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FIRST-ORDER NECESSARY CONDITION FOR THE EXISTENCE OF OPTIMAL POINT IN NONLINEAR PROGRAMMING PROBLEM

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In the theory of optimization we are often forced to make some simplifications in the face of great difficulties that arise in manipulating the given sets. For that reason we proceed by studying local properties of the sets with their (mostly conical) approximations (the question of the existence of local extremum in the nonlinear programming problem is equivalent to the question of the local disjointness of certain two sets). In [2] an approximation was defined by means of the so called contact cone for the purpose of building up the general theory of nonlinear optimization. In this paper we want to show some properties of the contact cone and, by means of this cone, to state a necessary condition for local disjointness of the sets in question at a suspicious point (also called candidate).

First we introduce some notation and recall the definition of the contact cone.

Let $E_n(n \geqq 1)$ be an n-dimensional Euclidean space, ${}_0\mathbf{x} \in E_n$ a fixed chosen point and $\mathbf{v}$ a nonzero vector. By smybol $p({}_0\mathbf{x}; \mathbf{v})$ we denote the halfline with description

$$\mathbf{x} = {}_0\mathbf{x} + t\mathbf{v}\,, \quad t \in (0, \infty)\,.$$

If a function $F(\mathbf{x})$, defined for the sake of simplicity in the whole space E_n, has the derivatives

$$\frac{\partial F}{\partial x^\alpha}\,, \quad \alpha = 1, \ldots, n\,,$$

et us set

$$F_\alpha({}_0\mathbf{x}) \equiv \left(\frac{\partial F}{\partial x^\alpha}\right)_{{}_0\mathbf{x}}\,, \quad \alpha = 1, \ldots, n\,,$$

$$\nabla F({}_0\mathbf{x}) \equiv \left(\frac{\partial F}{\partial \mathbf{x}}\right)_{{}_0\mathbf{x}}\,.$$

Definition 1. *Let $P({}_0\mathbf{x}; \mathbf{v})$ be an arbitrary polyhedral cone in E_n with the properties*

1) $\dim P({}_0\mathbf{x}; \mathbf{v}) = n$,

2) $p({}_0\mathbf{x}; \mathbf{v}) \subset \operatorname{int} P({}_0\mathbf{x}; \mathbf{v})$.

Then we call the set

$$U({}_0\mathbf{x}; \mathbf{v}) = \operatorname{int} P({}_0\mathbf{x}; \mathbf{v})$$

a polyhedral neighbourhood of the halfline $p({}_0\mathbf{x}; \mathbf{v})$ in E_n.

Definition 2. *Let $U({}_0\mathbf{x}; \mathbf{v})$ be an arbitrary polyhedral neighbourhood of the halfline $p({}_0\mathbf{x}; \mathbf{v})$ and H the open halfspace in E_n with the boundary R_H, and with the properties*

1) ${}_0\mathbf{x} \in H$,

2) *the set $U({}_0\mathbf{x}; \mathbf{v}) \cap R_H$ is non-empty and bounded.*

Then we call the set

$$\gamma(H; P) = U({}_0\mathbf{x}; \mathbf{v}) \cap R_H$$

a proper cut of the neighbourhood $U({}_0\mathbf{x}; \mathbf{v})$ of the halfline $p({}_0\mathbf{x}; \mathbf{v})$.

Definition 3. *A proper cut $\gamma'(H'; P)$ of a polyhderal neighbourhood $U({}_0\mathbf{x}; \mathbf{v})$ is called finer than another proper cut $\gamma(H; P)$ of the same neighbourhood $U({}_0\mathbf{x}; \mathbf{v})$, if it holds*

$$\gamma'(H'; P) \subset (H \cap U({}_0\mathbf{x}; \mathbf{v}))\,.$$

Remark 1. If $\gamma'(H'; P)$ is finer than $\gamma(H; P)$, we shall write

$$\gamma'(H'; P) \prec \gamma(H; P)\,.$$

Definition 4. *Given a set $A \subset E_n$ and ${}_0\mathbf{x} \in \bar{A}$, we call the halfline $p({}_0\mathbf{x}; \mathbf{v})$ with the property that to any its polyhedral neighbourhood $U({}_0\mathbf{x}; \mathbf{v})$ there is a proper cut $\gamma(H; P)$ such that*

$$\gamma'(H'; P) \cap A \neq \emptyset$$

for all proper cuts $\gamma'(H'; P) \prec \gamma(H; P)$, the σ-halfline of the set A at the point ${}_0\mathbf{x}$.

