Aplikace matematiky

Rolf Hünlich; Joachim Naumann

On general boundary value problems and duality in linear elasticity. II

Aplikace matematiky, Vol. 25 (1980), No. 1, 11–32

Persistent URL: http://dml.cz/dmlcz/103834

ON GENERAL BOUNDARY VALUE PROBLEMS AND DUALITY IN LINEAR ELASTICITY, II

ROLF HÜNLICH, JOACHIM NAUMANN

(Received January 19, 1978)

In the present part of our paper we complete the discussion presented in Part I in two directions.

Firstly, in Section 5 a number of existence theorems for a solution to Problem III (principle of minimum potential energy) is established. Here the particular interest is devoted to certain special cases of the functional φ which are suggested by the examples considered in Section 3.

Secondly, Sections 6 and 7 concern a relatively detailed discussion of the dual formulation of Problem III. In Section 6 we first generalize the classical approach to the dual problem and then put our discussion into the framework of the abstract duality theory. Finally, in Section 7 we introduce two problems which are conjugate to each other in the sense of [5]. By virtue of their equivalence to Problem I (boundary value problem) one obtains a transparent indication of the relationship between the solvability of Problem III and its dual one.

5. EXISTENCE THEOREMS

The aim of the present section is to prove the existence of a solution to Problem I by making use of its variational formulation (Problem III; cf. Theorem 4.1). For technical convenience, in what follows Problem III will be written in the equivalent form:

Principle of minimum potential energy. *Find $u \in \mathscr{V}$ such that*

$$F(v) \geqq F(u) \quad \forall v \in \mathscr{V} .$$

We are now going to present a number of conditions under which the functional F is increasing on $\mathscr{V}$. These conditions are mainly motivated by the examples discussed in Section 3.

For a more theoretical discussion of the existence of a solution to the variational inequality (4.2) (semi-coercive case) we only mention the papers [11], [6], [14]. Let us in particular refer to [3], [4] where a profound investigation of the Signorini problem may be found. ∎

1° Let us begin with the following simple case:

$$\text{(5.1)} \qquad \begin{cases} \mathscr{V}_{\text{ad}} \subseteq \{v_0\} + \mathscr{V}_0\,, \\ \textit{where } v_0 \in \mathscr{V} \textit{ is fixed, } \mathscr{V}_0 \textit{ is a closed subspace of } \mathscr{V} \textit{ such that} \\ \mathscr{V}_0 \cap \mathscr{R} = \{0\}. \end{cases}$$

We then have

Theorem 5.1. *Let condition* (5.1) *be satisfied. Then Problem III possesses exactly one solution.*

Proof. First of all, by the Hahn-Banach Theorem there exist constants $c_i \leqq 0$ $(i = 1, 2)$ such that

$$\varphi(h) \geqq c_1 \|h\|_V + c_2 \quad \forall h \in V.$$

Let $v \in \mathscr{V}_{\text{ad}}$, i.e. $v = v_0 + w$, $w \in \mathscr{V}_0$. Observing Lemma 1.1 we get

$$F(v) \geqq a(v_0, w) + \tfrac{1}{2}\, a(w, w) + \varphi(\gamma(v)) - (f, v) \geqq$$
$$\geqq k_1 \|w\|^2 + k_2 \|w\| + k_3 \|v\| + k_4 \geqq \tfrac{1}{2}\, k_1 \|v\|^2 + k_5$$

where $k_1 = \text{const} > 0$, $k_i = \text{const} \leqq 0$ $(i = 2, \ldots, 5)$. If $v \notin \mathscr{V}_{\text{ad}}$ then $F(v) = +\infty$.

Thus, the functional F being convex and lower semi-continuous, there exists at least one $u \in \mathscr{V}$ at which F attains its minimum on $\mathscr{V}$. If $\bar{u} \in \mathscr{V}$ is another function that renders the functional F its minimum on $\mathscr{V}$, we have $u - \bar{u} \in \mathscr{V}_0 \cap \mathscr{R} = \{0\}$ (cf. Section 4.2). ∎

Remark 5.1. Let us make some observations concerning the assumption $\mathscr{V}_0 \cap \mathscr{R} = \{0\}$. Suppose $\Omega \in C^{0,1}$.

(i) *Let $M \subset \Gamma$ be any non-empty set, open in Γ. Then*
(cf. [8; Lemma II.3]).

$$\varrho \in \mathscr{R}\;,\; \varrho = 0 \quad on \quad M \Rightarrow \varrho \equiv 0$$

(ii) *Under appropriate conditions on the shape of Γ there exist subsets $M \subset \Gamma$ such that*

$$\varrho \in \mathscr{R}\,, \quad \varrho_n = 0 \quad on \quad M \Rightarrow \varrho \equiv 0$$

(cf. [8]). ∎

The following two lemmas yield related results.

Lemma 5.1. *Let $\Omega \in C^{1,0}$. Then:*

$$\varrho \in \mathscr{R}, \quad \varrho_t = 0 \quad on \quad \Gamma \Rightarrow \varrho \equiv 0 .$$

Proof. Let us first of all note that the mapping $x \mapsto n(x)$ is continuous and surjective from Γ onto S^2 (the unit sphere in $\mathbb{R}^3$) [1]). Now, by our assumption,

$$a + b \times x = \varrho(x) = \varrho_n(x)\, n(x) \quad \forall x \in \Gamma . \tag{5.2}$$

Suppose $b \neq 0$. Then there exist a point $\bar{x} \in \Gamma$ with $n_i(\bar{x})\, b_i = 0$ and a non-void, open subset $M \subset \Gamma$ such that $n_i(x)\, b_i \neq 0$ for all $x \in M$. By virtue of (5.2), $a_i b_i = \varrho_n(\bar{x})\, n_i(\bar{x})\, b_i = 0$, and therefore $\varrho_n(x) = 0$ for all $x \in M$. But this means $\varrho(x) = 0$ for all $x \in M$, a contradiction to $b \neq 0$. Hence $b = 0$. We now find a point $\tilde{x} \in \Gamma$ such that $n_i(\tilde{x})\, a_i = 0$. Then (5.2) implies $a = 0$. ∎

Lemma 5.2. *Let Ω be a cube. Then:*

$$\varrho \in \mathscr{R}, \quad \varrho_t = 0 \quad on \quad \Gamma \Rightarrow \varrho \equiv 0 .$$

Proof. Without any loss of generality, let

$$\Omega = \{x \in \mathbb{R}^3 : |x_i| < d,\ i = 1, 2, 3\} .$$

Set

$$x^{(1)} = \{0, 0, d\}, \quad x^{(2)} = \{0, 0, -d\}, \quad x^{(3)} = \{0, d, 0\} .$$

It is then readily verified that the equations

$$\varrho_t(x^{(k)}) = 0 \quad (k = 1, 2, 3)$$

imply $a = b = 0$. ∎

The argument of the proof of Lemma 5.2 obviously applies to a wide range of other domains.

We are now going to illustrate some applications of Theorem 5.1. Let $\Omega \in C^{0,1}$, and let us consider Example 1 of Section 3. Set

$$\mathscr{V}_0 = \{v \in \mathscr{V} : \gamma(v) = 0 \ a.e. \ on \ \Gamma\} .$$

Then

$$\mathscr{V}_{\mathrm{ad}} = \{u_0\} + \mathscr{V}_0 ,$$

and Theorem 5.1 yields the existence and uniqueness of a solution to the displacement boundary value problem.

Let now $\Omega \in C^{1,1}$. It is then easily seen that the boundary value problems stated in Example 5 can be treated analogously (cf. Lemma 5.1 and (ii) of Remark 5.1).

