Alexander Ženíšek

Nonhomogeneous boundary conditions and curved triangular finite elements

Aplikace matematiky, Vol. 26 (1981), No. 2, 121–141

Persistent URL: http://dml.cz/dmlcz/103903

NONHOMOGENEOUS BOUNDARY CONDITIONS AND CURVED TRIANGULAR FINITE ELEMENTS

ALEXANDER ŽENÍŠEK

(Received May 18, 1979)

The majority of model problems for which the convergence of the finite element method has been analyzed is restricted to homogeneous Dirichlet problems (see, e.g., [3], [4], [6], [10], [11], [12]). There are only a few exceptions where nonhomogeneous Dirichlet boundary conditions have been treated (see, e.g., [1], [8], [9] where, however, only C^0-finite elements and second order elliptic equations are considered).

In this paper, both Dirichlet and Neumann nonhomogeneous boundary conditions are studied. In Section 1 the approximation of nonhomogeneous Dirichlet boundary conditions is analyzed in the case of elliptic equations of order $2m + 2$. This section is a generalization of the results presented in [12]. In Section 2 the approximation of boundary conditions is studied in the case of a mixed nonhomogeneous boundary value problem for second order elliptic equations.

The notation used in this paper is the same as in [12] and thus its explanation is omitted as far as standard symbols (e.g., derivatives $D^\alpha u$, spaces H^k, H_0^k, $W_\infty^{(k)}$, their norms, etc.) are concerned.

1. NONHOMOGENEOUS DIRICHLET PROBLEM FOR ELLIPTIC EQUATIONS OF ORDER $2m + 2$

Let Ω be a bounded and simply connected domain in the x,y-plane with a boundary Γ which is of class C^q with q sufficiently large to fulfil our requirements. We consider the following model problem

$$(1) \qquad (-1)^{m+1} \sum_{|\alpha|,|\beta|=m+1} D^\beta(a_{\alpha\beta} D^\alpha u) = f \quad \text{in} \quad \Omega\,,$$

$$(2) \qquad \left.\frac{\partial^j u}{\partial \nu^j}\right|_\Gamma = g_j \quad (j = 0, \ldots, m)$$

where ν is the outward normal to Γ and $a_{\alpha\beta}$, f, g_j sufficiently smooth functions (the smoothness will be specified later). The symbols α, β denote multiindices. We assume that there exists a constant $\mu > 0$ such that the inequality

$$\sum_{|\alpha|,|\beta|=m+1} a_{\alpha\beta}(x, y)\, \xi_\alpha \xi_\beta \geqq \mu \sum_{|\alpha|=m+1} \xi_\alpha^2 \tag{3}$$

holds for arbitrary $(x, y) \in \bar{\Omega}$ and for arbitrary values of ξ_α. Using (3) and Friedrichs' inequality we see that the bilinear form

$$a(v, w) = \sum_{|\alpha|,|\beta|=m+1} \iint_\Omega a_{\alpha\beta}(D^\alpha v)(D^\beta w)\, \mathrm{d}x\, \mathrm{d}y \tag{4}$$

is $H_0^{m+1}(\Omega)$-elliptic.

The weak solution of problem (1), (2) is a function $u \in V_g$ satisfying

$$a(u, v) = l(v) \quad \forall\, v \in V_0 = H_0^{m+1}(\Omega) \tag{5}$$

where

$$l(v) = \iint_\Omega f v\, \mathrm{d}x\, \mathrm{d}y\,, \tag{6}$$

$$V_g = \{v \in H^{m+1}(\Omega) : \partial^j v/\partial \nu^j = g_j \quad \text{on } \Gamma \text{ in the sense of traces} \quad (j = 0, \dots, m)\}\,. \tag{7}$$

We shall solve problem (1), (2) by the finite element method using curved triangular finite C^m-elements described in [12]. To this end let us triangulate the domain Ω, i.e. let us divide it into a finite number of triangles (the sides of which can be curved) in such a way that two arbitrary triangles are either disjoint, or have a common vertex, or a common side. Let the triangulation have the property that each interior triangle (i.e. a triangle having at most one point common with the boundary Γ) has straight sides and each boundary triangle has at most one curved side. Then this side lies on the boundary.

With every triangulation τ we associate two parameters h and ϑ defined by

$$h = \max_{T \in \tau} h_T\,, \quad \vartheta = \min_{T \in \tau} \vartheta_T\,, \tag{8}$$

where h_T and ϑ_T are the length of the largest side and the smallest angle, respectively, of the triangle with straight sides which has the same vertices as the triangle T. We restrict ourselves to such triangulations that ϑ is bounded away from zero as $h \to 0$, i.e.

$$\vartheta \geqq \vartheta_0\,, \quad \vartheta_0 = \text{const} > 0\,. \tag{9}$$

Let us replace the curved triangles T of the triangulation τ of Ω by the curved triangles T^* described in [12, Theorem 2] and denote the triangulation obtained in this way by τ_h. The union of the closed triangles of τ_h will be denoted by $\bar{\Omega}_h$ and the boundary of Ω_h by Γ_h. Let us note that if the curved side of a boundary triangle $T \in \tau$ has the parameteric representation

$$x = \varphi(s), \quad y = \psi(s), \quad s_2 \leqq s \leqq s_3 \tag{10}$$

then, according to [12], the parametric representation of the curved side of $T^* \in \tau_h$ is

$$x = \varphi^*(t), \quad y = \psi^*(t), \quad 0 \leqq t \leqq 1, \tag{11}$$

where $\varphi^*(t)$ and $\psi^*(t)$ are Hermite interpolation polynomials of degree $n = 2m + 1$ of the functions $\bar{\varphi}(t)$ and $\bar{\psi}(t)$, respectively. The polynomials $\varphi^*(t)$, $\psi^*(t)$ are uniquely determined by the derivatives $\bar{\varphi}^{(j)}(t_i)$, $\bar{\psi}^{(j)}(t_i)$, $j = 0, \ldots, m$; $i = 2,3$ ($t_2 = 0$, $t_3 = 1$). The functions $\bar{\varphi}(t)$, $\bar{\psi}(t)$ are defined by

$$\bar{\varphi}(t) = \varphi(s_2 + \bar{s}_{32}t), \quad \bar{\psi}(t) = \psi(s_2 + \bar{s}_{32}t) \tag{12}$$

where $\bar{s}_{32} = s_3 - s_2$.

At each vertex P_i of the triangles of the triangulation τ_h let us prescribe the parameters

$$D^\alpha w(P_i), \quad |\alpha| \leqq 2m. \tag{13}$$

At the centres of gravity P_0^T of the interior triangles T of τ_h let us prescribe the parameters

$$D^\alpha w(P_0^T), \quad |\alpha| \leqq m - 2 \tag{14}$$

and in the interiors of the boundary triangles T^* of τ_h let us prescribe the parameters

$$w(P_{0j}^T), \quad j = 1, \ldots, R \quad (R = mn(mn - 1)/2) \tag{15}$$

where $P_{01}^T, \ldots, P_{0R}^T$ are certain distinct points and $n = 2m + 1$ (for details see [12, p. 356]).

The parameters (13), (14) enable us to construct generalized Bell's C^m-elements on the interior triangles of τ_h. The parameters (13), (15) enable us to construct curved triangular C^m-elements on the boundary triangles of τ_h. (For details see [12, Section 2].)

