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A DEFLATION FORMULA FOR TRIADIAGONAL MATRICES

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A number of algorithms have been developed to find eigenvalues of a tridiagonal matrix numerically. A survey is given in [1], other appeared later [2], [3]. We shall present here a simple explicit formula which can be used to diminish the order of the tridiagonal matrix by one if one eigenvalue and the corresponding eigenvector are known (without destroying the tridiagonal shape). Combined with some algorithm for computing an eigenvalue and the eigenvector (in the symmetric case for the nonnegative tridiagonal matrix or an M-matrix), one obtains a method for theoretical computing all the eigenvalues and eigenvectors of such a matrix.

We shall present this formula in two cases: the symmetric case and the more general nonsymmetric case in which the non-zero entries above the diagonal are normalized to -1. To the latter case, any tridiagonal irreducible matrix can be brought by a diagonal similarity. We shall, in fact, start with this case and prove first a lemma.

Lemma. *Let*

$$\mathbf{A} = \begin{bmatrix} \alpha_1, & -1, & & & & \\ -\beta_1, & \alpha_2, & -1, & & & \\ & -\beta_2 & \alpha_3, & -1, & & \\ & & \cdots & \cdots & & \\ & & & \cdots & \cdots & \\ & & & & \cdots & \\ & & & -\beta_{n-2}, & \alpha_{n-1}, & -1 \\ & & & & -\beta_{n-1}, & \alpha_n \end{bmatrix} \tag{1}$$

be a (complex) tridiagonal matrix for which $\beta_i \neq 0$, $i = 1, \dots, n-1$. If $\mathbf{u} = (u_1, \dots, u_n)^{\mathsf{T}}$ is an eigenvector of $\mathbf{A}$ then $\mathbf{v} = (v_1, \dots, v_n)^{\mathsf{T}}$ with

$$v_k = \beta_k \beta_{k+1} \dots \beta_{n-1} u_k\,, \quad k = 1, \dots, n-1\,, \quad v_n = u_n \tag{2}$$

is the (up to a non-zero factor unique) eigenvector of $\mathbf{A}^{\mathsf{T}}$ corresponding to the same eigenvalue. This eigenvalue is simple if and only if

$$\sum_{k=1}^{n} \beta_k \beta_{k+1} \dots \beta_{n-1} u_k^2 \neq 0\,. \tag{3}$$

Proof. Let $\boldsymbol{u}$ correspond to the eigenvalue μ so that

$$\boldsymbol{A}\boldsymbol{u} = \mu \boldsymbol{u} .$$

The matrix $\boldsymbol{A} - \mu \boldsymbol{I}$ has rank $n - 1$ so that to μ, $\boldsymbol{u}$ is the only linearly independent eigenvector. Therefore, $\boldsymbol{A}^{\mathsf{T}}$ has also only one linearly independent eigenvector corresponding to μ. However, it is easily checked that the vector $\mathbf{v}$ defined by (2) is a non-zero vector which satisfies

$$\boldsymbol{A}^{\mathsf{T}}\mathbf{v} = \mu \mathbf{v} .$$

The well known Schur lemma [4] states then that in such a situation, μ is a simple eigenvalue of $\boldsymbol{A}$ if and only if $\mathbf{v}^{\mathsf{T}}\boldsymbol{u} \neq 0$. This yields exactly the condition (3).

Theorem 1. *Let $\mathbf{A}$ be the tridiagonal matrix from (1) for which $\beta_i \neq 0$, $i = 1, \ldots, n - 1$. Let $\mathbf{y} = (y_i)$ be a column eigenvector of $\mathbf{A}$ such that $y_i \neq 0$, $i = 1, \ldots, n$. Then all the remaining eigenvalues of $\mathbf{A}$ coincide with those of the $(n - 1) \times (n - 1)$ matrix*

$$\mathbf{B} = \begin{bmatrix} \hat{\alpha}_1, & 1, & & & & \\ -\hat{\beta}_1, & \hat{\alpha}_2, & -1, & & & \\ & \cdots & \cdots & & & \\ & & \cdots & \cdots & & \\ & & & \cdots & \cdots & \\ & & & -\hat{\beta}_{n-3}, & \hat{\alpha}_{n-2}, & -1, \\ & & & & -\hat{\beta}_{n-2}, & \hat{\alpha}_{n-1} \end{bmatrix} \tag{4}$$