[1]) This follows (e.g.) from Theorem VI of Hopf, H.: Vektorfelder in n-dimensionalen Mannigfaltigkeiten. Math. Ann., 96 (1927), 225–250. This theorem states that the degree of the mapping $x \mapsto n(x)$ with respect to the origin is equal to 1. Hence this mapping is not homotopic to zero and therefore it is surjective (this argument was submitted by T. Friedrich to the authors).

Maintaining the assumption $\Omega \in C^{1,1}$ we consider the following variant of the Signorini problem (cf. Example 6):

$$\text{(3.8)} \qquad \begin{cases} \gamma_n(u) \leqq 0 \quad a.e.\ on\ \Gamma\,, \quad \langle \pi_n(\sigma), \gamma_n(u) \rangle_{W_2^{1/2}(\Gamma)} = 0\,, \\ \langle \pi_n(\sigma), h \rangle_{W_2^{1/2}(\Gamma)} \geqq 0 \ \forall h \in W_2^{1/2}(\Gamma)\,, \quad h \leqq 0 \quad a.e.\ on\ \Gamma\,, \end{cases}$$

$$\text{(3.9')} \qquad \gamma_t(u) = k_0$$

where $k_0 \in V_t$ is fixed. In the present case, we define

$$v_0 \in \mathscr{V} \quad such\ that \quad \gamma_t(v_0) = k_0\,,$$

$$\mathscr{V}_0 = \{v \in \mathscr{V} : \gamma_t(v) = 0 \ a.e.\ on\ \Gamma\}$$

and $\varphi(h) = \varphi_n(h_n) + \varphi_t(h_t)$ for $h \in V(h = h_n n + h_t)$, where

$$\varphi_n(h) = \begin{cases} 0 & for \quad h \in W_2^{1/2}(\Gamma)\,, \quad h \leqq 0 \quad a.e.\ on\ \Gamma\,, \\ +\infty & for \quad h \in W_2^{1/2}(\Gamma)\,, \quad h > 0 \quad on\ a\ subset \\ & of\ positive\ measure\,, \end{cases}$$

$$\varphi_t(k) = \begin{cases} 0 & for \quad k = k_0\,, \\ +\infty & for \quad k \in V_t \setminus \{k_0\}\,. \end{cases}$$

Thus

$$\mathscr{V}_{\mathrm{ad}} = \{v \in V : \gamma_n(v) \leqq 0,\ \gamma_t(v) = k_0 \ a.e.\ on\ \Gamma\}$$

and therefore

$$\mathscr{V}_{\mathrm{ad}} \subseteq \{v_0\} + \mathscr{V}_0\,.$$

Observing that $\mathscr{V}_0 \cap \mathscr{R} = \{0\}$ (cf. Lemma 5.1) we obtain the existence and uniqueness of a solution to the above variant of the Signorini problem by applying Theorem 5.1.

Based on a similar device one can prove the existence and uniqueness of a solution to (2.1), (2.2) under the boundary conditions (3.10′), (3.11) (cf. Example 7) provided that there exists a subset $M \subset \Gamma$ satisfying condition (ii) in Remark 5.1, and (3.12), (3.13′) (cf. Example 8). Let us finally note that Theorem 5.1 also applies to certain mixed boundary conditions (cf. Example 4, the case $\Gamma_1 \neq \emptyset$). ∎

2° We now impose the following conditions upon the functional φ. Let

$$\varphi = \varphi_1 + \varphi_2\,,$$

where

$$\text{(5.3)} \qquad \begin{cases} \varphi_i \ is\ proper,\ convex\ and\ lower\ semi\text{-}continuous \\ (i = 1, 2)\,, \quad D(\varphi_1) \cap D(\varphi_2) \neq \emptyset\,; \end{cases}$$

$$\text{(5.4)} \qquad \begin{cases} \varphi_1(0) = 0\,, \quad \varphi_1(th) \leqq t^\alpha\, \varphi_1(h) \quad \forall t > 0\,, \quad \forall h \in D(\varphi_1) \\ where \quad \alpha > 1\,; \end{cases}$$

$$\text{(5.5)} \qquad \varrho \in \mathscr{R}\,, \quad \varrho \neq 0 \Rightarrow \varphi_1(\gamma(\varrho)) > 0\,. \quad ∎$$

Theorem 5.2. *Suppose that the functional φ admits the above decomposition where the conditions (5.3)–(5.5) are satisfied. Then Problem III has at least one solution.*

Proof. The functional F is proper, convex and lower semi-continuous on $\mathscr{V}$; it therefore remains to show that F is increasing on $\mathscr{V}$. Suppose the contrary holds, i.e. there exist a sequence $\{v_n\} \subset \mathscr{V}$ $(n = 1, 2, \dots)$ and a constant C_0 such that

$$\|v_n\| \to +\infty \quad as \quad n \to \infty\,, \qquad F(v_n) \leqq C_0 \quad for \quad n = 1, 2, \dots .$$

Setting $w_n = v_n/\|v_n\|$ (for n sufficiently large) we thus have

$$\tag{5.6} \tfrac{1}{2}\, a(w_n, w_n) + \frac{1}{\|v_n\|^2}\, \varphi(\gamma(v_n)) \leqq \frac{C_1}{\|v_n\|}\left(1 + \frac{1}{\|v_n\|}\right)$$

$(C_1 = \text{const})$.

From the Hahn-Banach Theorem one concludes the existence of constants $c_{ij} \leqq 0$ $(i, j = 1, 2)$ such that

$$\varphi_1(h) \geqq c_{11}\|h\|_V + c_{12} \quad \forall h \in V\,,$$

$$\varphi_2(h) \geqq c_{21}\|h\|_V + c_{22} \quad \forall h \in V\,.$$

Inserting these estimates into (5.6) we get

$$\tag{5.6$_1$} a(w_n, w_n) \leqq \frac{C_2}{\|v_n\|}\left(1 + \frac{1}{\|v_n\|}\right)$$

$(C_2 = \text{const};\ n$ sufficiently large).

Further, without any loss of generality, we may assume that $w_n \to w$ weakly in $\mathscr{V}$ and $w_n \to w$ strongly in $\mathscr{H}$ as $n \to \infty$. Thus, by virtue of (5.6$_1$),

$$a(w, w) \leqq \liminf a(w_n, w_n) \leqq 0\,,$$

i.e. $w \in \mathscr{R}$. On the other hand, Lemma 1.1(i) yields

$$a(w_n, w_n) + a_0|w_n|^2 \geqq a_0 c_1\,.$$

Taking the lim inf on the left hand side of this inequality one obtains $|w| \geqq \sqrt{c_1} > 0$.