Definition 5. *Let A be a given set and ${}_0\mathbf{x} \in \bar{A}$ a fixed chosen point. Let us denote by $\sum_\sigma p({}_0\mathbf{x}; \mathbf{v})$ the set of all σ-halflines of the set A at the point ${}_0\mathbf{x}$. Then the set*

$$S({}_0\mathbf{x}; A) = \{{}_0\mathbf{x}\} \cup \sum_\sigma p({}_0\mathbf{x}; \mathbf{v})$$

is called the contact cone of the set A at the point ${}_0\mathbf{x}$.

Before we proceed to the study of some properties of contact cones, we prove two lemmas.

Let $\varepsilon > 0$, $\mathbf{v} \neq \mathbf{o}$ and ${}_0\mathbf{x} \in E_n$ be arbitrary. We denote by $K_{\mathbf{v}}({}_0\mathbf{x}; \varepsilon)$ the rotary double-cone with the vertex ${}_0\mathbf{x}$, with the axis determined by the vector $\mathbf{v}$ and with the vertex-angle 2φ, where $\operatorname{tg} \varphi = 1/\varepsilon$.

Lemma 1. *Let $\partial M = \{\mathbf{x} \in E_n \mid F(\mathbf{x}) = 0\}$ be a smooth manifold in E_n*[1]*. Let ${}_0\mathbf{x} \in \partial M$. Then to any $\varepsilon > 0$ there is a neighbourhood $U({}_0\mathbf{x})$ of the point ${}_0\mathbf{x}$ such that*

$$\left(U({}_0\mathbf{x}) \cap \partial M \setminus \{{}_0\mathbf{x}\}\right) \subset \operatorname{ext} K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x}; \varepsilon)\,.$$

Proof. We find the description of the double-cone $K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x}; \varepsilon)$ first. If 2φ is its vertex angle, then any point $\mathbf{x}$ from its surface satisfies

$$\cos^2 \varphi = \left[\frac{\sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x})\,(x^\alpha - {}_0x^\alpha)}{\|\nabla F({}_0\mathbf{x})\|\;\|\mathbf{x} - {}_0\mathbf{x}\|}\right]^2 . \tag{1}$$

Setting

$$\begin{aligned} \mu &= \|\mathbf{x} - {}_0\mathbf{x}\| \cos \varphi\,,\\ \nu &= \|\mathbf{x} - {}_0\mathbf{x}\| \sin \varphi\,, \end{aligned} \tag{2}$$

we have

$$\frac{\nu}{\mu} = \frac{1}{\varepsilon} \tag{3}$$

and

$$\nu^2 = \|\mathbf{x} - {}_0\mathbf{x}\|^2 \left(1 - \left[\frac{\sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x})\,(x^\alpha - {}_0x^\alpha)}{\|\nabla F({}_0\mathbf{x})\|\;\|\mathbf{x} - {}_0\mathbf{x}\|}\right]^2\right).$$

From (1), (2) and (3) it follows

$$\|\mathbf{x} - {}_0\mathbf{x}\|^2 \left[\frac{\sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x})\,(x^\alpha - {}_0x^\alpha)}{\|\nabla F({}_0\mathbf{x})\|\;\|\mathbf{x} - {}_0\mathbf{x}\|}\right]^2 =$$

$$= \varepsilon^2 \left(\|\mathbf{x} - {}_0\mathbf{x}\|^2 - \left[\frac{\sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x})\,(x^\alpha - {}_0x^\alpha)}{\|\nabla F({}_0\mathbf{x})\|}\right]^2\right).$$

This yields

$$\|\mathbf{x} - {}_0\mathbf{x}\|^2\, \|\nabla F({}_0\mathbf{x})\| - \left(1 + \frac{1}{\varepsilon^2}\right) \left[\sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x})\,(x^\alpha - {}_0x^\alpha)\right]^2 = 0\,,$$

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[1]) *If the function $F(\mathbf{x})$ has continuous partial derivatives on some field $\Omega \subset E_n$ and if there is $\nabla F(\mathbf{x}) \neq \mathbf{o}$ at any point of the set Ω, then we call the set $\{\mathbf{x} \in E_n \mid F(\mathbf{x}) = 0\}$ smooth manifold in E_n.*

which is the equation of the surface of the double-cone $K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x}; \varepsilon)$. If we substitute $\mathbf{x} = {}_0\mathbf{x} + \nabla F({}_0\mathbf{x})$ in this equation, we obtain

$$\|\nabla F({}_0\mathbf{x})\|^4 - \left(1 + \frac{1}{\varepsilon^2}\right) \|\nabla F({}_0\mathbf{x})\|^4 \leqq 0$$