Let W_h denote the finite dimensional subspace of $C^m(\Omega_h)$ consisting of functions which we obtain by piecing together the curved triangular finite C^m-elements just mentioned with generalized Bell's C^m-elements. Further, let

$$V_{0h} = \{w \in W_h : \partial^j w/\partial \nu_h^j = 0 \text{ on } \Gamma_h, \ j = 0, \ldots, m\} \tag{16}$$

where ν_h is the outward normal to Γ_h. Finally, let V_{gh} be the subset of W_h consisting of those functions which at the nodal points lying on Γ_h satisfy the boundary conditions (2) and all consequences of these conditions containing the derivatives of order at most $2m$. E.g., in the case $m = 1$ we have at the nodal points (i.e. vertices) on Γ_h:

$$w = g_0, \ \frac{\partial w}{\partial x} = G_1, \ \frac{\partial w}{\partial y} = G_2, \tag{17}$$

$$(18) \qquad \varphi' \frac{\partial^2 w}{\partial x^2} + \psi' \frac{\partial^2 w}{\partial x \, \partial y} = G_1' , \quad \varphi' \frac{\partial^2 w}{\partial x \, \partial y} + \psi' \frac{\partial^2 w}{\partial y^2} = G_2'$$

where prime denotes the derivative with respect to s, g_0 and g_1 are the functions from (2), φ and ψ are the functions from (10) and

$$(19) \qquad G_1 = \pm \frac{\psi'}{\varrho} g_1 + \frac{\varphi'}{\varrho^2} g_0' , \quad G_2 = \mp \frac{\varphi'}{\varrho} g_1 + \frac{\psi'}{\varrho^2} g_0'$$

with

$$\varrho = \sqrt{[(\varphi')^2 + (\psi')^2]} .$$

If $v = (\psi'/\varrho, -\varphi'/\varrho)$ then we take the upper sign in (19), if $v = (-\psi'/\varrho, \varphi'/\varrho)$ then we take the lower sign. It should be noted that the relation $w'' = g_0''$ is a linear combination of (17) and (18).

According to [12, Lemma 2], we have

$$(20) \qquad v, w \in V_{gh} \Rightarrow v - w \in V_{0h} .$$

Now we can define the discrete problem for solving approximately problem (1), (2): Find $\tilde{u}_h \in V_{gh}$ such that

$$(21) \qquad \tilde{a}_h(\tilde{u}_h, v) = \tilde{l}_h(v) \quad \forall v \in V_{0h}$$

where

$$(22) \qquad \tilde{a}_h(v, w) = \sum_{|\alpha|,|\beta|=m+1} \int\int_{\Omega_h} \tilde{a}_{\alpha\beta}(D^\alpha v)(D^\beta w) \, dx \, dy$$

$$(23) \qquad \tilde{l}_h(v) = \int\int_{\Omega_h} \tilde{f} v \, dx \, dy .$$

The symbols $\tilde{a}_{\alpha\beta}$ denote continuous extensions of the functions $a_{\alpha\beta}$ to the plane E_2. The continuity of $\tilde{a}_{\alpha\beta}$ and inequality (3) imply the existence of a domain $\tilde{\Omega} \supset \Omega$ and of a constant $\tilde{\mu} > 0$ (dependent on $\tilde{\Omega}$) such that the inequality

$$(24) \qquad \sum_{|\alpha|,|\beta|=m+1} \tilde{a}_{\alpha\beta}(x, y) \, \xi_\alpha \xi_\beta \geqq \tilde{\mu} \sum_{|\alpha|=m+1} \xi_\alpha^2$$

holds for arbitrary $(x, y) \in \tilde{\Omega}$ and for arbitrary values of ξ_α.

Having established $\tilde{\Omega} \supset \Omega$ we can find $\tilde{h}$ (dependent on $\tilde{\Omega}$) such that

$$(25) \qquad \tilde{\Omega} \supset \Omega_h \quad \forall h < \tilde{h} .$$

Thus (24) holds for arbitrary $(x, y) \in \Omega_h$, $h < \tilde{h}$.

The symbol $\tilde{f}$ denotes an extension of the function f and will be specified in (32).

Finally, using quadrature formulas with integration points lying in $\bar{\Omega}$ we replace the forms $\tilde{a}_h(v, w)$ and $\tilde{l}_h(v)$ in the same way as in [12, p. 365] by the forms $a_h(v, w)$ and $l_h(v)$, respectively, and solve the following problem instead of problem (21):

Find $u_h \in V_{gh}$ such that

$$a_h(u_h, v) = l_h(v) \quad \forall v \in V_{0h}\,. \tag{26}$$

The estimate of the rate of convergence is based on the following abstract error theorem which is a modification of similar theorems from [3], [4]:

Theorem 1. *Let a family of discrete problems* (26) *be given and let* (25) *hold. Let there exist a constant $\gamma > 0$ independent on h such that for $h < \tilde{h}$ we have*

$$\gamma\|v\|^2_{m+1,\Omega_h} \leqq a_h(v, v) \quad \forall v \in V_{0h}\,. \tag{27}$$

Then for $h < \tilde{h}$ every problem (26) *has a unique solution u_h and*

$$\|\tilde{u} - u_h\|_{m+1,\Omega_h} \leqq C\left[\sup_{w\in V_{0h}} \frac{|\tilde{a}_h(\tilde{u}, w) - l_h(w)|}{\|w\|_{m+1,\Omega_h}} + \inf_{v\in V_{gh}}\left\{\|\tilde{u} - v\|_{m+1,\Omega_h} + \right.\right.$$
$$\left.\left. + \sup_{w\in V_{0h}} \frac{|\tilde{a}_h(v, w) - a_h(v, w)|}{\|w\|_{m+1,\Omega_h}}\right\}\right], \tag{28}$$

where $\tilde{u}$ is an arbitrary function in $H^{m+1}(\tilde{\Omega})$ and C is a constant independent on $\tilde{u}$ and h.

Proof. Assumption (27) implies that for $h < \tilde{h}$ every problem (26) has a unique solution u_h.

Let $v \in V_{gh}$ be an arbitrary function. Then, according to (20), $\bar{w} = u_h - v \in V_{0h}$ and relations (26), (27) imply

$$\gamma\|\bar{w}\|^2_{m+1,\Omega_h} \leqq a_h(\bar{w}, \bar{w}) = l_h(\bar{w}) - a_h(v, \bar{w}) +$$
$$+ [\tilde{a}_h(\tilde{u} - v, \bar{w}) - \tilde{a}_h(\tilde{u}, \bar{w}) + \tilde{a}_h(v, \bar{w})]\,. \tag{29}$$

The continuity of the functions $\tilde{a}_{\alpha\beta}$ in the domain $\tilde{\Omega}$ and inclusions (25) show that in the case $h < \tilde{h}$ there exists a constant $\tilde{M}$ independent on h such that

$$|\tilde{a}_h(v, w)| \leqq \tilde{M}\|v\|_{m+1,\Omega_h}\|w\|_{m+1,\Omega_h} \quad \forall v, w \in H^{m+1}(\Omega_h)\,.$$

Using this inequality we obtain from (29)

$$\|\bar{w}\|_{m+1,\Omega_h} \leqq \frac{1}{\gamma}\sup_{w\in V_{0h}} \frac{|\tilde{a}_h(\tilde{u}, w) - l_h(w)|}{\|w\|_{m+1,\Omega_h}} +$$
$$+ \frac{\tilde{M}}{\gamma}\|\tilde{u} - v\|_{m+1,\Omega_h} + \frac{1}{\gamma}\sup_{w\in V_{0h}} \frac{|\tilde{a}_h(v, w) - a_h(v, w)|}{\|w\|_{m+1,\Omega_h}}\,. \tag{30}$$

Combining (30) with the triangular inequality

$$\|\tilde{u} - u_h\|_{m+1,\Omega_h} \leqq \|\tilde{u} - v\|_{m+1,\Omega_h} + \|\bar{w}\|_{m+1,\Omega_h}$$

and taking the infimum with respect to $v \in V_{gh}$ we obtain inequality (28). Theorem 1 is proved.

In what follows the function $\tilde{u}$ will be a continuous extension of the exact solution of problem (1), (2) to the domain $\tilde{\Omega}$. In this case the first term on the right-hand side of (28) can be rewritten: Assumptions (36), (37) of Theorem 3 about the functions $\tilde{a}_{\alpha\beta}$, $\tilde{u}$ allow us to use Green's theorem and find

$$\tilde{a}_h(\tilde{u}, w) = \tilde{l}_h(w) \equiv \int\int_{\Omega_h} \tilde{f} w \, \mathrm{d}x \, \mathrm{d}y \quad (w \in V_{0h}) \tag{31}$$

with

$$\tilde{f} = (-1)^{m+1} \sum_{|\alpha|, |\beta| = m+1} D^\beta(\tilde{a}_{\alpha\beta} D^\alpha \tilde{u}) \,. \tag{32}$$

Equation (32) defines an extension of the function f because for $(x, y) \in \Omega$ the right-hand side of (32) is equal to f.