Finally, (5.6) implies

$$\varphi_1(\gamma(v_n)) \leqq C_1(\|v_n\| + 1)\,.$$

Hence

$$\varphi_1(\gamma(w_n)) \leqq \frac{C_1}{\|v_n\|^{\alpha-1}}\left(1 + \frac{1}{\|v_n\|}\right),$$

and therefore

$$\varphi_1(\gamma(w)) \leqq \liminf \varphi_1(\gamma(w_n)) \leqq 0\,,$$

a contradiction to (5.5). ∎

Theorem 5.2 may be used for proving the existence of a solution to (2.1), (2.2) under boundary conditions of elastic support type. Indeed, set for any $h \in V$

$$\varphi_1(h) = \frac{1}{2}\int_\Gamma a|h|^2 \, \mathrm{d}S \,, \quad \varphi_2(h) = -\int_\Gamma g_i h_i \, \mathrm{d}S$$

(cf. Example 3). The conditions (5.3)–(5.5) being satisfied, we obtain by the aid of Theorem 5.2 the existence of a solution to (2.1), (2.2) under the boundary condition

$$\pi(\sigma) = j^*(-a\,\gamma(u) + g)$$

(note that in the present case the solution is unique). Further, Theorem 5.2 also applies to certain mixed boundary conditions (cf. Example 4, the case $\Gamma_1 = \emptyset$, $\Gamma_3 \neq \emptyset$). ∎

3° In the present subsection we consider the following special decomposition of φ:

$$\varphi(h) = \varphi_0(h) - \langle g^*, h\rangle_V \quad \textit{for} \quad h \in V$$

($g^* \in V^*$ fixed) where

(5.7) φ_0 *is proper, convex and lower semi-continuous* ;

(5.8) $\varphi_0(0) = 0 \,, \quad \varphi_0(th) = t\varphi_0(h) \quad \forall t > 0 \,, \quad \forall h \in D(\varphi_0) \,.$

It is readily verified that a function $u \in \mathscr{V}$ is a solution to (4.2) if and only if

$$\begin{aligned} &a(u, u) + \varphi_0(\gamma(u)) = (f, u) + \langle g^*, \gamma(u)\rangle_V \,, \\ &a(u, v) + \varphi_0(\gamma(v)) \geqq (f, v) + \langle g^*, \gamma(v)\rangle_V \quad \forall v \in \mathscr{V} \,. \end{aligned} \tag{5.9}$$

Inserting $v = \varrho \in \mathscr{R}$ in to the inequality in (5.9) we obtain a necessary condition of solvability

$$\varphi_0(\gamma(\varrho)) \geqq (f, \varrho) + \langle g^*, \gamma(\varrho)\rangle_V \quad \forall \varrho \in \mathscr{R} \,. \tag{5.10}$$

If $\varphi_0 \equiv 0$ (traction boundary value problem, cf. Example 2), (5.10) turns into the well-known condition

$$(f, \varrho) + \langle g^*, \gamma(\varrho)\rangle_V = 0 \quad \forall \varrho \in \mathscr{R}$$

which in this case is also sufficient for the existence of a solution. ∎

We now prove

Theorem 5.3. *Suppose that the functional φ admits the above decomposition and satisfies the conditions* (5.7), (5.8). *Further, suppose that*

$$\varphi_0(\gamma(\varrho)) > (f, \varrho) + \langle g^*, \gamma(\varrho)\rangle_V \quad \forall \varrho \in \mathscr{R} \,, \quad \varrho \neq 0 \,. \tag{5.11}$$

Then Problem III has at least one solution.

Proof. We follow the reasoning of the proof of Theorem 5.2. Let us assume that there exist a sequence $\{v_n\} \subset \mathscr{V}$ ($n = 1, 2, \ldots$) and a constant C_0 such that

$$\|v_n\| \to +\infty \quad \text{as} \quad n \to \infty \,, \qquad F(v_n) \leqq C_0 \quad \text{for} \quad n = 1, 2, \ldots \,.$$

As above, setting $w_n = v_n/\|v_n\|$ (for n sufficiently large) we may assume that $w_n \to w$ weakly in $\mathscr{V}$. Repeating the corresponding arguments of the proof of Theorem 5.2 one easily obtains $w \in \mathscr{R}$ and $|w| \geqq \sqrt{c_1}$.

On the other hand, observing (5.8) we find

$$\varphi_0(\gamma(w_n)) \leqq (f, w_n) + \langle g^*, \gamma(w_n)\rangle_V + \frac{C_0}{\|v_n\|}.$$

Thus, by taking the lim inf on both sides of this inequality,

$$\varphi_0(\gamma(w)) \leqq (f, w) + \langle g^*, \gamma(w)\rangle_V .$$

This inequality contradicts (5.11). ■

Theorem 5.3 can be used for proving the existence of a solution to (2.1), (2.2) under the boundary conditions (3.8), (3.9) (cf. Example 6), (3.10), (3.11) (cf. Example 7) or (3.12), (3.13) (cf. Example 8), provided condition (5.11) is satisfied (cf. also [1], [3], [4]). ■

4° In conclusion of the present section we are going to consider the following situation (cf. also Section 3.3). Let $\varphi_0 : V \to (-\infty, +\infty]$ be a functional possessing the properties:

(5.12) $\qquad \varphi_0$ *satisfies* (5.7), (5.8); $\varphi_0(h) \geqq 0 \;\; \forall h \in D(\varphi_0)$.

Given $g^* \in V^*$ and $\mu \in \mathbb{R}$ $(\mu > 0)$ we introduce the functional

$$\varphi_\mu(h) = \mu\, \varphi_0(h) - \langle g^*, h\rangle_V \quad \textit{for} \quad h \in V$$

and discuss the limit cases $\mu \to \infty$ and $\mu \to 0$.

Firstly, let us define the functional

$$\varphi(h) = \lim_{\mu \to +\infty} \varphi_\mu(h) = \begin{cases} -\langle g^*, h\rangle_V & \textit{for} \quad h \in V \quad \textit{with} \quad \varphi_0(h) = 0\,, \\ +\infty & \textit{otherwise}\,. \end{cases}$$

Obviously, φ is proper, convex and lower semi-continuous. ■

We then have

Proposition 5.1. *Suppose*

(5.13) $\qquad \varrho \in \mathscr{R}, \quad \varrho \neq 0 \Rightarrow \varphi_0(\gamma(\varrho)) > 0$.

Then it holds:

(i) *For each* $\mu > \mu^*$ ($\mu^* = \text{const} > 0^2)$) *there exists a* $u_\mu \in \mathscr{V}$ *such that*

(5.14) $\qquad a(u_\mu, v - u_\mu) + \varphi_\mu(\gamma(v)) - \varphi_\mu(\gamma(u_\mu)) \geqq (f, v - u_\mu)$

for all $v \in \mathscr{V}$ ($f \in \mathscr{H}$ *arbitrary, fixed*).

(ii) *Let* $\{\mu_m\}$ $(m = 1, 2, \ldots; \mu_m > \mu^*)$ *be any sequence of reals such that* $\mu_m \to +\infty$ *as* $m \to \infty$, *and let* $u_m = u_{\mu_m} \in \mathscr{V}$ *be a function which satisfies* (5.14) *with* $\mu = \mu_m$. *Then the sequence* u_m $(m = 1, 2, \ldots)$ *is bounded, and the limit* $\bar{u}$ *of any weakly convergent subsequence satisfies the condition*

$$a(\bar{u}, v - \bar{u}) + \varphi(\gamma(v)) - \varphi(\gamma(\bar{u})) \geqq (f, v - \bar{u})$$

for all $v \in \mathscr{V}$.

Proof. (i) Hypothesis (5.13) implies the existence of a positive constant α_0 such that

$$\varphi_0(\gamma(\varrho)) \geqq \alpha_0 \|\varrho\| \quad \forall \varrho \in \mathscr{R} . \tag{5.15}$$

Set

$$\mu^* = \alpha_0^{-1}(c_0|f| + \|g^*\|_* \, \|\gamma\|_{\mathscr{L}(\mathscr{M}, V)}{}^{3)}) .$$

Observing (5.15) we obtain, for any $\mu > \mu^*$,

$$(f, \varrho) + \langle g^*, \gamma(\varrho)\rangle_V \leqq \alpha_0 \, \mu^* \|\varrho\| < \mu\varphi_0(\gamma(\varrho))$$

for all $\varrho \in \mathscr{R}$, $\varrho \neq 0$. Thus, the sufficient condition of solvability (5.11) is satisfied, and our assertion follows from Theorem 5.3 and Theorem 4.1.