and so we conclude that the inequality

$$\text{(4)} \qquad \|\mathbf{x} - {}_0\mathbf{x}\|^2 \, \|\nabla F({}_0\mathbf{x})\|^2 - \left(1 + \frac{1}{\varepsilon^2}\right) \left[\sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x}) \, (x^\alpha - {}_0x^\alpha)\right]^2 \leqq 0$$

characterizes the double-cone $K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x}; \varepsilon)$. Let now $\boldsymbol{h} \neq \mathbf{o}$ be any vector with the property ${}_0\mathbf{x} + \boldsymbol{h} \in T({}_0\mathbf{x})$, where $T({}_0\mathbf{x}) \equiv \{\mathbf{x} \in F_n \mid \sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x}) \, (x^\alpha - {}_0x^\alpha) = 0\}$ and let us choose $\varepsilon > 0$. We construct the straight-line p_h with the description

$$\mathbf{x} = {}_0\mathbf{x} + \boldsymbol{h} + \tau \, \nabla F({}_0\mathbf{x}) \,, \quad \tau \in (-\infty, \infty) \,.$$

The implicit function theorem implies that for $\|\boldsymbol{h}\|$ sufficiently small this line intersects the manifold ∂M at a single point $\mathbf{y}_t = {}_0\mathbf{x} + \boldsymbol{h} + t \, \nabla F({}_0\mathbf{x})$. Expanding the function $F(\mathbf{x})$ about ${}_0\mathbf{x}$, we get

$$F({}_0\mathbf{x} + \boldsymbol{h} + t \, \nabla F({}_0\mathbf{x})) - F({}_0\mathbf{x}) = \sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x}) \, (h^\alpha + tF_\alpha({}_0\mathbf{x})) + R \,,$$

where

$$\frac{R}{\|\boldsymbol{h} + t \, \nabla F({}_0\mathbf{x})\|} \to 0 \quad \text{as} \quad \|\boldsymbol{h} + t \, \nabla F({}_0\mathbf{x})\| \to 0 \,.$$

Hence we see that $\sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x}) \, h^\alpha + t\|\nabla F({}_0\mathbf{x})\|^2 + R = 0$ and referring to $\sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x}) \, h^\alpha = 0$, we have

$$t = - \frac{R}{\|\nabla F({}_0\mathbf{x})\|^2} \,.$$

Thus

$$\frac{t}{\|\mathbf{y}_t - {}_0\mathbf{x}\|} = \frac{t(\boldsymbol{h})}{\|\boldsymbol{h} + t(\boldsymbol{h}) \, \nabla F({}_0\mathbf{x})\|} \to 0 \quad \text{as} \quad \|\mathbf{y}_t - {}_0\mathbf{x}\| \to 0 \,,$$

i.e., sine of the angle of the vector $\mathbf{y}_t - {}_0\mathbf{x}$ with the tangent hyperplane $T({}_0\mathbf{x})$ of the manifold ∂M at the point ${}_0\mathbf{x}$ tends to zero as $\|\mathbf{y}_t - {}_0\mathbf{x}\| \to 0$. We also see that there exists a neighbourhood $U({}_0\mathbf{x})$ of the point ${}_0\mathbf{x}$ such that the vectors $\mathbf{y}_t - {}_0\mathbf{x}$ for $\mathbf{y}_t \in (U({}_0\mathbf{x}) \cap \partial M)$, include angles with the tangent hyperplane $T({}_0\mathbf{x})$, which are less than $\frac{1}{2}\pi - \varphi$. In other words, the points of the set $U({}_0\mathbf{x}) \cap \partial M \setminus \{{}_0\mathbf{x}\}$ belong to ext $K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x}; \varepsilon)$.

Lemma 2. *Let P be a polyhedral cone with a vertex ${}_0\mathbf{x}$ and $N = \{\mathbf{x} \in E_n \mid \sum_{\alpha=1}^{n} a^\alpha \,.$ $.\, (x^\alpha - {}_0x^\alpha) = 0\}$, $\|\mathbf{a}\| = 1$ a hyperplane in E_n with the properties*

$$P \cap N = \{{}_0\mathbf{x}\}\,,$$
$$P \setminus \{{}_0\mathbf{x}\} \subset H^+\,,$$

where $H^+ = \{\mathbf{x} \in E_n \mid \sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha) > 0\}$. Then there exists a rotary cone $K_{\mathbf{a}}^+({}_0\mathbf{x};$ $\varepsilon)$ with the vertex ${}_0\mathbf{x}$ and with the axis determined by vector $\mathbf{a}$, which fulfils

1) $P \subset K_{\mathbf{a}}^+({}_0\mathbf{x}; \varepsilon)$,

2) $K_{\mathbf{a}}^+({}_0\mathbf{x}; \varepsilon) \setminus \{{}_0\mathbf{x}\} \subset H^+$.