where

$$\begin{aligned} \hat{\alpha}_i &= \alpha_{i+1} + y_{i+1}/y_i - y_{i+2}/y_{i+1} , \quad && i = 1, \ldots, n - 2 \\ \hat{\alpha}_{n-1} &= \alpha_n + y_n/y_{n-1} , && \\ \hat{\beta}_i &= \beta_i y_i y_{i+2}/y_{i+1}^2 , && i = 1, \ldots, n - 2 . \end{aligned} \tag{5}$$

If $\mathbf{z} = (z_i)$ is an eigenvector of $\mathbf{A}$ linearly independent of $\mathbf{y}$ then the vector $\boldsymbol{p} = (p_i)$,

$$p_i = y_{i+1}(z_i/y_i - z_{i+1}/y_{i+1}) , \quad i = 1, \ldots, n - 1 \tag{6}$$

is a corresponding eigenvector of $\mathbf{B}$. If $\mathbf{y}$ corresponds to a simple eigenvalue of $\mathbf{A}$, i.e. if

$$\sum_{k=1}^{n} \beta_k \ldots \beta_{n-1} y_k^2 \neq 0 , \tag{7}$$

then conversely, to an eigenvector $\boldsymbol{p} = (p_1, \ldots, p_{n-1})^{\mathsf{T}}$ of $\mathbf{B}$, $\mathbf{z} = (z_1, \ldots, z_n)^{\mathsf{T}}$ is a corresponding eigenvector of $\mathbf{A}$ where

$$z_j = y_j \sum_{k=1}^{j-1} \varrho_k p_k + y_j \sum_{k=j}^{n-1} \sigma_k p_k , \quad j = 1, \ldots, n , \tag{8}$$

with

$$(9) \qquad \varrho_k = -y_{k+1}^{-1} \sum_{j=1}^{k} \beta_j \beta_{j+1} \dots \beta_{n-1} y_j^2 ,$$

$$\sigma_k = y_{k+1}^{-1} \sum_{j=k+1}^{n} \beta_j \beta_{j+1} \dots \beta_{n-1} y_j^2 , \quad k = 1, \dots, n .$$

Moreover, the common eigenvalues of both matrices $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *have the same multiplicities.*

Remark. The diagonal entries $\hat{\alpha}_i$ of the matrix $\boldsymbol{B}$ can also be expressed as

$$(10) \qquad \hat{\alpha}_i = y_{i+1}/y_i + \beta_i y_i/y_{i+1} + \lambda , \quad i = 1, \dots, n-1 ,$$

where λ is the eigenvalue of $\boldsymbol{A}$ corresponding to the eigenvector $\mathbf{y}$.

Proof. Let us show first the equivalence of (10) in the remark. If $i = 1, \dots, n-2$,

$$\alpha_{i+1} y_{i+1} - \beta_i y_i - y_{i+2} = \lambda y_{i+1}$$

whence

$$\alpha_{i+1} - y_{i+2}/y_{i+1} + y_{i+1}/y_i = y_{i+1}/y_i + \beta_i y_i/y_{i+1} + \lambda .$$

Since also

$$\alpha_n y_n - \beta_{n-1} y_{n-1} = \lambda y_n ,$$

we have

$$\alpha_n + y_n/y_{n-1} = y_n/y_{n-1} + \beta_{n-1} y_{n-1}/y_n + \lambda$$

as well and the assertion follows.