(ii) Let $\bar{\mu} > \mu^*$ such that $\mu_m \geqq \bar{\mu}$ for $m = 1, 2, \ldots$. By (5.9),

$$\begin{aligned} 0 = a(u_m, u_m) + \mu_m \, \varphi_0(\gamma(u_m)) - (f, u_m) - \langle g^*, \gamma(u_m)\rangle_V \geqq \\ \geqq \tfrac{1}{2}a(u_m, u_m) + \bar{\mu} \, \varphi_0(\gamma(u_m)) - (f, u_m) - \langle g^*, \gamma(u_m)\rangle_V \end{aligned} \tag{5.16}$$

for $m = 1, 2, \ldots$. Arguing as in the proof of Theorem 5.3 we get

$$\|u_m\| \leqq \text{const} \quad \textit{for} \quad m = 1, 2, \ldots .$$

Let now $\{u_{m_j}\}$ $(j = 1, 2, \ldots)$ be a subsequence of $\{u_m\}$ such that $u_{m_j} \to \bar{u}$ weakly in $\mathscr{V}$ as $j \to \infty$. The estimate $\mu_m \, \varphi_0(\gamma(u_m)) \leqq \text{const}$ $(m = 1, 2, \ldots)$ is readily deduced from (5.16). Thus

$$\varphi_0(\gamma(\bar{u})) \leqq \liminf \varphi_0(\gamma(u_{m_j})) \leqq 0 ,$$

i.e., $\varphi_0(\gamma(\bar{u})) = 0$.

Next, from (5.16) we easily conclude that

$$(f, \bar{u}) + \langle g^*, \gamma(\bar{u})\rangle_V \geqq a(\bar{u}, \bar{u}) . \tag{5.17$_1$}$$

On the other hand, the inequality

$$a(u_m, v) + \mu_m \, \varphi_0(\gamma(v)) \geqq (f, v) + \langle g^*, \gamma(v)\rangle_V$$

---

[2]) The constant μ^* will be specified in the course of the proof.

[3]) Here c_0 denotes the imbedding constant: $|v| \leqq c_0 \|v\|$ for $v \in \mathscr{V}$.

where $v \in \mathscr{V}$ (cf. (5.9)), yields

$$a(\bar{u}, v) \geqq (f, v) + \langle g^*, \gamma(v) \rangle_V \tag{5.17$_2$}$$

for any $v \in \mathscr{V}$ with $\gamma(v) \in D(\varphi)$.

Combining the inequalities (5.17$_1$) and (5.17$_2$) we get the result desired. ∎

Now, in addition to (5.12) let us suppose

$$D(\varphi_0) = V.$$

In this case we have

Proposition 5.2. *Let condition* (5.13) *be satisfied, and let*

$$(f, \varrho) + \langle g^*, \gamma(\varrho) \rangle_V = 0 \quad \forall \varrho \in \mathscr{R}.$$

Then it holds:

(i) *For each $\mu > 0$ there exists a $u_\mu \in \mathscr{V}$ which satisfies* (5.14) *for all $v \in \mathscr{V}$.*

(ii) *Let $\{\mu_n\}$ ($n = 1, 2, \ldots$; $\mu_n > 0$) be any sequence of reals such that $\mu_n \to 0$ as $n \to \infty$, and let $u_n = u_{\mu_n} \in \mathscr{V}$ be a function that fulfils* (5.14) *with $\mu = \mu_n$. Then the sequence $\{u_n - Pu_n\}$ ($n = 1, 2, \ldots$)* [4]) *is bounded, and the limit $\tilde{u}$ of any weakly convergent subsequence satisfies the conditions*

$$((\tilde{u}, \varrho)) = 0 \quad \forall \varrho \in \mathscr{R},$$
$$a(\tilde{u}, v) = (f, v) + \langle g^*, \gamma(v) \rangle_V \quad \forall v \in \mathscr{V}.$$

Proof. (i) The sufficient condition of solvability (5.11) being satisfied under the present hypotheses, the assertion follows from Theorem 5.3.

(ii) We have

$$\begin{aligned}(f, u_n - Pu_n) + \langle g^*, \gamma(u_n - Pu_n) \rangle_V &= (f, u_n) + \langle g^*, \gamma(u_n) \rangle_V = \\ &= a(u_n, u_n) + \mu_n\, \varphi_0(\gamma(u_n)) = \\ &= a(u_n - Pu_n, u_n - Pu_n) + \mu_n\, \varphi_0(\gamma(u_n)) \geqq c\|u_n - Pu_n\|^2\end{aligned}$$

(cf. Section 1, Lemma 1.1(ii)). Hence

$$\|u_n - Pu_n\| \leqq \text{const} \quad (n = 1, 2, \ldots).$$

Let $\{u_{n_k}\}$ ($k = 1, 2, \ldots$) be a subsequence of $\{u_n\}$ such that $(u_{n_k} - Pu_{n_k}) \to \tilde{u}$ weakly in $\mathscr{V}$ as $k \to \infty$. Clearly, $((\tilde{u}, \varrho)) = 0$ for all $\varrho \in \mathscr{R}$. Finally, observing that

$$a(u_\mu, v) + \mu\, \varphi_0(\gamma(v)) \geqq (f, v) + \langle g^*, \gamma(v) \rangle_V$$

for all $v \in \mathscr{V}$ and any $\mu > 0$ (cf. (5.9), (5.14)) we easily get the desired equation when setting $\mu = \mu_{n_k}$ in the latter inequality and then letting $k \to \infty$. ∎

[4]) P denotes the orthogonal projection onto $\mathscr{R}$ with respect to $\mathscr{V}$.

Let us consider Example 7 with $k = \mu$, and Example 8 with $-k_1 = k_2 = \mu$ ($\Omega \in C^{1,1}$) (cf. Section 3). In the first case we have

$$\varrho \in \mathscr{R}, \quad \varrho_t = 0 \quad on \quad \Gamma \Rightarrow \varrho \equiv 0,$$

while in the second the condition (ii) in Remark (5.1) is assumed to hold.

Then the functionals

$$\varphi_0(h) = \int_\Gamma |h_t| \, dS \quad and \quad \varphi_0(h) = \int_\Gamma |h_n| \, dS, \quad h \in V,$$

respectively, satisfy all the above conditions. For both the functionals

$$\varrho \in \mathscr{R}, \quad \varrho \neq 0 \Rightarrow \varphi_0(\gamma(\varrho)) > 0.$$

Thus, Propositions 5.1 and 5.2 can be applied. ∎

6. DUAL FORMULATION

The aim of this section is to establish a dual problem to the minimum problem III. First of all, this dual problem will be introduced on the pattern of classical linear elasticity. Then we show that essentially the same problem can be obtained when appropriately specializing the general abstract concept of duality (cf. [2], [9], [12], [13]). Our discussion thus presents a generalization of that in [7].