Proof. Let us choose $\varepsilon > 0$ arbitrary. By (4), the cone $K_{\mathbf{a}}^+({}_0\mathbf{x}; \varepsilon) = \{\mathbf{x} \in E_n \mid$ $\|\mathbf{x} - {}_0\mathbf{x}\| - \sqrt{(1 + 1/\varepsilon^2)} \sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha) \leqq 0\}$ has the property 2). Denote by $\mathfrak{M}$ the set of all halflines belonging to P and starting at the point ${}_0\mathbf{x}$. In the case $\mathfrak{M} = \emptyset$ the statement clearly holds. Let $\mathfrak{M} \neq \emptyset$ and define

$$Q = \{\mathbf{x} \in E_n \mid \|\mathbf{x} - {}_0\mathbf{x}\| = 1\}\,,$$
$$A_P = P \cap Q\,.$$

Since $P \setminus \{{}_0\mathbf{x}\} \subset H^+$, we have $\sum_{\alpha=1}^{n} a^\alpha(x - {}_0x^\alpha) > 0$ for all $\mathbf{x} \in P \setminus \{{}_0\mathbf{x}\}$ and thus also for $\mathbf{x} \in A_P$. Taking into account that the sets P and Q are closed, Q compact, we obtain

$$\mu = \inf_{\mathbf{x} \in A_P} \sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha) = \min_{\mathbf{x} \in A_P} \sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha) > 0\,.$$

In view of $|\sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha)| \leqq \|\mathbf{a}\| \, \|\mathbf{x} - {}_0\mathbf{x}\| = 1$ for all $\mathbf{x} \in A_P$, we find $\mu \leqq 1$. We introduce

$$\varepsilon_0 = \frac{\mu}{\sqrt{(2 - \mu^2)}}\,.$$

Hence $\varepsilon_0 > 0$ and from the definition of ε_0 it follows further that

$$1 + \frac{1}{\varepsilon^2} = \frac{2}{\mu^2} > \frac{1}{\mu^2} = \frac{1}{[\min_{\mathbf{x} \in A_P} \sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha)]^2} \geqq \frac{1}{[\sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha)]^2}$$
$$\text{for} \quad \mathbf{x} \in A_P\,,$$

which becomes

$$1 + \frac{1}{\varepsilon^2} \geqq \frac{\|\mathbf{x} - {}_0\mathbf{x}\|^2}{[\sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha)]^2} \quad \text{for} \quad \mathbf{x} \in P \setminus \{{}_0\mathbf{x}\} .$$

Therefore, it is

$$\|\mathbf{x} - {}_0\mathbf{x}\| - \sqrt{\left(1 + \frac{1}{\varepsilon_0^2}\right)} \sum_{\alpha=1}^{n} a^\alpha(x^\alpha - {}_0x^\alpha) \leqq 0 \quad \text{for} \quad \mathbf{x} \in P$$

and for that reason

$$P \subset K_{\boldsymbol{a}}^{+}({}_0\mathbf{x}; \varepsilon) .$$

Remark 2. Obviously, a neighbourhood $U({}_0\mathbf{x})$ with $(U({}_0\mathbf{x}) \cap \partial M \setminus \{{}_0\mathbf{x}\}) \subset \subset \operatorname{ext} K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x}; \varepsilon)$ fulfils $U({}_0\mathbf{x}) \cap M \cap \operatorname{int} K^{+}_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x}; \varepsilon) = \emptyset$, where $M = \{\mathbf{x} \in E_n \mid F(\mathbf{x}) \leqq 0\}$.

Theorem 1. *The contact cone of a smooth manifold $\partial M = \{\mathbf{x} \in E_n \mid F(\mathbf{x}) = 0\}$ at its point ${}_0\mathbf{x}$ is identical with the tangent hyperplane of this manifold at the point ${}_0\mathbf{x}$.*

Proof. Let $\xi^1, \ldots, \xi^n$ be the coordinates of the point $\mathbf{x}$ in the Cartesian coordinate system with the origin ${}_0\mathbf{x}$ and with axes $\boldsymbol{a}_i$, $i = 1, \ldots, n$, where $\boldsymbol{a}_i + {}_0\mathbf{x} \in T({}_0\mathbf{x}) = \{\mathbf{x} \in E_n \mid \sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x})(x^\alpha - {}_0x^\alpha) = 0\}$, $i = 1, \ldots, n - 1$, $\boldsymbol{a}_n = \nabla F({}_0\mathbf{x})$ i.e., $\mathbf{x} = {}_0\mathbf{x} + \sum_{\alpha=1}^{n} \xi^\alpha \boldsymbol{a}_\alpha$. By the implicit function theorem we can express $\xi^n = f(\xi^1, \ldots, \xi^{n-1})$ from the equation $F({}_0\mathbf{x} + \sum_{\alpha=1}^{n} \xi^\alpha \boldsymbol{a}_\alpha) = 0$ in a certain neighbourhood $U({}_0\mathbf{x})$. Let $p \subset T({}_0\mathbf{x})$ be the halfline with description

