 v \,\mathrm{d}s \right],$$

The unique solution of (4.15) is characterized by

$$\left\langle \operatorname{grad} \left[J(w_\lambda) - \int_{\partial\Omega} \lambda \gamma_1 w_\lambda \right], u \right\rangle_V = 0 \quad \text{for any} \quad u \in V\,.$$

Using Green's formula we obtain

$$\text{(4.16)} \qquad M_n(w_\lambda) = \lambda\,; \quad \nabla^4 w_\lambda = \tilde{f}\,, \quad \gamma_0 w_\lambda = 0\,.$$

Hence we deduce that $w_\lambda = w$, because the solution (4.5) in the form (4.11) satisfies (4.16) (taking into account the definition of λ).

Now we can write

$$\text{(4.17)} \qquad \mathscr{L}(w, \lambda) \leqq \mathscr{L}(v, \lambda) \quad \text{for any} \quad v \in V\,.$$

This (4.14) together implies that $\{w, M_n(w)\}$ is the saddle point of $\mathscr{L}$ on $V \times L_2^+(\partial\Omega)$.

Note that

$$\text{(4.18)} \qquad \int_{\partial\Omega} (v - \lambda)\, \gamma_1 w \,\mathrm{d}s \geqq 0 \quad \text{for any} \quad v \in L_2^+(\partial\Omega)\,.$$

In order to prove the uniqueness, let $\{w, \lambda\}$; $\{w^*, \lambda^*\}$ be two corresponding possible saddle points of $\mathscr{L}$ on $V \times L_2^+(\partial\Omega)$. In this case we have $\nabla^4 w^* = \tilde{f}$,

$$\text{(4.19)} \qquad \begin{aligned} \gamma_0 w^* &= 0\,, \\ M_n(w^*) &= \lambda^*\,, \end{aligned}$$

$$\gamma_1 w^* = 0\,; \quad M_n(w^*)\,\gamma_1(w^*) = 0\,,$$

$$(4.20) \qquad \int_{\partial\Omega} (v - \lambda^*)\,\gamma_1 w^*\,\mathrm{d}s \geqq 0 \quad \text{for every} \quad v \in L_2^+(\partial\Omega)\,.$$

We set $v = \lambda^*$ in the inequality (4.18) and $v = \lambda$ in (4.20). Then, adding (4.18) and (4.20), we obtain

$$(4.21) \qquad \int_{\partial\Omega} (\lambda^* - \lambda)\,(\gamma_1 w^* - \gamma_1 w)\,\mathrm{d}s \leqq 0\,.$$

If we set $y = w^* - w$, with Green's formula and (4.21) we conclude that

$$(4.22) \qquad a(y, y) = \int_{\partial\Omega} M_n(y)\,\gamma_1(y)\,\mathrm{d}s \leqq 0\,.$$

Then from V – ellipticity of the bilinear form $a(\cdot\,;\,\cdot)$ we obtain $y = 0$. Finally we have $\lambda^* - \lambda = M_n(w^* - w) = 0$, which implies $\lambda = \lambda^*$.

References

[1] *J. Cea:* Optimisation, thèorie et algorithmes, Dunod Paris, 1971.

[2] *R. Glowinski, J. L. Lions, Trémolièrs:* Analyse numérique des inéquations variationnelles. Tome 1, 2, Dunod Paris, 1976.

[3] *I. Ekeland, R. Temam:* Analyse convexe et problèmes variationnels, Dunod Paris, 1974.

[4] *J. L. Lions:* Quelques méthodes de résolution des problèmes aux limites non linéairs, Dunod Paris, 1969.

[5] *V. Mosco:* Dual variational inequalities, Jour. of Math., Anal. and Appl. 40, 202—206 (1972).

[6] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques, Academia Prague, 1967.

Súhrn

DUALITA V PREKÁŽKOVEJ A JEDNOSTRANNEJ ÚLOHE PRE BIHARMONICKÝ OPERÁTOR

JÁN LOVÍŠEK

V tejto práci je študovaný problém minimalizácie a problém maximalizácie (duálny problém) ako aj ich vzájomný vzťah pri rovnováhe tenkej dosky. Je odvodený duálny tvar variačnej nerovnosti na základe konjugovaných funkcií (v zmysle Fenchela-Rockaffelara), pre prekážku vo vnútri oblasti ako aj na hranici.

Author's address: Doc. Ing. RNDr. *Ján Lovíšek,* CSc., SVŠT, Stavebná fakulta, Radlinského 11, 884 20 Bratislava.