If we do not use numerical integration then condition (27) takes the form

$$K \|v\|^2_{m+1, \Omega_h} \leqq \tilde{a}_h(v, v) \quad \forall v \in V_{0h} \,, \quad h < \tilde{h} \tag{33}$$

where the constant $K > 0$ is independent on h and inequality (28) reduces to the inequality

$$\|\tilde{u} - \tilde{u}_h\|_{m+1, \Omega_h} \leqq C \inf_{v \in V_{gh}} \|\tilde{u} - v\|_{m+1, \Omega_h} \tag{34}$$

which is a generalization of a similar inequality derived for second order nonhomogeneous Dirichlet problems in the case of polygonal domains by Strang [9]. Let us note that inequality (33) follows immediately from (24) by means of Friedrichs' inequality (121). (The independence of the constant K on h is a consequence of the fact that in the case of $V_{0h} \subset H^1_0(\Omega_h)$ we can set $C = b^2$ in (121) where b is the length of the side of a square containing $\tilde{\Omega}$ – see [7, pp. 13–14].)

It follows from the construction of V_{gh} that the interpolate $\Pi\tilde{u}$ of the function $\tilde{u}$ belongs to V_{gh}. Thus, setting $v = \Pi\tilde{u}$ and using the interpolation theorem from [12] we obtain:

Theorem 2. *Let $\tilde{u} \in H^{3m+2}(\tilde{\Omega})$. Then*

$$\inf_{v \in V_{gh}} \|\tilde{u} - v\|_{m+1, \Omega_h} \leqq C h^{2m+1} \|\tilde{u}\|_{3m+2, \tilde{\Omega}} \tag{35}$$

where C is a constant independent on h and $\tilde{u}$.

Thus owing to a sufficiently smooth approximation of the boundary Γ by the curve Γ_h consisting of arcs (11) the problem of convergence of a finite element procedure using curved triangular finite C^m-elements reduces also in the case of nonhomogeneous boundary conditions to an interpolation problem and to an analysis of the effect of numerical integration.

Inspecting the proofs of [12, Theorems 7, 8, 9] we see that they are valid also in the case of boundary conditions (2) (i.e. for $v, w \in V_{0h}$ in the case of Theorem 7 and for $v \in V_{gh}$, $w \in V_{0h}$ in the case of Theorems 8, 9). Thus the following theorem can be proved in the same way as [12, Theorem 10]:

Theorem 3. *Let inequality* (3) *hold for arbitrary* $(x, y) \in \bar{\Omega}$ *and for arbitrary values of* ξ_α. *Let*

$$\tilde{u} \in H^{3m+2}(\tilde{\Omega}), \tag{36}$$

$$\tilde{a}_{\alpha\beta} \in W_\infty^{(2m^2+3m+1)}(\tilde{\Omega}), \quad |\alpha| = |\beta| = m + 1, \tag{37}$$

$$\tilde{f} \in H^{2m+1}(\tilde{\Omega}) \tag{38}$$

where $\tilde{u}$ *is an extension of the solution of problem* (1), (2) *to the domain* $\tilde{\Omega}$, $\tilde{a}_{\alpha\beta}$ *are extensions of the coefficients* $a_{\alpha\beta}$ *to* $\tilde{\Omega}$ *and* $\tilde{f}$ *is defined by* (32). *Let the degree of arcs from which* Γ_h *consists be equal to* $2m + 1$. *Let the numerical quadrature scheme over the unit triangle* T_0 *be of degree of precision* $2(n + 2)\, m$,

$$E^*(p^*) = 0 \quad \forall p^* \in P(2(n + 2)\, m), \tag{39}$$

with $n = 1$ *for generalized Bell's* C^m*-elements and* $n = 2m + 1$ *for curved triangular finite* C^m*-elements. Then for sufficiently small* h *the solution* u_h *of the discrete problem* (26) *exists and is unique and the following estimate holds*

$$\|\tilde{u} - u_h\|_{m+1,\Omega_h} \leqq C h^{2m+1} [\|\tilde{f}\|_{2m+1,\tilde{\Omega}} + \|\tilde{u}\|_{3m+2,\tilde{\Omega}} (1 + \sum_{|\alpha|,|\beta|=m+1} \|\tilde{a}_{\alpha\beta}\|_{2m+1,\infty,\tilde{\Omega}})], \tag{40}$$

where C *is a constant independent on* h, $\tilde{u}$ *and* $\tilde{a}_{\alpha\beta}$.

Remark 1. Instead of (36) it suffices to assume that $u \in H^{3m+2}(\Omega)$. Then, according to Calderon's extension theorem, there exists an extension $\tilde{u}$ of u for which $\tilde{u} \in H^{3m+2}(\tilde{\Omega})$, $\|\tilde{u}\|_{3m+2,\tilde{\Omega}} \leqq C\|u\|_{3m+2,\Omega}$.

Remark 2. The error functional $E^*(p^*)$ appearing in (39) is defined in [12, p. 368].

2. NONHOMOGENEOUS MIXED BOUNDARY VALUE PROBLEM FOR SECOND ORDER EQUATIONS

In Section 1 we considered only main (stable) boundary conditions. In this section we study natural boundary conditions. Their approximation and analysis is different from the approximation and analysis of Dirichlet boundary conditions. For simplicity, we restrict ourselves to second order elliptic equations and consider the following model problem:

$$-\frac{\partial}{\partial x}\left(k_1 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_2 \frac{\partial u}{\partial y}\right) = f \text{ in } \Omega, \tag{41}$$

$$u|_{\Gamma_1} = g, \quad \text{mes } \Gamma_1 > 0, \tag{42}$$

$$k_1 \frac{\partial u}{\partial x} v_1 + k_2 \frac{\partial u}{\partial y} v_2 |_{\Gamma_2} = Q \,, \tag{43}$$

where

$$k_i(x, y) \geqq \mu > 0 \,, \quad i = 1, 2, \quad (x, y) \in \bar{\Omega} \,. \tag{44}$$

The domain Ω satisfies the same assumptions as in Section 1, the symbols Γ_1, Γ_2 denote disjoint parts of the boundary Γ of Ω; it holds $\Gamma = \Gamma_1 + \Gamma_2$. The vector (v_1, v_2) is the unit vector of the outward normal to the curve Γ_2 and k_1, k_2, f, g, Q are sufficiently smooth functions (their smoothness will be specified later).

Equation (41) is a special case of equation (1) for $m = 0$. Inequalities (44) imply that inequality (3) is satisfied. In this case the bilinear form (4) takes the form

$$a(u, v) = \iint_\Omega \left(k_1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + k_2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y \tag{45}$$

and is V_0-elliptic where

$$V_0 = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1 \text{ in the sense of traces} \} \,. \tag{46}$$

The weak solution of problem (41)–(43) is a function $u \in V_g$ satisfying

$$a(u, v) = l(v) \quad \forall v \in V_0 \,, \tag{47}$$

where

$$l(v) = l^\Omega(v) + l^\Gamma(v) \,, \tag{48}$$

$$l^\Omega(v) = \iint_\Omega f v \, \mathrm{d}x \, \mathrm{d}y \,, \quad l^\Gamma(v) = \int_{\Gamma_2} Q v \, \mathrm{d}s \,, \tag{49}$$

$$V_g = \{ v \in H^1(\Omega) : v = g \text{ on } \Gamma_1 \text{ in the sense of traces} \} \,. \tag{50}$$

We shall solve problem (41)–(43) by the finite element method: Let us approximate the domain Ω by a domain Ω_h in the same way as in Section 1. Let the functions (11) be now Hermite interpolation polynomials of degree $2k + 1$ of functions (12) where k is a given integer. (In the case of second order problems we usually choose $k = 1$.) Then on the interior triangles of the triangulation τ_h we shall use Koukal's polynomials of degree $2k + 1$ [5, Theorem 5] which are uniquely determined by the parameters

$$D^\alpha w(P_i) \,, \quad |\alpha| \leqq k \quad (i = 1, 2, 3) \tag{51}$$

$$D^\alpha w(P_0) \,, \quad |\alpha| \leqq k - 1 \tag{52}$$

where P_1, P_2, P_3 are vertices of a triangle and P_0 its centre of gravity denoted in a local notation.