Let us show now that the matrix $\boldsymbol{A} - \lambda \boldsymbol{I}$ can be written in the form

$$(11) \qquad \boldsymbol{A} - \lambda \boldsymbol{I} = \boldsymbol{PQ}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 1, & 0, & \dots & 0, & 0 \\ -\beta_1 \dfrac{y_1}{y_2}, & 1, & \dots & 0, & 0 \\ 0, & -\beta_2 \dfrac{y_2}{y_3}, & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & -\beta_{n-2} \dfrac{y_{n-2}}{y_{n-1}}, & 1 \\ 0, & 0, & \dots & 0, & -\beta_{n-1} \dfrac{y_{n-1}}{y_n} \end{bmatrix},$$

$$
\mathbf{Q} = \begin{bmatrix} \frac{y_2}{y_1}, & -1, & 0, & \dots & 0, & 0 \\ 0, & \frac{y_3}{y_2}, & -1, & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots & -1, & \\ 0, & 0, & 0, & \dots & \frac{y_n}{y_{n-1}} & -1 \end{bmatrix};
$$

thus $\boldsymbol{P}$ is $n \times (n-1)$, $\mathbf{Q}$ is $(n-1) \times n$. All off-diagonal entries of the matrix $\boldsymbol{PQ}$ are evidently equal to the corresponding off-diagonal entries of $\boldsymbol{A} - \lambda \boldsymbol{I}$. However, since $(\boldsymbol{A} - \lambda \boldsymbol{I})\,\mathbf{y} = \boldsymbol{0}$ as well as $\mathbf{Qy} = \boldsymbol{0}$ and thus $\boldsymbol{PQ}\mathbf{y} = \boldsymbol{0}$, the diagonal entries of both matrices also coincide.

It is easily seen by (10) that the product $\boldsymbol{R} = \mathbf{Q}\boldsymbol{P}$ is equal to the matrix $\boldsymbol{B} - \lambda \boldsymbol{I}$:

$$
\boldsymbol{R} = \boldsymbol{B} - \lambda \boldsymbol{I}\,. \tag{12}
$$

Let now μ be an eigenvalue of $\boldsymbol{A}$ corresponding to a vector $\mathbf{z}$, linearly independent of $\mathbf{y}$:

$$
\boldsymbol{A}\mathbf{z} = \mu \mathbf{z}\,.
$$

By (11),

$$
\boldsymbol{P}\mathbf{Q}\mathbf{z} = (\mu - \lambda)\,\mathbf{z}\,.
$$

The vector $\boldsymbol{p}$ defined in (6) satisfies

$$
\boldsymbol{p} = \mathbf{Q}\mathbf{z}
$$

and is different zero. Therefore

$$
\boldsymbol{P}\boldsymbol{p} = (\mu - \lambda)\,\mathbf{z}\,. \tag{13}
$$

It follows

$$
\boldsymbol{R}\boldsymbol{p} = \mathbf{Q}\boldsymbol{P}\boldsymbol{p} = (\mu - \lambda)\,\mathbf{Q}\mathbf{z} = (\mu - \lambda)\,\boldsymbol{p}\,,
$$

so that, by (12),

$$
\boldsymbol{B}\boldsymbol{p} = \mu \boldsymbol{p}\,.
$$

The proof of the first part is complete.

To prove the second part, assume first that all eigenvalues of $\boldsymbol{A}$ are simple. Then it is easily seen that the formula (6) yields $n-1$ linearly independent eigenvectors of $\boldsymbol{B}$ corresponding to the remaining (linearly independent) eigenvectors of $\boldsymbol{A}$. Moreover, the correspondence between the eigenvectors $\mathbf{z}$ of $\boldsymbol{A}$ and $\boldsymbol{p}$ of $\boldsymbol{B}$ is one-to-one. To compute the vector $\mathbf{z}$ from the vector $\boldsymbol{p}$, observe that $\mathbf{z}$ satisfies the condition

$$
\mathbf{v}^{\mathrm{T}}\mathbf{z} = \boldsymbol{0}
$$

where $\boldsymbol{v}$ is the eigenvector of $\boldsymbol{A}^{\mathsf{T}}$ corresponding to the same eigenvalue as $\boldsymbol{y}$. By the Lemma, $\boldsymbol{v} = (v_1, \ldots, v_n)^{\mathsf{T}}$ is given by (2) so that $\boldsymbol{z}$ satisfies

$$\sum_{k=1}^{n} \beta_k \beta_{k+1} \ldots \beta_{n-1} y_k z_k = 0 .$$

Now the following proposition is easily proved.