1° The inversion of Hooke's law (cf. Section 2) is given by

$$\varepsilon_{ij} = b_{ijkl}\sigma_{kl} \quad a.e. \ in \ \Omega$$

where the coefficients b_{ijkl} possess the following properties:

$$\begin{aligned} &b_{ijkl} \text{ is measurable and bounded on } \Omega, \\ &b_{ijkl} = b_{jikl} = b_{klij} \quad \text{for a.a.} \quad x \in \Omega, \\ &b_{ijkl}\sigma_{ij}\sigma_{kl} \geqq b_0 \sigma_{ij}\sigma_{ij} \quad \text{for all symmetric} \\ &\text{tensors } \sigma_{ij} \text{ and a.a.} \quad x \in \Omega; \quad b_0 = \text{const} > 0. \end{aligned}$$

Given any $f \in \mathscr{H}$ (fixed) we introduce the functional

$$G(\tau) = \beta(\tau) + \varphi^*(-\pi(\tau)) + I_f(\tau), \quad \tau \in \boldsymbol{T}$$

where

$$\beta(\sigma, \tau) = \int_\Omega b_{ijkl}\sigma_{ij}\tau_{kl} \, dx, \quad \beta(\tau) = \tfrac{1}{2}\beta(\tau, \tau), \quad \sigma, \tau \in \boldsymbol{S},$$

$$I_f(\tau) = \begin{cases} 0 & \text{if} \quad \tau_{ij,j} + f_i = 0 \quad \text{a.e. in } \Omega, \\ +\infty & \text{otherwise} \end{cases}$$

(cf. also Section 1 for the definition of $\boldsymbol{S}$, $\boldsymbol{T}$, π and φ^*). Further, let

$$\boldsymbol{T}_{\text{ad}} = \{\tau \in \boldsymbol{T} : \tau \in D(I_f) \quad and \quad -\pi(\tau) \in D(\varphi^*)\}.$$

In the case $\boldsymbol{T}_{\mathrm{ad}} \neq \emptyset$ the elements in $\boldsymbol{T}_{\mathrm{ad}}$ will be called "statically admissible stress fields". Clearly, $D(G) = \boldsymbol{T}_{\mathrm{ad}}$. The set $D(I_f)$ being convex and closed in $\boldsymbol{T}$, the functional G is convex and lower semi-continuous on $\boldsymbol{T}$. Adopting the classical terminology, $G(\tau)$ may be called the "complementary energy" of the body for the stress field $\tau \in \boldsymbol{T}_{\mathrm{ad}}$. ∎

As the first result we have

Theorem 6.1. *Let $u \in \mathscr{V}$, $\sigma \in \boldsymbol{T}$. Then it holds:*

(i) $F(u) + G(\sigma) \geqq 0$.

(ii) $F(u) + G(\sigma) = 0$ *if and only if* $\{u, \sigma\}$ *is a solution to Problem I.*

Proof. Observing the generalized Green formula (cf. Lemma 1.3(ii)) one easily finds

$$
\begin{aligned}
(6.1) \qquad & F(u) + G(\sigma) = \\
&= \tfrac{1}{2}\, a(u, u) + \beta(\sigma) - \int_\Omega \sigma_{ij} u_{i,j}\, \mathrm{d}x + \\
&\quad + \varphi(\gamma(u)) + \varphi^*(-\pi(\sigma)) + \langle \pi(\sigma), \gamma(u) \rangle_V + I_f(\sigma) = \\
&= \tfrac{1}{2}\, \beta(\sigma - \bar{\sigma}, \sigma - \bar{\sigma}) + \varphi(\gamma(u)) + \varphi^*(-\pi(\sigma)) + \langle \pi(\sigma), \gamma(u) \rangle_V + I_f(\sigma)
\end{aligned}
$$

where we have used the notation $\bar{\sigma}_{ij} = a_{ijkl}\, \varepsilon_{kl}(u)$.

The inequality in (i) is now seen at once. The assertion (ii) is readily checked when comparing (6.1) with (2.1)–(2.3). ∎

Theorem 6.1 suggests the introduction of the following problems.

Problem II* (*principle of virtual stresses*).

Find $\sigma \in \boldsymbol{T}_{\mathrm{ad}}$ such that

$$
\beta(\sigma, \tau - \sigma) + \varphi^*(-\pi(\tau)) - \varphi^*(-\pi(\sigma)) \geqq 0 \quad \forall \tau \in \boldsymbol{T}_{\mathrm{ad}}\,.
$$

Problem III* (*principle of minimum of complementary energy*).

Find $\sigma \in \boldsymbol{T}_{\mathrm{ad}}$ such that

$$
G(\tau) \geqq G(\sigma) \quad \forall \tau \in \boldsymbol{T}_{\mathrm{ad}}\,. \quad ∎
$$

Our discussion in Subsection 2° will show that Problem III* represents a dual formulation of Problem III.

The following two theorems yield a first information about the solvability of Problem II* and III*.

Theorem 6.2. *It holds:*

(i) *Let $\{u, \sigma\}$ be a solution to Problem I. Then σ is a solution to Problem II*.*

(ii) *σ is a solution to Problem II* if and only if σ is a solution to Problem III*.*

Proof. We have $\sigma \in \boldsymbol{T}_{\text{ad}}$ and

$$\varphi^*(-\pi(\tau)) - \varphi^*(-\pi(\sigma)) \geqq -\langle \pi(\tau) - \pi(\sigma), \gamma(u) \rangle_V \quad \forall \tau \in \boldsymbol{T}_{\text{ad}}$$

(cf. (2.3″)). On the other hand, the generalized Green formula takes the form

$$\beta(\sigma, \tau - \sigma) = \langle \pi(\tau) - \pi(\sigma), \gamma(u) \rangle_V \quad \forall \tau \in \boldsymbol{T}_{\text{ad}} .$$

Adding the last two relations we obtain the assertion.

The proof of (ii) parallels that of Theorem 4.1(iii) and may therefore be omitted. ■

Theorem 6.3. *Let $\boldsymbol{T}_{\text{ad}} \neq \emptyset$. Then Problem III* possesses exactly one solution.*

Proof. Let $\tau \in \boldsymbol{T}_{\text{ad}}$. Observing Lemma 1.3 (i), (iii) we get

$$\begin{aligned} G(\tau) &\geqq \tfrac{1}{2} b_0 \|\tau\|_{\boldsymbol{S}}^2 - c_1 \|\pi(\tau)\|_{V^*} - c_2 \geqq \\ &\geqq \tfrac{1}{2} b_0 (\|\tau\|_{\boldsymbol{S}}^2 + |f|^2) - c_1 \|\pi\|_{(\mathscr{L}\boldsymbol{T}V^*)} \|\tau\|_{\boldsymbol{T}} - \tfrac{1}{2} b_0 |f|^2 - c_2 \geqq \\ &\geqq \tfrac{1}{4} b_0 \|\tau\|_{\boldsymbol{T}}^2 - c_3 \end{aligned}$$

where $c_i = \text{const} \geqq 0$ $(i = 1, 2, 3)$.

Thus, by a standard argument, there exists at least one solution to Problem III*.

Let $\sigma_1, \sigma_2 \in \boldsymbol{T}_{\text{ad}}$ be two solutions to Problem III*. Then

$$0 \geqq \beta(\sigma_1 - \sigma_2, \sigma_1 - \sigma_2) \geqq b_0 \|\sigma_1 - \sigma_2\|_{\boldsymbol{S}}^2 ,$$

and therefore $\sigma_1 = \sigma_2$. ■

Remark 6.1. Observing Theorems 4.1 and 6.2 it is easy to see that $\boldsymbol{T}_{\text{ad}} \neq \emptyset$ provided Problem III possesses a solution. ■

Theorem 6.4. *Let $u \in \mathscr{V}_{\text{ad}}$ be a solution to Problem III, and let $\sigma \in \boldsymbol{T}_{\text{ad}}$ be a solution to Problem III*. Then it holds:*

(i) *$\{u, \sigma\}$ is a solution to Problem I.*

(ii) $F(u) = \min\limits_{v \in \mathscr{V}} F(v) = -\min\limits_{\tau \in \boldsymbol{T}} G(\tau) = -G(\sigma)$.