$$\xi^1 = \lambda\bar{\xi}^1, \ldots, \xi^{n-1} = \lambda\bar{\xi}^{n-1}, \quad \xi^n = 0, \quad \lambda \geqq 0,$$

where

$$(\bar{\xi}^1, \ldots, \bar{\xi}^{n-1}) \neq (0, \ldots, 0) .$$

Let $U({}_0\mathbf{x}; p)$ be an arbitrary polyhedral neighbourhood of the halfline p. As the intersection of the plane R rectangular to $T({}_0\mathbf{x})$ and containing p with the manifold ∂M, we obtain the curve k with description

$$\xi^1 = \mu\bar{\xi}^1, \ldots, \xi^{n-1} = \mu\bar{\xi}^{n-1}, \quad \xi^n = f(\mu\bar{\xi}^1, \ldots, \mu\bar{\xi}^{n-1}),$$

$$\mu \in \langle \mu_1, \mu_2 \rangle, \quad \mu_1 < \mu_2$$

(we consider the points of the neighbourhood $U({}_0\mathbf{x})$ only). Then $R \cap U({}_0\mathbf{x}; p)$ is a two-dimensional cone with vertex ${}_0\mathbf{x}$ in the plane R and the halfline p is contained in its relative interior. Since the straightline described by

$$\xi^1 = t\bar{\xi}^1, \ldots, \xi^{n-1} = t\bar{\xi}^{n-1}, \quad \xi^n = 0, \quad t \in (-\infty, \infty),$$

is tangent to the curve k at the point ${}_0\mathbf{x}$, there exists such a proper cut of the neighbourhood $U({}_0\mathbf{x};\, p)$ that any finer cut intersects the curve k. This means that p is a σ-halfline of the set ∂M at the point ${}_0\mathbf{x}$. So we have established the inclusion $T({}_0\mathbf{x}) \subset S({}_0\mathbf{x};\, \partial M)$. It remains to show that p cannot be a σ-halfline of the set ∂M at the point ${}_0\mathbf{x}$, supposing p does not belong to the tangent hyperplane $T({}_0\mathbf{x})$. When $p \not\subset T({}_0\mathbf{x})$, we easily find a double-cone $K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x};\, \varepsilon)$ with the property $(p \setminus \{{}_0\mathbf{x}\}) \subset \operatorname{int} K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x};\, \varepsilon)$. In virtue of Lemma 1, there is $U({}_0\mathbf{x})$ with $(U({}_0\mathbf{x}) \cap \partial M \setminus \{{}_0\mathbf{x}\}) \subset \operatorname{ext} K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x};\, \varepsilon)$. Now it suffices to choose such a polyhderal neighbourhood of the halfline p that it may be included inside $K_{\nabla F({}_0\mathbf{x})}({}_0\mathbf{x};\, \varepsilon)$ and we see that p cannot be a σ-halfline of the set ∂M at ${}_0\mathbf{x}$, for no proper cut of this neighbourhood intersects the set ∂M locally.

Theorem 2. *Let the boundary ∂M of the set $M = \{\mathbf{x} \in E_n \mid F(\mathbf{x}) \leqq 0\}$ be a smooth manifold in E_n. Then the contact cone of the set M at the point ${}_0\mathbf{x} \in \partial M$ is the halfspace $\overline{H^-} = \{\mathbf{x} \in E_n \mid \sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x})\,(x^\alpha - {}_0x^\alpha) \leqq 0\}$.*

Proof. By the preceding theorem $T({}_0\mathbf{x}) \subset S({}_0\mathbf{x};\, M)$. Let $\mathbf{y} \neq {}_0\mathbf{x}$ and suppose $p_y \equiv \{\mathbf{x} \in E_n \mid \mathbf{x} = {}_0\mathbf{x} + \lambda(\mathbf{y} - {}_0\mathbf{x}), \lambda > 0\} \subset H^-$, where $H^- = \{\mathbf{x} \in E_n \mid \sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x}) \,.\, (x^\alpha - {}_0x^\alpha) < 0\}$. By Taylor's theorem we have $F({}_0\mathbf{x} + \lambda(\mathbf{y} - {}_0\mathbf{x})) < 0$ for $\lambda > 0$ sufficiently small. Thus there is a neighbourhood $U({}_0\mathbf{x})$ of the point ${}_0\mathbf{x}$ with $(U({}_0\mathbf{x}) \cap p) \subset M$ and therefore p is a σ-halfline of the set M at the point ${}_0\mathbf{x}$.