Proposition 1. *Let n be an integer, $n \geqq 2$, let the numbers $a_1, \ldots, a_n$ satisfy $\sum_{j=1}^{n} a_j \neq 0$. Then the system of linear equations*

$$x_i - x_{i+1} = u_i , \quad i = 1, \ldots, n - 1 ,$$

$$\sum_{i=1}^{n} a_i x_i = 0$$

has the unique solution

$$x_j = \sum_{k=1}^{j-1} r_k u_k + \sum_{k=j}^{n-1} s_k u_k$$

where

$$r_k = -\sum_{j=1}^{k} a_j \Big/ \sum_{j=1}^{n} a_j , \quad s_k = \sum_{j=k+1}^{n} a_j \Big/ \sum_{j=1}^{n} a_j ,$$

$$k = 1, \ldots, n - 1 .$$

Setting $a_k = \beta_k \ldots \beta_{n-1} y_k^2$, $k = 1, \ldots, n$, $u_i = p_i / y_{i+1}$, $\varrho_i = r_i / y_{i+1}$, $\sigma_i = s_i / y_{i+1}$, $i = 1, \ldots, n - 1$, the condition $\sum_{j=1}^{n} a_j \neq 0$ being satisfied, we obtain that (8) and (9) yields the unique (up to a factor) and non-zero vector $\boldsymbol{z}$ corresponding to $\boldsymbol{p}$.

It remains to prove the second part for the case that not all eigenvalues of $\boldsymbol{A}$ are simple. It follows from the Lemma that there exists a sequence of matrices $\{\boldsymbol{A}_i\}_{i=1}^{\infty}$ such that $\boldsymbol{A}$ is the limit of $\boldsymbol{A}_i$, each matrix $\boldsymbol{A}_i$ is tridiagonal of the form (1), has simple eigenvalues and an eigenvector $\boldsymbol{y}^{(i)}$ with non-zero coordinates which satisfies (8) and converges to $\boldsymbol{y}$ if $i \to \infty$.

It follows from (5) that the corresponding matrices $\boldsymbol{B}_i$ converge to the matrix $\boldsymbol{B}$ corresponding to $\boldsymbol{A}$. Moreover, the vectors $\boldsymbol{p}^{(i)}$ from (6) are defined and converge as soon as $\boldsymbol{z}^{(i)}$ converge. Therefore, the multiplicities of the common eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$ coincide and even the formulas (8) and (9) hold. The proof is complete.

In the sequel, we shall use the notion of an M-matrix (or, equivalently, of the matrix of class $\boldsymbol{K}$ [5]). As is well known [5], such a matrix is characterized by the fact that all the off-diagonal entries are nonpositive and one of the following properties holds:

1° $\boldsymbol{Ax} > \boldsymbol{0}$ for some nonnegative vector $\boldsymbol{x}$;

2° all real eigenvalues of $\boldsymbol{A}$ are positive;

3° $\boldsymbol{A}^{-1}$ exists and is nonnegative;

in the case that $\boldsymbol{A}$ is irreducible, we have another condition:

4° $\boldsymbol{Ax} \geqq \boldsymbol{0}$, $\boldsymbol{Ax} \neq \boldsymbol{0}$ for some positive vector $\boldsymbol{x}$.

To an M-matrix, a positive eigenvalue ω exists such that $\operatorname{Re} \lambda > \omega$ for any eigenvalue and ω corresponds to a nonnegative eigenvector. If $\boldsymbol{A}$ is an irreducible M-matrix then ω is simple and the corresponding nonnegative eigenvector is even positive. Moreover, this is the only nonnegative eigenvector of $\boldsymbol{A}$. Now we are able to prove the following theorem.