(iii) (*a-posteriori-estimates*):

$$F(v) + G(\tau) \geqq \tfrac{1}{2} a_0 \|\varepsilon(v) - \varepsilon(u)\|_{\boldsymbol{S}}^2 ,$$

$$F(v) + G(\tau) \geqq \tfrac{1}{2} b_0 \|\tau - \sigma\|_{\boldsymbol{S}}^2$$

for all $v \in \mathscr{V}$ and all $\tau \in \boldsymbol{T}$.

Proof. Set $\bar{\sigma}_{ij} = a_{ijkl}\, \varepsilon_{kl}(u)$. Then $\{u, \bar{\sigma}\}$ is a solution to Problem I (cf. Theorem 4.1), and it holds $F(u) = -G(\bar{\sigma})$ (cf. Theorem 6.1). Thus, $\bar{\sigma}$ is a solution to Problem III* (cf. Theorem 6.2), hence $\bar{\sigma} = \sigma$ (cf. Theorem 6.3). The assertions (i) and (ii) are now seen at once.

The estimates in (iii) can be obtained by the same argument as that which led us to (6.1):

$$F(v) + G(\tau) \geqq F(v) + G(\sigma) \geqq$$
$$\geqq \tfrac{1}{2}\, a(v, v) + \tfrac{1}{2}\, \beta(\sigma, \sigma) - (\sigma, \varepsilon(v))_{\mathbf{S}} =$$
$$= \tfrac{1}{2}\, a(v - u, v - u) \geqq \tfrac{1}{2}\, a_0 \|\varepsilon(v) - \varepsilon(u)\|_{\mathbf{S}}^2 \,,$$

and

$$F(v) + G(\tau) \geqq F(u) + G(\tau) \geqq$$
$$\geqq \tfrac{1}{2}\, a(u, u) + \tfrac{1}{2}\, \beta(\tau, \tau) - (\tau, \varepsilon(u))_{\mathbf{S}} =$$
$$= \tfrac{1}{2}\, \beta(\tau - \sigma, \tau - \sigma) \geqq \tfrac{1}{2}\, b_0 \|\tau - \sigma\|_{\mathbf{S}}^2$$

where $v \in \mathscr{V}$ and $\tau \in \boldsymbol{T}$ are arbitrary. ∎

2° We are now going to derive the dual problem to Problem III from the general theory of duality.

To this end, define

$$K_1 v = \varepsilon(v) = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \,, \quad v \in \mathscr{V} \,,$$

$$\alpha(\sigma, \tau) = \int_\Omega a_{ijkl} \sigma_{ij} \tau_{kl} \, \mathrm{d}x \,, \quad \alpha(\tau) = \tfrac{1}{2}\, \alpha(\tau, \tau) \,, \quad \sigma, \tau \in \mathbf{S} \,,$$

$$g(v) = \varphi(\gamma(v)) - (f, v) \,, \quad v \in \mathscr{V} \,.$$

Obviously, $K_1 \in \mathscr{L}(\mathscr{V}, \mathbf{S})$. Then the functional F takes the form

$$F(v) = \alpha(K_1 v) + g(v) \,, \quad v \in \mathscr{V} \,.$$

Following the pattern of the general theory of duality (cf. e.g. [2], [13]) we now introduce the functional

$$\Phi(v, \tau) = \alpha(K_1 v + \tau) + g(v) \,, \quad v \in \mathscr{V} \,, \quad \tau \in \mathbf{S} \,,$$

and instead of Problem III we consider the more general problem

$$(\mathscr{P}) \qquad \inf_{v \in \mathscr{V}} \Phi(v, 0) = \inf_{v \in \mathscr{V}} [\alpha(K_1 v) + g(v)] \,.$$

The functional Φ is proper, convex and lower semi-continuous on $\mathscr{V} \times \mathbf{S}$. Further, for any $\sigma \in \mathbf{S}$ [5]) it holds

$$\Phi^*(0, \sigma) = \sup_{\{v, \tau\} \in \mathscr{V} \times S} [(\sigma, \tau)_{\mathbf{S}} - \alpha(K_1 v + \tau) - g(v)] = \alpha^*(\sigma) + g^*(-K_1^* \sigma) \,.$$

Here $K_1^* \in \mathscr{L}(\mathbf{S}\ \mathscr{V}^*)$ denotes the adjoint of K_1 while α^* and g^* are the conjugate functionals to α and g, respectively (cf. [2; Chap. III, 1]).

[5]) In what follows, the space $\mathbf{S}$ will be identified with its dual.

According to [2], [13] the dual problem to $(\mathscr{P})$ is given by

$$(\mathscr{P}^*) \qquad \sup_{\tau\in \mathbf{S}} [-\Phi^*(0, \tau)] = \sup_{\tau\in \mathbf{S}} [-\alpha^*(\tau) - g^*(-K_1^*\tau)] .$$

Let us now calculate $\alpha^*(\tau)$ and $g^*(-K_1^*\tau)$ $(\tau \in \mathbf{S})$ explicitly. Firstly, set $\bar{\eta}_{ij} = b_{ijkl}\tau_{kl}$. Then

$$(\tau, \eta)_{\mathbf{S}} - \alpha(\eta) =$$
$$= \int_\Omega a_{ijkl}\bar{\eta}_{ij}\eta_{kl}\, dx - \frac{1}{2}\int_\Omega a_{ijkl}\eta_{ij}\eta_{kl}\, dx =$$
$$= -\frac{1}{2}\int_\Omega a_{ijkl}(\bar{\eta}_{ij} - \eta_{ij})(\bar{\eta}_{kl} - \eta_{kl})\, dx + \frac{1}{2}\beta(\tau, \tau)$$

for any $\eta \in \mathbf{S}$. Thus

$$\alpha^*(\tau) = \sup_{\eta\in\mathbf{S}} [(\tau,\eta)_{\mathbf{S}} - \alpha(\eta)] = \beta(\tau) .$$

Secondly, we have

$$g^*(-K_1^*\tau) = \sup_{v\in\mathscr{V}_{\mathrm{ad}}} [\langle -K_1^*\tau, v\rangle_{\mathscr{V}} - g(v)] = {}^{6)}$$
$$= \sup_{v\in\mathscr{V}_{\mathrm{ad}}} \left[-\int_\Omega \tau_{ij}v_{i,j}\, dx - \varphi(\gamma(v)) + (f, v)\right].$$

Suppose $\tau \notin D(I_f)$. Then there exists a function $\varphi \in [\mathscr{D}(\Omega)]^3$ such that

$$\int_\Omega (f_i\varphi_i - \tau_{ij}\varphi_{i,j})\, dx > 0 .$$

Set $v = \bar{v} + t\varphi$ where $\bar{v} \in \mathscr{V}_{\mathrm{ad}}$ is fixed, while $t > 0$ is arbitrary. Then $v \in \mathscr{V}_{\mathrm{ad}}$ and

$$g^*(-K_1^*\tau) \geqq \int_\Omega (f_i\bar{v}_i - \tau_{ij}\bar{v}_{i,j})\, dx + t\int_\Omega (f_i\varphi_i - \tau_{ij}\varphi_{i,j})\, dx - \varphi(\gamma(\bar{v})) ,$$

i.e. $g^*(-K_1^*\tau) = +\infty$.