On the curved triangles T^* of the triangulation τ_h we shall use Zlámal's curved triangular finite C^0-elements [11]. These finite elements are uniquely determined by parameters (51), (52) where P_1, P_2, P_3 denote the vertices of T^* in a local notation and P_0 is the image of the point R_0 in transformation [12, (23)]. R_0 is the centre of gravity of the triangle T_0 which lies in the ξ,η-plane and has the vertices $R_1(0, 0)$, $R_2(1, 0)$, $R_3(0, 1)$.

In this section the symbol W_h will denote the finite dimensional subspace of $C^0(\Omega_h)$ consisting of functions which we obtain by piecing together Koukal's and Zlámal's C^0-elements. Further, we define the space V_{0h} by

$$V_{0h} = \{w \in W_h : w = 0 \text{ on } \Gamma_{h1}\} \tag{53}$$

where Γ_{h1} is the part of Γ_h approximating Γ_1. Finally, let V_{gh} be the subset of W_h consisting of functions which at the nodal points lying on Γ_{h1} satisfy the boundary condition (42) and all consequences of this condition containing the derivatives of order at most k. E.g., in the case $k = 2$ we have at the nodal points (i.e. vertices) on Γ_{h1}:

$$w = g\,, \quad \varphi' \frac{\partial w}{\partial x} + \psi' \frac{\partial w}{\partial y} = g'\,, \tag{54}$$

$$(\varphi')^2 \frac{\partial^2 w}{\partial x^2} + 2\varphi'\psi' \frac{\partial^2 w}{\partial x\, \partial y} + (\psi')^2 \frac{\partial^2 w}{\partial y^2} + \varphi'' \frac{\partial w}{\partial x} + \psi'' \frac{\partial w}{\partial y} = g'' \tag{55}$$

where $\varphi(s)$, $\psi(s)$ are the functions from (10) and the prime denotes the derivative with respect to s.

The relations of the type (54), (55) indicate how to specify the smoothness of the function g: We assume $g(x, y) \in C^k(U)$ where U is a domain containing the curve Γ_1.

Let us note that implication (20) holds where the symbols V_{gh}, V_{0h} have the meaning defined in this section and where v, w are arbitrary functions from V_{gh}.

Now we can define the discrete problem for solving approximately problem (41)–(43): Find $\tilde{u}_h \in V_{gh}$ such that

$$\tilde{a}_h(\tilde{u}_h, v) = \tilde{l}_h(v) \quad \forall v \in V_{0h} \tag{56}$$

where

$$\tilde{a}_h(v, w) = \iint_{\Omega_h} \left(\tilde{k}_1 \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + \tilde{k}_2 \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y\,, \tag{57}$$

$$\tilde{l}_h(v) = \tilde{l}_h^{\Omega}(v) + \tilde{l}_h^{\Gamma}(v)\,, \tag{58}$$

$$\tilde{l}_h^{\Omega}(v) = \iint_{\Omega_h} \tilde{f} v \, \mathrm{d}x\, \mathrm{d}y\,, \tag{59}$$

$$\tilde{l}_h^{\Gamma}(v) = \int_{\Gamma_{h2}} Q_h v \, \mathrm{d}s \tag{60}$$

where $\Gamma_{h2} = \Gamma_h - \Gamma_{h1}$. The symbols $\tilde{k}_i$ denote continuous extensions of the functions k_i to the plane E_2. Similarly as in Section 1, using the continuity of $\tilde{k}_i$ we can establish a domain $\tilde{\Omega} \supset \Omega$ and find $\tilde{\mu} > 0$ and $\tilde{h}$ (dependent on $\tilde{\Omega}$) such that (24) and (25) hold.

The symbol $\tilde{f}$ denotes an extension of the function f to the domain $\tilde{\Omega}$ and will be specified in (68).

The symbol Q_h denotes the function which we obtain by "transferring" the function Q from the curve Γ_2 onto the curve Γ_{h2}: Let $c(P_2, P_3)$ be an arc lying on Γ_2 which has the parametric representation (10); P_2 and P_3 are its end points denoted in a local notation. Let $c_h(P_2, P_3) \subset \Gamma_{h2}$ be the approximation of $c(P_2, P_3)$. Let

$$(61) \qquad (x, y) \equiv (\varphi^*(t), \psi^*(t)) \in c_h(P_2, P_3), \quad 0 \leqq t \leqq 1$$

where $x = \varphi^*(t)$, $y = \psi^*(t)$ is the parametric representation of $c_h(P_2, P_3)$ (cf. (11)). Then we set

$$(62) \qquad Q_h(x, y) = Q(\varphi(s_2 + \bar{s}_{32}t), \psi(s_2 + \bar{s}_{32}t)) \equiv Q(\bar{\varphi}(t), \bar{\psi}(t)).$$

According to the definition of the line integral, we have

$$(63) \qquad \int_{c_h(P_2,P_3)} Q_h w \, \mathrm{d}s = \int_0^1 Q(\bar{\varphi}(t), \bar{\psi}(t)) \, w(\varphi^*(t), \psi^*(t)) \, \varrho^*(t) \, \mathrm{d}t,$$

where the functions $\bar{\varphi}(t)$, $\bar{\psi}(t)$ are defined in (12) and where

$$(64) \qquad \varrho^*(t) = \sqrt{\{[\varphi^{*\prime}(t)]^2 + [\psi^{*\prime}(t)]^2\}}.$$

Using quadrature formulas with integration points lying in $\bar{\Omega}$ we replace the forms $\tilde{a}_h(v, w)$ and $\tilde{l}_h^{\Omega}(v)$ in the same way as in [12, p. 365] by the forms $a_h(v, w)$ and $l_h^{\Omega}(v)$, respectively. Further, computing numerically the integral on the right-hand side of (63) for each $c_h \subset \Gamma_{h2}$ we obtain a linear form $l_h^{\Gamma}(v)$. We solve the following problem instead of problem (56): Find $u_h \in V_{gh}$ such that

$$(65) \qquad a_h(u_h, v) = l_h(v) \quad \forall v \in V_{0h},$$

where

$$(66) \qquad l_h(v) = l_h^{\Omega}(v) + l_h^{\Gamma}(v).$$

The estimate of the rate of convergence is based again on Theorem 1, where $m = 0$ and where V_{gh}, V_{0h}, $\tilde{a}_h(v, w)$, $a_h(v, w)$ and $l_h(w)$ have the meaning introduced in this section.

In what follows the function $\tilde{u}$ from Theorem 1 will be a continuous extension of the exact solution u of problem (41)–(43) to the domain $\tilde{\Omega}$. In this case the first term on the right-hand side of (28) can be rewritten: Assumptions (70), (71) of Theorem 4 about the functions $\tilde{u}$, $\tilde{k}_i$ allow us to use Green's theorem and find

$$(67) \quad \tilde{a}_h(\tilde{u}, w) = \iint_{\Omega_h} \tilde{f} w \, \mathrm{d}x \, \mathrm{d}y + \int_{\Gamma_{h2}} \left(\tilde{k}_1 \frac{\partial \tilde{u}}{\partial x} \nu_{h1} + \tilde{k}_2 \frac{\partial \tilde{u}}{\partial y} \nu_{h2} \right) w \, \mathrm{d}s \quad (w \in V_{0h}),$$

where ν_{h1}, ν_{h2} are the direction cosines of the outward normal to the curve Γ_{h2} and where

(68) $$\tilde{f} = -\frac{\partial}{\partial x}\left(\tilde{k}_1 \frac{\partial \tilde{u}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\tilde{k}_2 \frac{\partial \tilde{u}}{\partial y}\right).$$

Equation (68) defines an extension of the function f because for $(x, y) \in \Omega$ the right-hand side of (68) is equal to f – cf. (41).

Using (59), (66) and (67) we can write

(69) $$\frac{|\tilde{a}_h(\tilde{u}, w) - l_h(w)|}{\|w\|_{1,\Omega_h}} \leqq \frac{|\tilde{l}_h^\Omega(w) - l_h^\Omega(w)|}{\|w\|_{1,\Omega_h}} + \\ + \frac{\left|\int_{\Gamma_{h2}} \left(\tilde{k}_1 \frac{\partial \tilde{u}}{\partial x} \nu_{h1} + \tilde{k}_2 \frac{\partial \tilde{u}}{\partial y} \nu_{h2}\right) w \, ds - l_h^\Gamma(w)\right|}{\|w\|_{1,\Omega_h}}.$$

The following theorem is a consequence of [12, Theorems 8, 9] with $r = N^* = 2k + 1$, $m = 0$ and of the interpolation theorems for Koukal's and Zlámal's C^0-elements.