Theorem 2. *Let $\boldsymbol{A}$ be defined by (1), let $\beta_i > 0$, $i = 1, \ldots, n - 1$ and let $\boldsymbol{y} = (y_i)$ be a positive eigenvector of $\boldsymbol{A}$ corresponding to a positive eigenvalue ω. Then $\boldsymbol{A}$ is an M-matrix and the matrix $\boldsymbol{B}$ defined by (4) and (5) has the property that $\boldsymbol{B} - \omega\boldsymbol{I}$ is an M-matrix as well. In fact, $(\boldsymbol{B} - \omega\boldsymbol{I})\,\boldsymbol{u} \geqq \boldsymbol{0}$ where $\boldsymbol{u}$ is the positive vector $\boldsymbol{u} = (1/y_1, \beta_1/y_2, \beta_1\beta_2/y_3, \ldots, \beta_1\beta_2 \ldots \beta_{n-2}/y_{n-1})^{\mathrm{T}}$.*

Proof. It follows from condition 1° that $\boldsymbol{A}$ is a (nonsingular) M-matrix since $\boldsymbol{Ay} = \omega\boldsymbol{y}$, $\omega > 0$, $\boldsymbol{y} > \boldsymbol{0}$. The matrix $\boldsymbol{B} - \omega\boldsymbol{I}$ is then also a nonsingular M-matrix since it has also all off-diagonal entries nonpositive and all its real eigenvalues – being equal to $\alpha_i - \omega$ where α_i are eigenvalues of $\boldsymbol{A}$ different from ω – are positive.

Another method how to prove this fact is by 4°, to show the last assertion. Let $\boldsymbol{u}$ be the vector defined there. Observe that the k-th diagonal entry of $\boldsymbol{B}$ satisfies not only

$$\hat{\alpha}_k = \alpha_{k+1} + y_{k+1}/y_k - y_{k+2}/y_{k+1}\,,$$

$$\hat{\alpha}_{n-1} = \alpha_n + y_n/y_{n-1}\,,$$

but also

$$\hat{\alpha}_k = \omega + y_{k+1}/y_k + \beta_k y_k/y_{k+1}\,, \quad k = 1, \ldots, n - 1\,.$$

It follows that, for the vector $\boldsymbol{u}$ defined above,

$$(\boldsymbol{B} - \omega\boldsymbol{I})\,\boldsymbol{u} = \begin{bmatrix} y_2/y_1 + \beta_1 y_1/y_2, & -1, & \\ -\beta_1 y_1 y_3/y_2^2, & y_3/y_2 + \beta_2 y_2/y_3, & \\ \multicolumn{3}{c}{\dotfill} \\ & -\beta_{n-2} y_{n-2} y_n/y_{n-1}^2, & y_n/y_{n-1} + \beta_{n-1} y_{n-1}/y_n \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1/y_1 \\ \beta_1/y_2 \\ \vdots \\ \beta_1 \ldots \beta_{n-2}/y_{n-1} \end{bmatrix} = \begin{bmatrix} y_2/y_1^2 \\ 0 \\ \vdots \\ \beta_1 \ldots \beta_{n-1}/y_n \end{bmatrix}$$

which is nonnegative as asserted ($\boldsymbol{B}$ is irreducible).

Theorem 3. *Let*

$$\mathbf{A} = \begin{bmatrix} \alpha_1, & -\gamma_1, & & & \\ -\gamma_1, & \alpha_2, & -\gamma_2, & & \\ & -\gamma_2, & \ddots & \ddots & \\ & & \ddots & \alpha_{n-1}, & -\gamma_{n-1} \\ & & & -\gamma_{n-1}, & \alpha_n \end{bmatrix} \tag{14}$$

be a tridiagonal matrix, $\gamma_i > 0$, $i = 1, \ldots, n - 1$. Let $\mathbf{y} = (y_1, \ldots, y_n)^\mathsf{T}$ be a positive eigenvector of $\mathbf{A}$ corresponding to the smallest eigenvalue of $\mathbf{A}$. Then the remaining eigenvalues of $\mathbf{A}$ coincide with the eigenvalues of the tridiagonal matrix

$$\mathbf{B} = \begin{bmatrix} \hat{\alpha}_1, & -\hat{\gamma}_1, & & & \\ -\hat{\gamma}_1, & \hat{\alpha}_2, & -\hat{\gamma}_2, & & \\ & -\hat{\gamma}_2, & \ddots & \ddots & \\ & & \ddots & \hat{\alpha}_{n-2}, & -\hat{\gamma}_{n-2} \\ & & & -\hat{\gamma}_{n-2}, & \hat{\alpha}_{n-1} \end{bmatrix} \tag{15}$$

where $\hat{\alpha}_i = \alpha_{i+1} + \gamma_i y_{i+1}/y_i - \gamma_{i+1} y_{i+2}/y_{i+1}$, $i = 1, \ldots, n - 2$,

$$\hat{\alpha}_{n-1} = \alpha_n + \gamma_{n-1} y_n / y_{n-1} , \tag{16}$$

$$\hat{\gamma}_i = (\gamma_i \gamma_{i+1} y_i y_{i+2} / y_{i+1}^2)^{1/2} , \quad i = 1, \ldots, n - 2 . \tag{17}$$