If $\tau \in D(I_f)$ we get by the aid of the Green formula

$$g^*(-K_1^*\tau) = \sup_{v\in\mathscr{V}} [\langle -\pi(\tau), \gamma(v)\rangle_V - \varphi(\gamma(v))] =$$
$$= \sup_{h\in D(\varphi)} [\langle -\pi(\tau), h\rangle_V - \varphi(h)] = \varphi^*(-\pi(\tau)) .$$

We thus obtain

$$\Phi^*(0, \tau) = \alpha^*(\tau) + g^*(-K_1^*\tau) =$$
$$= \begin{cases} \beta(\tau) + \varphi^*(-\pi(\tau)) + I_f(\tau) & \textit{if} \quad \tau \in \mathbf{T} , \\ +\infty & \textit{if} \quad \tau \in \mathbf{S} \setminus \mathbf{T} \end{cases}$$

---

[6]) $\langle v^*, v\rangle_{\mathscr{V}}$ denotes the dual pairing between $v^* \in \mathscr{V}^*$ and $v \in \mathscr{V}$.

and therefore

$$\Phi^*(0, \tau) = G(\tau), \quad \tau \in \boldsymbol{T}.$$

Finally, we mention a special result of the duality theory. Let $v_0 \in \mathscr{V}_{ad}$ (i.e. $\varphi(\gamma(v_0)) < +\infty$) be fixed. Then

(2.8) $$\Phi(v_0, \tau) = \alpha(K_1 v_0 + \tau) + g(v_0) < +\infty \quad \forall \tau \in \boldsymbol{S}.$$

Further, it is easily seen that

(6.3) *the function* $\tau \mapsto \Phi(v_0, \tau)$ *is continuous on* $\boldsymbol{S}$.

Proposition 6.1. *Suppose that*

(6.4) $$\inf_{v \in \mathscr{V}} \Phi(v, \tau) > -\infty \quad \forall \tau \in \boldsymbol{S}.$$

Then there exists exactly one element $\sigma \in \boldsymbol{S}$ *with*

$$-\Phi^*(0, \sigma) = \max_{\tau \in \boldsymbol{S}} [-\Phi^*(0, \tau)] = \inf_{v \in \mathscr{V}} \Phi(v, 0).$$

In other words, if condition (6.4) is fulfilled then there exists exactly one stress field $\sigma \in \boldsymbol{T}_{ad}$ such that

(6.5) $$-G(\sigma) = -\min_{\tau \in \boldsymbol{T}} G(\tau) = \inf_{v \in \mathscr{V}} F(v)\cdot$$

The proposition stated above is easily deduced from [2; Chap. III, Prop. 2.3] when observing the properties (6.2) and (6.3). The uniqueness of the solution σ is guaranteed by Theorem 6.3. ∎

3° We turn once more to the class of functionals considered in Section 5.3:

$$\varphi(h) = \varphi_0(h) - \langle g^*, h \rangle_V, \quad h \in V,$$

where

(6.6) $$g^* \in V^* \textit{ fixed}; \quad \varphi_0 \textit{ fulfils } (5.7), (5.8).$$

Let us recall that under these assumptions the condition

(5.10) $$\varphi_0(\gamma(\varrho)) \geqq (f, \varrho) + \langle g^*, \gamma(\varrho) \rangle_V \quad \forall \varrho \in \mathscr{R}$$

is necessary for Problem III to have a solution. *This condition is also necessary for the solvability of Problem III**. Indeed, let $\sigma \in \boldsymbol{T}_{ad}$ be a solution to Problem III*. Observing Theorem (6.1) (i) we get

(6.7) $$F(v) \geqq -G(\sigma) > -\infty \quad \forall v \in \mathscr{V}.$$

Suppose there exists $\varrho_0 \in \mathscr{R}$ such that

$$\varphi_0(\gamma(\varrho_0)) < (f, \varrho_0) + \langle g^*, \gamma(\varrho_0) \rangle_V.$$

Hence

$$F(t\varrho_0) = t[\varphi_0(\gamma(\varrho_0)) - (f, \varrho_0) - \langle g^*, \gamma(\varrho_0)\rangle_V] < 0$$

for any $t > 0$, a contradiction to (6.7). ∎

We are now going to prove that condition (5.10) is even sufficient for Problem III* to have a solution when imposing an additional condition on φ_0.

Theorem 6.5. *Let*

$$\varphi(h) = \varphi_0(h) - \langle g^*, h\rangle_V\,, \quad h \in V\,,$$

where condition (6.6) *is satisfied. In addition, let there exist* $c_0 = \text{const} > 0$ *such that*

$$\varphi_0(h_1 + h_2) \geqq \varphi_0(h_2) - c_0\|h_1\|_V \quad \forall h_1, h_2 \in V\,. \tag{6.8}$$

Let (5.10) *be fulfilled. Then there exists exactly one solution to Problem III* and* (6.5) *holds.*

Proof. Let $\tau \in \mathbf{S}$. Given any $v \in \mathscr{V}$ we set $v = w + \varrho$ where $w = v - Pv$, $\varrho = Pv \in \mathscr{R}$ (cf. footnote 4). By (6.8) and Lemma 1.1 (ii)

$$\Phi(v, \tau) \geqq \alpha(K_1 v + \tau) + \varphi_0(\gamma(w) + \gamma(\varrho)) -$$
$$- \varphi_0(\gamma(\varrho)) - (f, w) - \langle g^*, \gamma(w)\rangle_V \geqq -c_1(1 + \|\tau\|^2_{\mathbf{S}})$$

where the positive constant c_1 does not depend on v. Hence, condition (6.4) is satisfied. The assertion follows now from Proposition 6.1. ∎

Theorem 6.5 applies to the friction problems considered in Section 3. Indeed, in the case of friction along any tangential direction we have

$$\varphi(h) = \varphi_0(h) - \langle h_0^*, h_n\rangle\, W_2^{1/2}(\Gamma)\,, \quad h \in V\,,$$

where

$$h_0^* \in W_2^{-1/2}(\Gamma), \quad \varphi_0(h) = \int_\Gamma k|h_t|\, \mathrm{d}S \quad \text{fot} \quad h \in V$$

(cf. Example 7). Thus, if the condition

$$\int_\Gamma k|\gamma_t(\varrho)|\, \mathrm{d}S \geqq (f, \varrho) + \langle h_0^*, \gamma_n(\varrho)\rangle_{W_2{}^{1/2}(\Gamma)} \quad \forall \varrho \in \mathscr{R}$$

is satisfied then Problem III* possesses exactly one solution (in particular $\mathbf{T}_{\text{ad}} \neq \emptyset$) and (6.5) holds.

An analogous observation is true with respect to the other friction problem (cf. Example 8). ∎

7. FORMULATION IN TERMS OF CONJUGATE PROBLEMS

The aim of this last section of our paper is to complete the preceding discussion by formulating equivalently Problem I in terms of conjugate problems in the sense of [5]. In connection with our results obtained in Sections 4 and 6 this approach will make clearer the relation between the solvability of the primal minimum problem and its dual one (cf. also [12]).

1° First of all, let us define the space

$$\boldsymbol{T}_0 = \{\tau \in \boldsymbol{S} : \tau_{ij,j} = 0 \ a.e. \ in \ \Omega\}$$

as a closed subspace of $\boldsymbol{S}$ and the mappings

$$Kv = \{K_1 v, K_2 v\}, \quad v \in \mathscr{V},$$

$$L\tau = \{L_1\tau, L_2\tau\}, \quad \tau \in \boldsymbol{T}_0,$$

where

$$K_1 v = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \quad K_2 v = \gamma(v),$$

$$L_1\tau = \tau \ (injection \ from \ \boldsymbol{T}_0 \ in \ \boldsymbol{S}), \quad L_2\tau = -\pi(\tau).$$

Clearly, $K \in \mathscr{L}(\mathscr{V}, \boldsymbol{S} \times V)$, $L \in \mathscr{L}(\boldsymbol{T}, \boldsymbol{S} \times V^*)$ (the product spaces being furnished with the usual Hilbert space structure)[7]).