Theorem 4. *Let*

(70) $$\tilde{u} \in H^{2k+2}(\tilde{\Omega}),$$

(71) $$\tilde{k}_i \in W_\infty^{(2k+1)}(\tilde{\Omega}), \quad i = 1, 2,$$

(72) $$\tilde{f} \in H^{2k+1}(\tilde{\Omega}),$$

where $\tilde{u}$ is an extension of the solution u of problem (41)–(43) to the domain $\tilde{\Omega}$, $\tilde{k}_i$ are extensions of the coefficients k_i to $\tilde{\Omega}$ and the function $\tilde{f}$ is defined by (68). Let the degree of arcs (11) from which Γ_h consists be equal to $2k + 1$. Let the numerical quadrature scheme over the unit triangle T_0 be of degree of precision $4k$, i.e.,

(73) $$E^*(p^*) = 0 \quad \forall p^* \in P(4k)$$

for both Koukal's and Zlámal's C^0-elements. Then for $h < \tilde{h}$ the sum of the second and third terms on the right-hand side of (28) (with $m = 0$) and of the first term on the right-hand side of (69) is bounded by

(74) $$Ch^{2k+1}\left[\|\tilde{f}\|_{2k+1,\tilde{\Omega}} + \|\tilde{u}\|_{2k+2,\tilde{\Omega}}\left(1 + \sum_{i=1}^{2} \|\tilde{k}_i\|_{2k+1,\infty,\tilde{\Omega}}\right)\right],$$

where C is a constant independent on h, $\tilde{u}$ and $\tilde{k}_i$.

It remains to estimate the second term on the right-hand side of (69) and to find a sufficient condition for the validity of inequality (27). In solving the first problem we start from the inequality

(75) $$\left|\int_{\Gamma_{h2}} \left(\tilde{k}_1 \frac{\partial \tilde{u}}{\partial x} \nu_{h1} + \tilde{k}_2 \frac{\partial \tilde{u}}{\partial y} \nu_{h2}\right) w \, ds - l_h^\Gamma(w)\right| \leqq$$

$$\leqq \left| \int_{\Gamma_{h2}} \left(\tilde{k}_1 \frac{\partial \tilde{u}}{\partial x} v_{h1} + \tilde{k}_2 \frac{\partial \tilde{u}}{\partial y} v_{h2} - Q_h \right) w \, \mathrm{d}s \right| + \left| \int_{\Gamma_{h2}} Q_h w \, \mathrm{d}s - l_h^\Gamma(w) \right| .$$

The first term on the right-hand side of (75) depends on the error of approximation of the curve Γ_2 by Γ_{h2}, the second term divided by $\|w\|_{1,\Omega_h}$ is less or equal to the error of numerical integration on Γ_{h2}. Both terms are estimated in Theorem 5. Before formulating and proving Theorem 5 we must make some notes on numerical integration on Γ_{h2} and establish some lemmas.

Let us have at our disposal a numerical quadrature scheme over the segment [0, 1]

$$\int_0^1 G^*(t) \, \mathrm{d}t \doteq \sum_{j=1}^{J} \omega_j^* \, G^*(t_j) , \tag{76}$$

where ω_j^* are the coefficients and t_j the integration points of the formula. According to the definition of the line integral, we have

$$\int_{c_h} F(x, y) \, \mathrm{d}s = \int_0^1 F(\varphi^*(t), \psi^*(t)) \, \varrho^*(t) \, \mathrm{d}t = \int_0^1 F^*(t) \, \varrho^*(t) \, \mathrm{d}t , \tag{77}$$

where the function $\varrho^*(t)$ is defined by (64). Relations (76) and (77) imply

$$\int_{c_h} F(x, y) \, \mathrm{d}s \doteq \sum_{j=1}^{J} \omega_{j,c_h} \, F(B_{j,c_h}) \tag{78}$$

with

$$\omega_{j,c_h} = \omega_j^* \varrho^*(t_j) , \quad B_{j,c_h} = (\varphi^*(t_j), \psi^*(t_j)) . \tag{79}$$

Both ω_{j,c_h} and B_{j,c_h} depend on $\varphi^*(t)$, $\psi^*(t)$ and thus on c_h. As the curve Γ_{h2} is a union of arcs c_h the linear form $l_h^\Gamma(w)$ is of the form

$$l_h^\Gamma(w) = \sum_{c_h} \sum_{j=1}^{J} \omega_{j,c_h} \, Q_h(B_{j,c_h}) \, w(B_{j,c_h}) . \tag{80}$$

Let us define the error functionals

$$E_{c_h}(F) = \int_{c_h} F(x, y) \, \mathrm{d}s - \sum_{j=1}^{J} \omega_{j,c_h} \, F(B_{j,c_h}) , \tag{81}$$

$$E_I^*(F^*) = \int_0^1 F^*(t) \, \mathrm{d}t - \sum_{j=1}^{J} \omega_j^* \, F^*(t_j) . \tag{82}$$

According to (76)–(79), the following identity holds:

$$E_{c_h}(F) = E_I^*(F^* \varrho^*) . \tag{83}$$

With respect to (83) we have

$$E_{c_h}(Q_h w) = E_I^*(Q_h^* w^* \varrho^*) \tag{84}$$

where, according to (62), (63),

(85) $$Q_h^*(t) = Q(\bar{\varphi}(t), \bar{\psi}(t)),$$

(86) $$w^*(t) = w(\varphi^*(t), \psi^*(t)).$$

The functions $\bar{\varphi}(t), \bar{\psi}(t)$ are given by (12). It follows from the construction of the curved triangular finite elements (see [10], [11], [12]) that the function $w^*(t)$ is a polynomial of degree $2k + 1$ in one variable t.

Lemma 1. *Let r be a given integer and $I = [0, 1]$. There exists a constant C independent on $v^* \in P(r)$ such that*

(87) $$\max_I |v^{*(j)}| \leqq C|v^*|_{j,I}, \quad j \geqq 0 \quad \forall v^* \in P(r),$$

(88) $$|v^*|_{j,I} \leqq C|v^*|_{i,I}, \quad 0 \leqq i \leqq j \quad \forall v^* \in P(r),$$

$P(r)$ being the space of all polynomials of degree not greater than r.

Relations (87), (88) are one-dimensional analogies of relations (24), (25) from [11].

Lemma 2. *Let the boundary Γ be of class C^{2k+2}. Let the functions $\varphi^*(t)$, $\psi^*(t)$ defining the arc $c_h(P_2, P_3) \subset \Gamma_{h2}$ be Hermite interpolation polynomials of degree $2k + 1$ of the functions $\bar{\varphi}(t), \bar{\psi}(t)$ (see (12)) on the interval $I = [0, 1]$. If h is sufficiently small then the following estimates hold:*

(89) $$a_1 h_T \leqq \bar{s}_{32} \leqq a_2 h_T, \quad a_i = \text{const} > 0,$$

(90a) $$|\varphi^{*(j)}(t) - \bar{\varphi}^{(j)}(t)| \leqq C h_T^{2k+2}, \quad j = 0, 1, \ldots, 2k + 1,$$

(90b) $$|\psi^{*(j)}(t) - \bar{\psi}^{(j)}(t)| \leqq C h_T^{2k+2}, \quad j = 0, 1, \ldots, 2k + 1,$$

(91) $$|\varphi^{*(j)}(t)| \leqq C h_T^j, \quad |\psi^{*(j)}(t)| \leqq C h_T^j, \quad j = 1, 2, \ldots,$$

(92) $$|\varrho^{*(j)}(t)| \leqq C^* h_T^{j+1}, \quad j = 0, 1, \ldots,$$

where $\bar{s}_{32} = s_3 - s_2$, the function $\varrho^*(t)$ is defined by (64), the constants a_1, a_2, C depend only on Γ and the constant C^* depend on Γ and j.