If $\mathbf{z} = (z_1, \ldots, z_n)^\mathsf{T}$ is an eigenvector of $\mathbf{A}$ linearly independent of y then $\mathbf{p} = (p_1, \ldots, p_{n-1})^\mathsf{T}$ is the eigenvector of $\mathbf{B}$ corresponding to the same eigenvalue where

$$p_i = (\gamma_i y_i y_{i+1})^{1/2} \cdot (z_i y_i^{-1} - z_{i+1} y_{i+1}^{-1}) , \quad i = 1, \ldots, n - 1 . \tag{18}$$

Conversely, to an eigenvector $\mathbf{p} = (p_1, \ldots, p_{n-1})^\mathsf{T}$ of $\mathbf{B}$, $\mathbf{z} = (z_1, \ldots, z_n)^\mathsf{T}$ is a corresponding eigenvector of $\mathbf{A}$ where

$$z_j = -y_j \sum_{k=1}^{j-1} \frac{p_k}{\sqrt{(\gamma_k y_k y_{k+1})}} \sum_{t=1}^{k} y_t^2 + y_j \sum_{k=j}^{n-1} \frac{p_k}{\sqrt{(\gamma_k y_k y_{k+1})}} \sum_{t=k+1}^{n} y_t^2 . \tag{19}$$

Proof. Let us find a diagonal matrix $\mathbf{D} = \operatorname{diag}\{d_1, \ldots, d_n\}$ such that $\mathbf{DAD}^{-1}$ is normalized as in (1).

Since the entries $(1, 2), (2, 3), \ldots, (n - 1, n)$ of $\mathbf{DAD}^{-1}$ should be -1, we have

$$d_i \gamma_i d_{i+1}^{-1} = 1 , \quad i = 1, \ldots, n - 1 \tag{20}$$

or, if we set $d_1 = 1$,

$$d_i = \gamma_1 \ldots \gamma_{i-1} , \quad i = 2, \ldots, n . \tag{21}$$

Considering the entries $(2, 1), (3, 2), \ldots, (n, n - 1)$, we obtain $d_{i+1} \gamma_i d_i^{-1} = \beta_i$, whence

$$\gamma_i^2 = \beta_i , \quad i = 1, \ldots, n - 1 .$$

Moreover, the eigenvector $\mathbf{y}$ of $\boldsymbol{A}$ corresponds to the eigenvector $\mathbf{Dy}$ of $\mathbf{DAD}^{-1}$, and similarly for $\mathbf{z}$. By Theorem 1, the remaining eigenvalues of $\boldsymbol{A}$ except that corresponding to $\mathbf{y}$ coincide with those of the matrix corresponding to $\mathbf{DAD}^{-1}$ in (2). Denote by $\tilde{\mathbf{B}}$ the matrix from (4) and (5). Since $\hat{\beta}_i$ is positive, we can find a diagonal $(n-1)$ by $(n-1)$ matrix $\mathbf{H} = \operatorname{diag}\{h_1, \ldots, h_{n-1}\}$ such that $\boldsymbol{B} = \boldsymbol{H}\tilde{\boldsymbol{B}}\boldsymbol{H}^{-1}$ is already symmetric. These numbers h_i should satisfy

$$h_i h_{i+1}^{-1} = h_{i+1}\hat{\beta}_i h_i^{-1} .$$

If we denote this common value by $\hat{\gamma}_i$ and set $h_{n-1} = 1$, we obtain

$$h_k = \hat{\gamma}_k \hat{\gamma}_{k+1} \ldots \hat{\gamma}_{n-2} \tag{22}$$

and

$$\hat{\beta}_k = \hat{\gamma}_k^2 .$$

We shall thus set $\hat{\gamma}_k$ as the positive square root:

$$\hat{\gamma}_k = \sqrt{(\hat{\beta}_k)} . \tag{23}$$

Let us show now that the matrix $\boldsymbol{B} = \boldsymbol{H}\tilde{\boldsymbol{B}}\boldsymbol{H}^{-1}$ coincides with the matrix $\boldsymbol{B}$ in (15). From (5) we obtain, having in mind that the y_i's from (5) are coordinates of the vector $\mathbf{Dy}$, d_i given by (21):

$$\begin{aligned}\hat{\gamma}_k = \sqrt{(\hat{\beta}_k)} &= (\beta_k d_k y_k d_{k+2} y_{k+2}/(d_{k+1} y_{k+1})^2)^{1/2} = \\ &= (\gamma_k \gamma_{k+1} y_k y_{k+2}/y_{k+1}^2)^{1/2} ,\end{aligned}$$

i.e. (17).

The diagonal entries $\hat{\alpha}_k$ of $\boldsymbol{B}$ are the same as the diagonal entries of $\boldsymbol{B}$. Using again the formulae (5), we obtain

$$\hat{\alpha}_k = \alpha_{k+1} + d_{i+1} y_{i+1}/(d_i y_i) - d_{i+2} y_{i+2}/(d_{i+1} y_{i+1}) , \quad k = 1, \ldots, n-2 ,$$

$$\hat{\alpha}_{n-1} = \alpha_n + d_n y_n/(d_{n-1} y_{n-1}) .$$

Using (21), this yields (16).

Now let $\mathbf{z}$ be an eigenvector of $\boldsymbol{A}$ linearly independent of $\mathbf{y}$. Then $\mathbf{Dz}$ is the corresponding eigenvector of the matrix $\mathbf{DAD}^{-1}$. By (6), the eigenvector $\boldsymbol{p} = (p_i)$ of $\boldsymbol{B}$ has coordinates

$$\tilde{p}_i = d_{i+1} y_{i+1}(z_i/y_i - z_{i+1}/y_{i+1}) .$$

Since $\boldsymbol{B} = \boldsymbol{H}\tilde{\boldsymbol{B}}\boldsymbol{H}^{-1}$ has eigenvector $\boldsymbol{p} = \boldsymbol{H}\tilde{\boldsymbol{p}}$, we obtain from (21) and (22) for $\boldsymbol{p} = (p_i)$

$$p_i = h_i \tilde{p}_i = \gamma_1 \gamma_2 \ldots \gamma_i \hat{\gamma}_i \hat{\gamma}_{i+1} \ldots \hat{\gamma}_{n-2} y_{i+1}(z_i/y_i - z_{i+1}/y_{i+1}) ,$$

which yields, after dividing by $\gamma_1 \ldots \gamma_{n-2} \sqrt{(\gamma_{n-1} y_n/y_{n-1})}$, the formula (18).

The converse follows from (18). All eigenvalues being simple by Lemma, the correspondence between the eigenvectors $\mathbf{z}$ of $\boldsymbol{A}$ and $\boldsymbol{p}$ of $\boldsymbol{B}$ is one-to-one.

If we denote, for a moment, by $\boldsymbol{p} = (p_i)$ the eigenvector of the normalized matrix $\tilde{\boldsymbol{B}} = \boldsymbol{H}^{-1}\boldsymbol{B}\boldsymbol{H}$, we have

(24)
$$\tilde{p}_i = \frac{1}{h_i} p_i .$$

Similarly, if $\tilde{\boldsymbol{y}} = (\tilde{y}_i)$ and $\tilde{\boldsymbol{z}} = (\tilde{z}_i)$ are eigenvectors of the normalized matrix $\boldsymbol{DAD}^{-1}$, we have

$$\tilde{y}_j = d_j y_j ,$$
$$\tilde{z}_j = d_j z_j .$$

Therefore, using (8), we obtain

$$\tilde{z}_j = \tilde{y}_j \sum_{k=1}^{j-1} \varrho_k \tilde{p}_k + \tilde{y}_j \sum_{k=j}^{n-1} \sigma_k \tilde{p}_k$$

where

$$\varrho_k = -\frac{1}{\tilde{y}_{k+1}} \sum_{j=1}^{k} \gamma_j^2 \dots \gamma_{n-1}^2 \gamma_1^2 \dots \gamma_{j-1}^2 y_j^2$$

which can be written as

$$\varrho_k = C\left(-\frac{1}{d_{k+1} y_{k+1}} \sum_{j=1}^{k} y_j^2\right), \quad k = 1, \dots, n-1 ,$$