On the other hand, if follows from the proof of Theorem 1 that for $p \subset H^+$, where $H^+ = \{\mathbf{x} \in E_n \mid \sum_{\alpha=1}^{n} F_\alpha({}_0\mathbf{x})\,(x^\alpha - {}_0x^\alpha) > 0\}$, there exists a polyhedral neighbourhood $U({}_0\mathbf{x};\, p)$ of the halfline p and a neighbourhood $U({}_0\mathbf{x})$ of the point ${}_0\mathbf{x}$ with $U({}_0\mathbf{x};\, p) \cap U({}_0\mathbf{x}) \cap M = \emptyset$, which shows that p cannot be a σ-halfline of the set M at the point ${}_0\mathbf{x}$.

Remark 3. Theorem 2 for the set int M, with M given as in the formulation of Theorem 2, and closedness of any contact cone (see [2]) yield the identity $S({}_0\mathbf{x};\, \operatorname{int} M) = S({}_0\mathbf{x};\, M)$.

Let $F(\mathbf{x})$, $G_i(\mathbf{x})$, $(i = 1, \ldots, m)$ be hereafter continuously differentiable functions in E_n and let the boundaries ∂M and ∂N_i of the sets

$$M = \{\mathbf{x} \in E_n \mid F(\mathbf{x}) \leqq 0\}\,,$$

$$N_i = \{\mathbf{x} \in E_n \mid G_i(\mathbf{x}) \leqq 0\}\,, \quad i = 1, \ldots, m\,,$$

respectively, be smooth manifolds in E_n. We put

$$N = \bigcap_{i=1}^{m} N_i\,.$$

For the sake of convenience we shall assume that no constraint determining the set N in the neighbourhood of the point ${}_0\mathbf{x} \in \partial M \cap \partial N$, is redundant. Denote by I the set of such $i \in \{1, \ldots, m\}$ for which $G_i({}_0\mathbf{x}) = 0$, while for $i \in \{1, \ldots, m\} \setminus I$ it is $G_i({}_0\mathbf{x}) < 0$.

Theorem 3. *The contact cone $S({}_0\mathbf{x}; N)$ of the set N at the point ${}_0\mathbf{x}$ satisfies the relation*

$$S({}_0\mathbf{x}; N) \subset \bigcap_{i \in I} S({}_0\mathbf{x}; N_i)\,.$$

Proof. We consider the non-trivial case $S({}_0\mathbf{x}; N) \neq \{{}_0\mathbf{x}\}$. Let p be a σ-halfline of the set N at the point ${}_0\mathbf{x}$ and $U({}_0\mathbf{x}; p)$ its arbitrary polyhedral neighbourhood. Then there is such a proper cut of this neighbourhood that all finer proper cuts have non-empty intersections with the set N. Hence p is a σ-halfline of each set N_i, $i \in I$.

Remark 4. The identity $S({}_0\mathbf{x}; N) = \bigcap_{i \in I} S({}_0\mathbf{x}; N_i)$ does not hold generally. E. G., for $N_1 = \{(x^1, x^2) \in E_2 \mid -x^2 + (x^1)^2 \leqq 0\}$, $N_2 = \{(x^1, x^2) \in E_2 \mid x^2 + (x^1)^2 \leqq 0\}$ and ${}_0\mathbf{x} = (0, 0)$ we have $S({}_0\mathbf{x}; N) = \{{}_0\mathbf{x}\}$ and $S({}_0\mathbf{x}; N_1) \cap S({}_0\mathbf{x}; N_2) = \{(x^1, x^2) \in E_2 \mid x^2 = 0\}$.

In Theorem 4 a sufficient condition is given for the identity mentioned above to be satisfied. Before formulating this theorem we shall need to introduce the following definition.