Proof. If Γ is of class C^{2k+2} then there exists a parametric representation of Γ

$$x = \varphi(s), \quad y = \psi(s), \quad s \in [A, B]$$

such that $\varphi \in C^{2k+2}(A, B)$, $\psi \in C^{2k+2}(A, B)$ and

(93) $$|\varphi^{(j)}(s)| \leqq M, \quad |\psi^{(j)}(s)| \leqq M, \quad s \in [A, B],$$
$$j = 0, 1, \ldots, 2k + 2,$$

where M is a constant. The segment $[A, B]$ can be divided into a finite number of segments $[A_i, B_i]$ such that at least one of inequalities

(94) $$|\varphi'(s)| \geqq \beta > 0, \quad |\psi'(s)| \geqq \beta > 0, \quad s \in [A_i, B_i]$$

holds, where β is a constant. As in [12] the triangulation is chosen in such a way that each segment $[s_2, s_3]$ (a local notation) is a subsegment of a certain segment $[A_i, B_i]$.

First we prove inequalities (89), We have

$$\text{mes } c(P_2, P_3) = \int_{s_2}^{s_3} \sqrt{[(\varphi')^2 + (\psi')^2]} \, \mathrm{d}s \,. \tag{95}$$

It follows from (9) and from the sine theorem that the length of the smallest side of the triangle $P_1P_2P_3$ is greater than or equal to $h_T \sin \vartheta_0$. Thus $h_T \sin \vartheta_0 \leqq$ mes $c(P_2, P_3)$. The first inequality (89) then follows from (93) and (95).

If h is sufficiently small then there exists a constant $K > 0$ independent on the triangulation τ of Ω such that $Kh_T \geqq$ mes $c(P_2, P_3)$. The second inequality (89) then follows from (94) and (95).

As mes $I = 1$ we obtain from the remainder theorem for the Hermite interpolation

$$\left|\varphi^{*(j)}(t) - \bar{\varphi}^{(j)}(t)\right| \leqq C_1 \max_I \left|\bar{\varphi}^{(2k+2)}(t)\right| ,$$

where $j = 0, 1, \ldots, 2k + 1$. According to (12) and (93), we have $\max_I \left|\bar{\varphi}^{(2k+2)}(t)\right| \leqq \leqq M\bar{s}_{32}^{2k+2}$ and (90a) follows from (89). Estimate (90b) can be proved in the same way.

Relations (12), (89), (93) imply $\left|\bar{\varphi}^{(j)}(t)\right| \leqq Ch_T^j$, $\left|\bar{\psi}^{(j)}(t)\right| \leqq Ch_T^j$, $j = 0, 1, \ldots, 2k + 2$. Hence $\left|\varphi^{*(j)}(t)\right| \leqq Ch_T^j + \left|\bar{\varphi}^{(j)}(t) - \varphi^{*(j)}(t)\right|$ and (90a) implies the first inequality (91) for $j = 1, \ldots, 2k + 1$. In the case $j \geqq 2k + 2$ this inequality is satisfied automatically. The second inequality (91) can be proved in the same way.

Estimate (92) can be obtained by differentiating relation (64) and using (91) and the relation

$$\varrho^*(t) \equiv \sqrt{\{[\varphi^{*\prime}(t)]^2 + [\psi^{*\prime}(t)]^2\}} \geqq Ch_T \,. \tag{96}$$

Inequality (96) follows from (90) and (94): If the first inequality (94) holds then $\left|\bar{\varphi}'(t)\right| \geqq \bar{s}_{32}\beta > 0$. Then, according to (89), $a_1\beta h_T \leqq \left|\varphi^{*\prime}(t)\right| + \left|\varphi^{*\prime}(t) - \bar{\varphi}'(t)\right|$. This implies, with respect to (90a), $\left|\varphi^{*\prime}(t)\right| \geqq Ch_T$ for sufficiently small h and (96) follows. Lemma 2 is proved.

Lemma 3. *If h is sufficiently small then*

$$\left|w^*\right|_{0,I} \leqq Ch_T^{-1/2}\left|w\right|_{0,c_h} \tag{97}$$

where the constant C depends only on Γ.

Proof. According to (96), we have

$$\left|w\right|_{0,c_h}^2 = \int_{c_h} w^2 \, \mathrm{d}s = \int_0^1 (w^*)^2 \, \varrho^* \, \mathrm{d}t \geqq Ch_T\left|w^*\right|_{0,I}^2$$

and (97) follows.

Lemma 4. *Let the boundary Γ of Ω be of class C^{2k+2}. If h is sufficiently small then*

(98) $$\int_{\Delta\Gamma_h} w^2 \, \mathrm{d}s \leqq C\|w\|^2_{1,\Omega_h} \quad \forall w \in H^1(\Omega_h)\,,$$

where $\Delta\Gamma_h$ is an arbitrary part of the boundary Γ_h of Ω_h and C is a constant depending only on $\Delta\Gamma$, i.e. on the part of Γ which is approximated by $\Delta\Gamma_h$.

Inspecting the proof of the trace theorem (see [7, pp. 15–16]) we see that the independence of the constant C on h follows from (90).

Theorem 5. *Let the part Γ_2 of the boundary Γ of Ω be of class C^{2k+2}, let the extension $\tilde{u}$ of the solution of problem (41)–(43) be twice continuously differentiable on $\tilde{\Omega}$ with derivatives bounded by a constant K_2,*

$$|D^\alpha \tilde{u}(x, y)| \leqq K_2\,, \quad |\alpha| \leqq 2\,, \quad (x, y) \in \tilde{\Omega}\,,$$

let the extensions $\tilde{k}_1$, $\tilde{k}_2$ of the functions k_1, k_2 be once continuously differentiable on $\tilde{\Omega}$ with derivatives bounded by a constant K_1,

$$|D^\alpha \tilde{k}_i(x, y)| \leqq K_1\,, \quad |\alpha| \leqq 1\,, \quad (x, y) \in \tilde{\Omega}\,, \quad i = 1, 2,$$

and let the function $Q(x, y)$ belong to the space $C^{2k+1}(U)$ where U is a domain containing Γ_2. Let the functions $\varphi^*(t)$, $\psi^*(t)$ defining the arcs c_h of Γ_h be Hermite interpolation polynomials of degree $2k+1$ of the functions $\bar{\varphi}(t)$, $\bar{\psi}(t)$. In computing the integrals (77) let us use a quadrature formula of degree of precision $4k + 1$, i.e. let

(99) $$E_I^*(v^*) = 0 \quad \forall v^* \in P(4k + 1)\,.$$

Then for sufficiently small h the second term on the right-hand side of (69) is bounded by

(100) $$Ch^{2k+1}\,,$$

where C is a constant independent on h, $C = C(K_1, K_2, Q, \Gamma_2)$.

P r o o f. a) First we estimate the first term on the right-hand side of (75). Let us denote for simplicity

(101) $$\sigma = \tilde{k}_1 \frac{\partial \tilde{u}}{\partial x} \nu_{h1} + \tilde{k}_2 \frac{\partial \tilde{u}}{\partial y} \nu_{h2} - Q_h\,.$$

The Cauchy inequality and Lemma 4 imply

(102) $$\left|\int_{\Gamma_{h2}} \sigma w \, \mathrm{d}s\right| \leqq |\sigma|_{0,\Gamma_{h2}} |w|_{0,\Gamma_{h2}} \leqq C_1 \sqrt{(\operatorname{mes} \Gamma_{h2})} \max_{\Gamma_{h2}} |\sigma| \,.\, \|w\|_{1,\Omega_h}\,,$$
$$C_1 = C_1(\Gamma_2)\,.$$

Let us set

$$(103) \qquad \Delta_1 = \varphi^*(t) - \bar{\varphi}(t)\,, \quad \Delta_2 = \psi^*(t) - \bar{\psi}(t)\,.$$

According to (90) we have

$$(104) \qquad \Delta_i = O(h_T^{2k+2})\,, \quad \Delta_i' = O(h_T^{2k+2})\,, \quad i = 1, 2\,.$$

Using the Taylor formula we obtain

$$\tilde{k}_i(\varphi^*(t), \psi^*(t)) = \tilde{k}_i(\bar{\varphi}(t) + \Delta_1, \bar{\psi}(t) + \Delta_2) =$$

$$= \tilde{k}_i(\bar{\varphi}(t), \bar{\psi}(t)) + O(\Delta_1) + O(\Delta_2) = \tilde{k}_i(\bar{\varphi}(t), \bar{\psi}(t)) + O(h_T^{2k+2})\,.$$