C being independent of k; similarly

$$\sigma_k = C\left(\frac{1}{d_{k+1} y_{k+1}} \sum_{j=k+1}^{n-1} y_j^2\right)$$

with the same C. By (24), leaving out the constant C,

$$z_j = -y_j \sum_{k=1}^{j-1} \frac{p_k}{h_k d_{k+1} y_{k+1}} \sum_{t=1}^{k} y_t^2 + y_j \sum_{k=j}^{n-1} \frac{p_k}{h_k d_{k+1} y_{k+1}} \sum_{t=k+1}^{n-1} y_t^2 ,$$
$$j = 1, \dots, n-1 .$$

From (22), (21) and (17), one gets (leaving out another factor independent of j) (19). The proof is complete.

Let us conclude with an analogue of Theorem 2 for symmetric tridiagonal matrices.

Theorem 4. *Let* $\boldsymbol{A}$ *be given by* (14), *let* $\gamma_i > 0$, $i = 1, \dots, n-1$ *and let* $\boldsymbol{y} = (y_j)$ *be a positive eigenvector of* $\boldsymbol{A}$ *corresponding to a positive eigenvalue* ω. *Then* $\boldsymbol{A}$ *is an M-matrix and the matrix* $\boldsymbol{B}$ *defined by* (15), (16) *and* (17) *has the property that* $\boldsymbol{B} - \omega\boldsymbol{I}$ *is an M-matrix as well. Moreover,* $(\boldsymbol{B} - \omega\boldsymbol{I})\,\boldsymbol{u} \geqq \boldsymbol{0}$ *where* $\boldsymbol{u}$ *is the positive vector* $\boldsymbol{u} = ((\gamma_1 y_1 y_2)^{-1/2}, (\gamma_2 y_2 y_3)^{-1/2}, \dots, (\gamma_{n-1} y_{n-1} y_n)^{-1/2})^{\mathrm{T}}$.

The proof follows for instance from Theorem 2 by transforming the vectors u, y and the numbers β_i using the formulae $\beta_i = \gamma_i^2$, (21), (22) and (17).

Bibliography

[1] *J. H. Wilkinson:* The algebraic eigenvalue problem. Clarendon Press, Oxford 1965.
[2] *P. A. Businger:* Numerically stable deflation of Hessenberg and symmetric tridiagonal matrices. BIT 11 (1971), 262—270.
[3] *W. Gander:* Stationärer Quotienten-Differenzen Algorithmus. Prozedur qdstat. In: Numerische Prozeduren aus Nachlass und Lehre von Prof. Heinz Rutishauser. ISNM Vol. 33. Birkhäuser Verlag 1977.
[4] *W. Jentzsch:* Über Integralgleichungen mit positivem Kern. J. für Math. 141 (1912), 235—244.
[5] *M. Fiedler, Vl. Pták:* On matrices with non-positive off-diagonal elements and positive principal minors. Czech. Math. J. 87 (1962), 382—400.

Souhrn

VZOREC PRO DEFLACI TŘÍDIAGONÁLNÍCH MATIC

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Jsou uvedeny explicitní vzorce, jimiž ze znalosti jednoho jednoduchého vlastního čísla a odpovídajícího vlastního vektoru třídiagonální matice řádu n lze vypočíst opět třídiagonální matici řádu $n - 1$, jejíž vlastní čísla jsou totožná se zbylými vlastními čísly původní matice. Rovněž jsou uvedeny vzorce pro vzájemné převádění zbylých vlastních vektorů původní matice a vlastních vektorů nové matice. Pro speciální případ třídiagonální M-matice a kladného vlastního vektoru je získaná matice opět M-maticí.

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