Definition 6. *Let $K_1, \ldots, K_l$ be convex cones in E_n with a common vertex ${}_0\mathbf{x}$. The system of cones $K_1, \ldots, K_l$ will be said to be separable, if there is a hyperplane in E_n containing the point ${}_0\mathbf{x}$ and separating one arbitrary cone from the intersection of the others (i.e., for some $i_0 \in \{1, \ldots, l\}$ the cone K_{i_0} belongs to one of the closed halfspaces determined by this hyperplane and the intersection of the others to the opposite one).*

Lemma 3. (See [1].) *Let convex sets $M_1, \ldots, M_l$ in E_n satisfy*

$$\operatorname{rel\,int} M_1 \cap \ldots \cap \operatorname{rel\,int} M_l \neq \emptyset\,.$$

Then

$$\operatorname{rel\,int}(M_1 \cap \ldots \cap M_l) = \operatorname{rel\,int} M_1 \cap \ldots \cap \operatorname{rel\,int} M_l\,.$$

Lemma 4. (See [1].) *When a system of convex cones $K_1, \ldots, K_l$ with a common vertex is not separable in E_n, then*

$$\operatorname{rel\,int} K_1 \cap \ldots \cap \operatorname{rel\,int} K_l \neq \emptyset\,.$$

Theorem 4. *If a collection of contact cones $S({}_0\mathbf{x}; N_i)$, $i \in I$, is not separable in E_n, then*

$$S({}_0\mathbf{x}; N) = \bigcap_{i \in I} S({}_0\mathbf{x}. N_i)\,.$$

Proof. We use induction.

In this proof we shall assume without loss of generality that $I = \{1, \ldots, k\}$.

1) Let $k = 2$. It is sufficient to show $S({}_0\mathbf{x}; N_1) \cap S({}_0\mathbf{x}; N_2) \subset S({}_0\mathbf{x}; N)$. According to Lemmas 3 and 4 we have $\operatorname{int} (S({}_0\mathbf{x}; N_1) \cap S({}_0\mathbf{x}; N_2)) = \operatorname{int} S({}_0\mathbf{x}; N_1) \cap \operatorname{int} S({}_0\mathbf{x}; N_2) \neq \emptyset$. Let p be a σ-halfline of the sets N_1 and N_2 at the point ${}_0\mathbf{x}$. There is always a halfline p' with $p' \subset (\operatorname{int} S({}_0\mathbf{x}\ N_1) \cap \operatorname{int} S({}_0\mathbf{x}; N_2))$, in any one of its polyhedral neighbourhoods $U({}_0\mathbf{x}\ p)$. From the proof of Theorem 2 we see that in a certain neighbourhood $P({}_0\mathbf{x})$ of the point ${}_0\mathbf{x}$ the points of the halfline $p' \setminus \{{}_0\mathbf{x}\}$ belong to the set $\operatorname{int} N_1 \cap \operatorname{int} N_2$. Hence it is obvious that necessarily $p \subset S({}_0\mathbf{x}; N)$, which is caused by the existence of such a proper cut γ of the polyhedral neighbourhood $U({}_0\mathbf{x}\ p)$ that any finer proper cut intersects the set $P({}_0\mathbf{x}) \cap p'$.

2) Let the statement holds for $k > 2$. Let the system $S({}_0\mathbf{x}; N_1), \ldots, S({}_0\mathbf{x}; N_{k+1})$ be not separable in E_n. Then the system $S({}_0\mathbf{x}; N_1), \ldots, S({}_0\mathbf{x}; N_k)$ is not separable in E_n either. Now part 1) of this proof can be repeated in terms of $M_1 = \bigcap_{i=1}^{k} N_i$, $M_2 = N_{k+1}$.

Theorem 5. *Let $M \cap N \neq \emptyset$. Then the following implication is valid: ${}_0\mathbf{x} \in M \cap N$, $\operatorname{int} S({}_0\mathbf{x}; M) \cap S({}_0\mathbf{x}; N) \neq \emptyset \Rightarrow U({}_0\mathbf{x}) \cap \operatorname{int} M \cap N \neq \emptyset$ for any neighbourhood $U({}_0\mathbf{x})$ of the point ${}_0\mathbf{x}$.*

Proof. In virtue of Theorem 2, $S({}_0\mathbf{x}; M)$ is, for ${}_0\mathbf{x} \in \partial M$, a closed halfspace whose boundary we denote by R. Introducing a Cartesian coordinate system with origin ${}_0\mathbf{x}$ and axes $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}$, $\nabla F({}_0\mathbf{x})$, where ${}_0\mathbf{x} + \mathbf{a}_i \in R$, $i = 1, \ldots, n-1$ and $\mathbf{a}_i$, $(i = 1, \ldots, n-1)$ are orthogonal vectors, we can write any point $\mathbf{x} \in E_n$ uniquely in the form $\mathbf{x} = {}_0\mathbf{x} + \sum_{i=1}^{n-1} \xi^i \mathbf{a}_i + \xi^n \nabla F({}_0\mathbf{x})$. According to the implicit function theorem, in a certain neighbourhood of the point ${}_0\mathbf{x}$ it is possible to transcribe the equation $F({}_0\mathbf{x} + \sum_{i=1}^{n-1} \xi^i \mathbf{a}_i + \xi^n \nabla F({}_0\mathbf{x})) = 0$ equivalently in the form $\xi^n = f(\xi^1, \ldots, \xi^{n-1})$, where

$$\frac{\|f(\xi^1, \ldots, \xi^{n-1})\|}{\|(\xi^1, \ldots, \xi^{n-1})\|} \to 0 \quad \text{as} \quad \|(\xi^1, \ldots, \xi^{n-1})\| \to 0\,. \tag{5}$$