As $\tilde{k}_i(\bar{\varphi}(t), \bar{\psi}(t)) = k_i(\bar{\varphi}(t), \bar{\psi}(t))$ we can write

$$(105) \qquad \tilde{k}_i(\varphi^*(t), \psi^*(t)) = k_i(\bar{\varphi}(t), \bar{\psi}(t)) + O(h_T^{2k+2})\,.$$

Similarly we obtain

$$(106) \qquad \frac{\partial \tilde{u}}{\partial x}(\varphi^*(t), \psi^*(t)) = \frac{\partial u}{\partial x}(\bar{\varphi}(t), \bar{\psi}(t)) + O(h_T^{2k+2})\,,$$

$$(107) \qquad \frac{\partial \tilde{u}}{\partial y}(\varphi^*(t), \psi^*(t)) = \frac{\partial u}{\partial y}(\bar{\varphi}(t), \bar{\psi}(t)) + O(h_T^{2k+2})\,.$$

Further,

$$(108) \qquad v_{hi} = v_i + O(h_T^{2k+1})\,, \quad i = 1, 2.$$

To prove (108) let us realize that

$$v_{h1} = \psi^{*\prime}(t)/\varrho^*(t)\,, \quad v_{h2} = -\varphi^{*\prime}(t)/\varrho^*(t)\,,$$

$$v_1 = \bar{\psi}'(t)/\bar{\varrho}(t)\,, \quad v_2 = -\bar{\varphi}'(t)/\bar{\varrho}(t)$$

where $\varrho^*(t)$ is defined by (64) and where

$$\bar{\varrho}(t) = \sqrt{\{[\bar{\varphi}'(t)]^2 + [\bar{\psi}'(t)]^2\}}\,.$$

Let us set

$$\delta^2(t) = 1 + \frac{2}{\bar{\varrho}(t)}(v_1\Delta_2' - v_2\Delta_1') + \left(\frac{\Delta_1'}{\bar{\varrho}(t)}\right)^2 + \left(\frac{\Delta_2'}{\bar{\varrho}(t)}\right)^2\,.$$

Then we have

$$v_{h1} = (\bar{\psi}'(t) + \Delta_2')/(\bar{\varrho}(t)\,\delta(t)) = v_1 + O(h_T^{2k+1})$$

because (104) holds and $\bar{\varrho}(t) \geqq a_1\beta h_T$. The second estimate (108) can be proved similarly.

Estimates (105)–(108) imply

$$\tilde{k}_1 \frac{\partial \tilde{u}}{\partial x} v_{h1} + \tilde{k}_2 \frac{\partial \tilde{u}}{\partial y} v_{h2}\Big|_{c_h} = k_1 \frac{\partial u}{\partial x} v_1 + k_2 \frac{\partial u}{\partial y} v_2\Big|_c + O(h_T^{2k+1})\,.$$

As, according to (43) and (61), (62),

$$k_1 \frac{\partial u}{\partial x} v_1 + k_2 \frac{\partial u}{\partial y} v_2 \Big|_c = Q(\bar{\varphi}(t), \bar{\psi}(t)),$$

$$Q_h(\varphi^*(t), \psi^*(t)) \equiv Q_h\Big|_{c_h} = Q(\bar{\varphi}(t), \bar{\psi}(t)),$$

we obtain with respect to (101)

$$\text{(109)} \qquad \max_{c_h} |\sigma| = O(h_T^{2k+1})$$

or in more detail,

$$\text{(110)} \qquad \max_{c_h} |\sigma| \leqq C_2 h_T^{2k+1}, \quad C_2 = C_2(K_1, K_2, \Gamma_2).$$

As mes $\Gamma_{h2} < 2$ mes Γ_2 and $h_T \leqq h$ relations (102) and (110) imply

$$\text{(111)} \qquad \left| \int_{\Gamma_{h2}} \sigma w \, \mathrm{d}s \right| \leqq C_3 h^{2k+1} \|w\|_{1,\Omega_h}, \quad C_3 = C_3(K_1, K_2, \Gamma_2).$$

b) Now we estimate the second term on the right-hand side of (75). According to (80) and (81), we have

$$\text{(112)} \qquad \left| \int_{\Gamma_{h2}} Q_h w \, \mathrm{d}s - l_h^\Gamma(w) \right| \leqq \sum_{c_h \subset \Gamma_{h2}} |E_{c_h}(Q_h w)| .$$

Taking into account relation (84) we shall estimate the term $E_I^*(Q_h^* w^* \varrho^*)$. Let us consider the form

$$\text{(113)} \qquad E_I^*(u^* w^*), \quad u^* \in W_\infty^{(2k+1)}(I), \quad w^* \in P(2k+1).$$

According to (82) and (113) we have

$$\text{(114)} \qquad |E_I^*(u^* w^*)| \leqq C_4 |u^*|_{0,\infty,I} \max_I |w^*| .$$

Using (87) and the inequality $|u^*|_{0,\infty,I} \leqq \|u^*\|_{2k+1,\infty,I}$ we obtain from (114)

$$|E_I^*(u^* w^*)| \leqq C_5 \|u^*\|_{2k+1,\infty,I} |w^*|_{0,I} .$$

For a given $w^* \in P(2k+1)$ let us define a linear form $f(u^*)$ on $W_\infty^{(2k+1)}(I)$ by

$$f(u^*) = E_I^*(u^* w^*) \quad \forall u^* \in W_\infty^{(2k+1)}(I).$$

The linear functional $f(u^*)$ is continuous with the norm less than or equal to $C_5 |w^*|_{0,I}$ on the one hand, and vanishes over $P(2k)$ on the other hand, by virtue of assumption (99). Therefore, using the Bramble-Hilbert lemma (see [2] or [4], [10], [12]), we obtain

$$\text{(115)} \qquad |E_I^*(u^* w^*)| \leqq C_6 |u^*|_{2k+1,\infty,I} |w^*|_{0,I}$$

$$\forall u^* \in W_\infty^{(2k+1)}(I), \quad \forall w^* \in P(2k+1).$$

Relations (12) and (85) give

$$\text{(116)} \qquad Q_h^{*(r)}(t) = \bar{s}_{32}^r \left[\frac{\partial^r Q}{\partial x^r} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}s} \right)^r + \ldots + \frac{\partial Q}{\partial y} \frac{\mathrm{d}^r \psi}{\mathrm{d}s^r} \right].$$

The assumptions of Theorem 5 about Γ_2 and Q together with (116) show that $Q_h^* \in C^{2k+1}(I)$. Thus, setting $u^* = Q_h^* \varrho^*$ we obtain from estimate (115)

$$|E_I^*(Q_h^* \varrho^* w^*)| \leqq C_6 \max_I |(Q_h^* \varrho^*)^{(2k+1)}| \cdot |w^*|_{0,I} . \tag{117}$$

As

$$|(Q_h^* \varrho^*)^{(2k+1)}| \leqq \sum_{r=0}^{2k+1} \binom{2k+1}{r} |Q_h^{*(r)}| \cdot |\varrho^{*(2k+1-r)}| ,$$

we get from (116) and Lemma 2

$$\max_I |(Q_h^* \varrho^*)^{(2k+1)}| \leqq C_7 h_T^{2k+2}, \quad C_7 = C_7(Q, \Gamma_2) . \tag{118}$$

Relations (84), (117) and (118) together with Lemma 3 give

$$|E_{c_h}(Q_h w)| \leqq C_8 h^{2k+3/2} |w|_{0,c_h} . \tag{119}$$

Let us combine estimates (112) and (119) and use the Cauchy inequality and the fact that the number of boundary triangles is $O(h^{-1})$. Then we obtain with respect to Lemma 4

$$\left| \int_{\Gamma_{h2}} Q_h w \, ds - l_h^\Gamma(w) \right| \leqq C_9 h^{2k+1} \|w\|_{1,\Omega_h} \tag{120}$$

with $C_9 = C_9(Q, \Gamma_2)$.

c) Combining inequalities (75), (111) and (120) we obtain the bound (100) for the second term on the right-hand side of (69). Theorem 5 is proved.