In our new coordinate system the tangent hyperplane R of the manifold ∂M at the point ${}_0\mathbf{x}$ has the description $\xi^n > 0$. If $\mathbf{y}$ is a fixed chosen point with $\mathbf{y} \notin R$, then the halfline $p = \{\mathbf{x} \in E_n \mid \mathbf{x} = {}_0\mathbf{x} + \lambda(\mathbf{y} - {}_0\mathbf{x}), \lambda > 0\}$ does not intersect the manifold ∂M in a certain neighbourhood $\tilde{U}({}_0\mathbf{x})$. This is caused by (5) and by the fact that the points $(\xi^1, \ldots, \xi^n) \in p$ satisfy $\xi^n / \|(\xi^1, \ldots, \xi^{n-1})\| = \text{konst} \neq 0$. Thus we have established the existence of a neighbourhood $\tilde{U}({}_0\mathbf{x})$ such that either $p \cap \tilde{U}({}_0\mathbf{x}) \subset \operatorname{int} M$, or $p \cap \tilde{U}({}_0\mathbf{x}) \subset \operatorname{ext} M$ (see also the proof of Theorem 2). Therefore $(p \cap \tilde{U}({}_0\mathbf{x})) \subset \operatorname{int} M$, provided that $p \subset \operatorname{int} S({}_0\mathbf{x}; M) \cap S({}_0\mathbf{x}; N)$.

Let $U({}_0\mathbf{x};\, p)$ be an arbitrary polyhedral neighbourhood of the half-line p with the property $U({}_0\mathbf{x};\, p) \subset \operatorname{int} S({}_0\mathbf{x};\, M)$. Lemmas 1 and 2 imply the existence of a neighbourhood $O({}_0\mathbf{x})$ of the point ${}_0\mathbf{x}$ with $(O({}_0\mathbf{x}) \cap U({}_0\mathbf{x};\, p)) \subset \operatorname{int} M$. As p is a σ-halfline of N, there exist points $\mathbf{z} \in N$ in $U({}_0\mathbf{x};\, p) \cap U({}_0\mathbf{x})$ for any neighbourhood $U({}_0\mathbf{x})$. This completes the proof, since it has been shown that there are points $\mathbf{z} \in \operatorname{int} M \cap N$ in any neighbourhood $U({}_0\mathbf{x})$ of the point ${}_0\mathbf{x}$.

Remark 5. We emphasize that no special expression for the set N was needed in the proof of Theorem 5 and hence this theorem remains true for a general set N.

Definition 7. *The sets A and B are said to be locally disjoint at a point ${}_0\mathbf{x} \in \bar{A} \cap \bar{B}$, if there is a neighbourhood $U({}_0\mathbf{x})$ with*

$$U({}_0\mathbf{x}) \cap A \cap B \setminus \{{}_0\mathbf{x}\} = \emptyset\,.$$

The nonlinear programming problem:

Find minimal point of a function $F(\mathbf{x})$ over a set N under the same assumptions on the function $F(\mathbf{x})$ as above, where N can be an arbitrary non-empty set, is reduced to finding the point ${}_0\mathbf{x} \in N$ with $U({}_0\mathbf{x}) \cap N \cap \operatorname{int} M = \emptyset$ for a certain neighbourhood $U({}_0\mathbf{x})$ (we suppose without loss of generality that $F({}_0\mathbf{x}) = 0$). For ${}_0\mathbf{x}$ to be a solution of this problem it is necessary that the sets $\operatorname{int} M$ and N are locally disjoint at the point ${}_0\mathbf{x}$.

Concluding remark. Theorem 5 gives a necessary condition for the existence of solution of the nonlinear programming problem mentioned above. An analogous necessary condition formulated by means of so called tent of the set, has derived by Boltjanskij in [3].

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Souhrn

NUTNÁ PODMÍNKA PRVNÍHO ŘÁDU PRO EXISTENCI LOKÁLNÍHO EXTRÉMU V ÚLOZE NELINEÁRNÍHO PROGRAMOVÁNÍ

JAN PALATA

V článku je podána nutná podmínka pro existenci lokálního extrému jedenkrát spojitě diferencovatelné (obecně nekonvexní) funkce na obecné množině.

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