It remains to find a sufficient condition for the validity of inequality (27). Inspecting the proofs of such sufficient conditions in the case of Dirichlet boundary conditions on Γ (see, e.g., [3], [4], [12]) we see that in virtue of [12, Theorem 7] the problem reduces to establishing the inequality

$$|v|_{0,\Omega_h}^2 \leqq C |v|_{1,\Omega_h}^2 \quad \forall v \in V_{0h} , \tag{121}$$

where C is a constant independent on v and h. The solution of this problem follows from the following theorem which is proved in [13].

Theorem 6. *Let the boundary Γ of a bounded domain Ω be of class C^{2k+2} and let S be an arbitrary but fixed part of Γ such that* mes $S > 0$. *Let every triangulation τ of $\bar{\Omega}$ satisfy the condition*

$$\bar{h}/h \geqq c_0 \quad (c_0 = \text{const} > 0, \; \bar{h} = \min_{T \in \tau} h_T) \tag{122}$$

and have the property that S is a union of the curved sides of some boundary triangles. Let S_h be the part of Γ_h which approximates S. Then the constant $C(\Omega_h)$ appearing in the inequality

$$\|v\|_{1,\Omega_h}^2 \leqq C(\Omega_h) \left(\int_{S_h} v^2 ds + |v|_{1,\Omega_h}^2 \right) \quad \forall v \in W_h \tag{123}$$

can be chosen in such a way that

$$C(\Omega_h) \to K(\Omega) \quad \text{if} \quad h \to 0\,, \tag{124}$$

where $K(\Omega)$ *is an arbitrary constant which can occur in Friedrichs' inequality*

$$\|v\|^2_{1,\Omega} \leqq K(\Omega)\left(\int_S v^2 \, \mathrm{d}s + |v|^2_{1,\Omega}\right) \quad \forall v \in H^1(\Omega)\,. \tag{125}$$

Corollary. *Let the assumptions of Theorem* 6 *be satisfied with* $S = \Gamma_1$. *Then inequality* (121) *holds for* $h < \bar{h}$, *where* $\bar{h}$ *is sufficiently small.*

Theorem 7. *Let* $\tilde{k}_i(x, y) \in W^{(2k+1)}_\infty(\tilde{\Omega})$ *and let inequalities* (44) *hold. Let the assumptions of Theorem* 6 *be satisfied with* $S = \Gamma_1$. *In computing the bilinear forms* $a_h(v, w)$ *let us use a quadrature formula of degree of precision* $4k$, *i.e. let* (73) *hold for both Koukal's and Zlámal's* C^0*-elements. Then inequality* (27) (*with* $m = 0$) *holds for* $h < \bar{h}$, *where* $\bar{h}$ *is sufficiently small.*

The proof of Theorem 7 follows the same lines as that of [12, Corollary 1]. We use the corollary of Theorem 6 together with [12, Theorem 7] where we set $N^* = 2k + 1$, $m = 0$ for both the curved and the interior triangles. (It should be noted that in the case $m = 0$ the proof of [12, Theorem 7] does not depend on the boundary conditions prescribed on Γ.)

The results of this section are summarized in the following theorem:

Theorem 8. *Let the assumptions of Theorems* 4, 5 *and* 7 *be satisfied. If* h *is sufficiently small then the solution* u_h *of the discerte problem* (65) *exists and is unique and*

$$\|\tilde{u} - u_h\|_{1,\Omega_h} = O(h^{2k+1})\,, \tag{126}$$

where $\tilde{u}$ *is an extension of the solution* u *of problem* (41)–(43) *to the domain* $\tilde{\Omega}$.

The assertion of Theorem 8 follows from Theorem 1 with $m = 0$, from (67), (69) and from the assertions of Theorems 4, 5 and 7.

Remark. If we consider the Newton boundary condition

$$bu + k_1 \frac{\partial u}{\partial x} v_1 + k_2 \frac{\partial u}{\partial y} v_2 \bigg|_{\Gamma_2} = Q \quad (b \geqq b_0 > 0\,, \quad b_0 = \text{const}) \tag{127}$$

instead of the Neumann condition (43) we can obtain similar results. The bilinear form $\tilde{a}_h(v, w)$ has in this case the form

$$\tilde{a}_h(v, w) = \int_{\Gamma_{h2}} b_h v w \, \mathrm{d}s + \iint_{\Omega_h} \left(\tilde{k}_1 \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \tilde{k}_2 \frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y\,, \tag{128}$$

where b_h is the function obtained by "transferring" the function b from Γ_2 onto Γ_{h2} (cf. (61), (62)). In proving condition (27) we use again Theorem 6. In the case of the boundary condition (127) we can also consider the situation $\Gamma = \Gamma_2$.

The results obtained in this section can be extended to the case of fourth order elliptic equations with various combinations of boundary conditions.

References

[1] *J. J. Blair:* Higher order approximations to the boundary conditions for the finite element method. Math. Comp. 30 (1976), 250—262.

[2] *J. H. Bramble, S. R. Hilbert:* Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation. SIAM J. Numer. Anal. 7 (1970), 112—124.

[3] *P. G. Ciarlet, P. A. Raviart:* The combined effect of curved boundaries and numerical integration in isoparametric finite element methods. In: The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A. K. Aziz, Editor), pp. 409—474, Academic Press, New York 1972.

[4] *P. G. Ciarlet:* The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam 1978.

[5] *S. Koukal:* Piecewise polynomial interpolations in the finite element method. Apl. Mat. 18 (1973), 146—160.

[6] *L. Mansfield:* Approximation of the boundary in the finite element solution of fourth order problems. SIAM J. Numer. Anal. 15 (1978), 568—579.

[7] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague 1967.

[8] *R. Scott:* Interpolated boundary conditions in the finite element method. SIAM J. Numer. Anal. 12 (1975), 404—427.

[9] *G. Strang:* Approximation in the finite element method. Numer. Math. 19 (1972), 81—98.

[10] *M. Zlámal:* Curved elements in the finite element method. I. SIAM J. Numer. Anal. 10 (1973), 229—240.

[11] *M. Zlámal:* Curved elements in the finite element method. II. SIAM J. Numer. Anal. 11 (1974), 347—362.

[12] *A. Ženíšek:* Curved triangular finite C^m-elements. Apl. Mat. 23 (1978), 346—377.

[13] *A. Ženíšek:* Discrete forms of Friedrichs' inequalities in the finite element method. (To appear.)

Souhrn

NEHOMOGENNÍ OKRAJOVÉ PODMÍNKY A ZAKŘIVENÉ TROJÚHELNÍKOVÉ KONEČNÉ PRVKY

ALEXANDER ŽENÍŠEK

V článku je navržen způsob aproximace nehomogenních okrajových podmínek Dirichletova a Neumannova typu při řešení okrajových úloh eliptických rovnic metodou konečných prvků. V případě Dirichletových podmínek splňují parametry, které jednoznačně určují testovací funkce, v uzlových bodech ležících na hranici podmínky typu (17), (18), resp. (54), (55). V případě Neumannových podmínek předepsaných na Γ_2 je křivkový integrál podél křivky Γ_2 aproximován křivkovým integrálem podél aproximující křivky Γ_{h2}.

V první části článku je studována konvergence metody konečných prvků při řešení nehomogenního Dirichletova problému eliptických rovnic řádu $2m + 2$. Tato část článku zobecňuje výsledky získané v [12]: při použití zakřivených trojúhelníkových

konečných C^m-prvků popsaných v [12] je rychlost konvergence v normě prostoru $H^{m+1}(\Omega_h)$ opět $O(h^{2m+1})$.

V druhé části článku je analyzována konvergence metody konečných prvků v případě nehomogenního smíšeného okrajového problému eliptických rovnic druhého řádu při použití Koukalových polynomů stupně $2k + 1$ [5, Věta 5] a Zlámalových zakřivených trojúhelníkových konečných C^0-prvků [11], které lze na Koukalovy prvky napojit. Rychlost konvergence v normě prostoru $H^1(\Omega_h)$ je $O(h^{2k+1})$.

V obou částech článku je studován vliv numerické integrace na rychlost konvergence.

Author's address: Doc. RNDr. *Alexander Ženíšek*, CSc., Laboratoř počítacích strojů VUT, Obránců míru 21, 602 00 Brno.