Lemma 7.1. *There exists a positive constant c_0 such that*

$$\|Kv\|^2_{\boldsymbol{S}\times V} = \|K_1 v\|^2_{\boldsymbol{S}} + \|\gamma(v)\|^2_V \geqq c_0\|v\|^2 \quad \forall v \in \mathscr{V}.$$

Proof. Suppose, contrary to our assertion, that there exists a sequence $\{v_n\} \subset \mathscr{V}$ such that $\|v_n\| = 1$ $(n = 1, 2, \ldots)$ and

$$\|K_1 v_n\|^2_{\boldsymbol{S}} + \|\gamma(v_n)\|^2_V \leqq \frac{1}{n} \quad (n = 1, 2, \ldots). \tag{7.1}$$

Without any loss of generality, one may assume that $v_n \to v$ weakly in $\mathscr{V}$ as $n \to \infty$. We then infer from (7.1) that $K_1 v = 0$ and $\gamma(v) = 0$; thus $v \in \mathscr{R}$ and therefore $v = 0$ (cf. Remark 5.1).

On the other hand, setting $v_n = w_n + \varrho_n$ where $w_n = v_n - Pv_n$, $\varrho_n = Pv_n$ (cf. footnote 4) we get with the aid of Korn's inequality (cf. Lemma 1.1 (ii)) that $w_n \to 0$ strongly in $\mathscr{V}$ as $n \to \infty$. The space $\mathscr{R}$ being finite-dimensional it follows $\varrho_n \to 0$ strongly in $\mathscr{V}$, and thus $v_n \to 0$ strongly in $\mathscr{V}$ as $n \to \infty$, a contradiction. ∎

The adjoint operators $K^* \in \mathscr{L}(\boldsymbol{S} \times V^*, \mathscr{V}^*)$, $L^* \in \mathscr{L}(\boldsymbol{S} \times V, \boldsymbol{T}_0^*)$ are given by

$$\langle K^*\{\tau, h^*\}, v\rangle_{\mathscr{V}} = \langle K_1^*\tau, v\rangle_{\mathscr{V}} + \langle K_1^* h^*, v\rangle_{\mathscr{V}} = (\tau, K_1 v)_{\boldsymbol{S}} + \langle h^*, K_2 v\rangle_V$$

[7]) Recall that we identify $\boldsymbol{S}^*$ with $\boldsymbol{S}$, and V^{**} with V. The dual pairing between $\tau^* \in \boldsymbol{T}_0^*$ and $\tau \in \boldsymbol{T}_0$ will be denoted by $\langle \tau^*, \tau\rangle_{\boldsymbol{T}_0}$.

for all $\{\tau, h^*\} \in \mathbf{S} \times V^*$ and any $v \in \mathscr{V}$, and

$$\langle L^*\{\sigma, h\}, \tau\rangle_{\mathbf{T}_0} = \langle L_1^*\sigma, \tau\rangle_{\mathbf{T}_0} + \langle L_2^* h, \tau\rangle_{\mathbf{T}_0} = (\sigma, L_1\tau) + \langle L_2\tau, h\rangle_V$$

for all $\{\sigma, h\} \in \mathbf{S} \times V$ and any $\tau \in \mathbf{T}_0$.

We then have

Lemma 7.2. *It holds*

(i) $\operatorname{Im} L = \operatorname{Ker} K^*$,

(ii) $\operatorname{Im} K = \operatorname{Ker} L^*$.

Proof. (i) The inclusion $\operatorname{Im} L \subset \operatorname{Ker} K^*$ is an immediate consequence of the generalized Green formula (cf. Lemma 1.3 (ii)).

Let $\{\tau, h^*\} \in \operatorname{Ker} K^*$, i.e.

$$(\tau, K_1 v)_{\mathbf{S}} + \langle h^*, K_2 v\rangle_V = 0 \quad \forall v \in \mathscr{V} .$$

Setting $v = \varphi \in [\mathscr{D}(\Omega)]^3$ gives

$$\int_\Omega \tau_{ij}\varphi_{i,j}\, \mathrm{d}x = 0 \quad \forall \varphi \in [\mathscr{D}(\Omega)]^3 ,$$

i.e. $\tau \in \mathbf{T}_0$. Again using the generalized Gıeen formula we find

$$\int_\Omega \tau_{ij} v_{i,j}\, \mathrm{d}x + \langle -\pi(\tau), K_2 v\rangle_V = 0 \quad \forall v \in \mathscr{V} .$$

Thus $h^* = -\pi(\tau) = L_2\tau$, and therefore $\{\tau, h^*\} \in \operatorname{Im} L$.

In order to prove (ii) we first of all note that $\operatorname{Im} K$ is closed in $\mathbf{S} \times V$ (cf. Lemma 7.1). Therefore

$$\operatorname{Im} K = {}^\perp(\operatorname{Ker} K^*) = \{\{\sigma, h\} \in \mathbf{S} \times V : (\sigma, \tau)_{\mathbf{S}} + \langle h^*, h\rangle_V = 0$$
$$\forall \{\tau, h^*\} \in \operatorname{Ker} K^*\}$$

(cf. e.g. [10; Theorem 3.2]). We then easily find by virtue of (i)

$$\{\sigma, h\} \in \operatorname{Im} K \Leftrightarrow (\sigma, \tau)_{\mathbf{S}} + \langle h^*, h\rangle_V = 0 \quad \textit{for all} \quad \{\tau, h^*\} \in \operatorname{Ker} K^*$$
$$\Leftrightarrow (\sigma, L_1\bar\tau)_{\mathbf{S}} + \langle L_2\bar\tau, h\rangle_V = 0 \quad \forall \bar\tau \in \mathbf{T}_0$$
$$\Leftrightarrow \{\sigma, h\} \in \operatorname{Ker} L^* . \quad \blacksquare$$

Let $f \in \mathscr{H}$ be fixed. Then there exists (at least one) $\sigma^{(f)} \in \mathbf{T}$ such that $\sigma_{ij,j}^{(f)} + f_i = 0$ a.e. in Ω. Define $g^{(f)} = \pi(\sigma^{(f)})$.

We now introduce the mapping

$$\mathfrak{M} = \{(\operatorname{grad} \alpha)(\cdot) - \sigma^{(f)},\ \partial\varphi(\cdot) + g^{(f)}\} .$$

$\mathfrak{M}$ is a multivalued mapping from $\mathbf{S} \times V$ into $\mathbf{S} \times V^*$ with the effective domain

$D(\mathfrak{M}) = \boldsymbol{S} \times D(\partial\varphi)$ [8]). Observing that

$$\operatorname{grad} \beta = (\operatorname{grad} \alpha)^{-1},$$
$$h \in \partial\varphi^*(h^*) \Leftrightarrow h^* \in \partial\varphi(h)$$

(cf. Section 2.1) one gets

$$\mathfrak{M}^{-1} = \{(\operatorname{grad} \beta)(\cdot + \sigma^{(f)}), \partial\varphi^*(\cdot - g^{(f)})\} .$$

By Lemma 7.2, the mappings

$$K^*\mathfrak{M}K \quad \text{and} \quad L^*\mathfrak{M}^{-1}L$$

are conjugate to each other in the sense of [5]. Now we can introduce the corresponding conjugate problems:

Problem IV. *Find $u \in \mathscr{V}$ such that*

$$\exists h^* \in \partial\varphi(K_2 u) : K_1^*[(\operatorname{grad} \alpha)(K_1 u) - \sigma^{(f)}] + K_2^*(h^* + g^{(f)}) = 0 .$$

Problem IV*. *Find $\sigma \in \boldsymbol{T}_0$ such that*

$$\exists h \in \partial\varphi^*(L_2\sigma - g^{(f)}) : L_1^*[(\operatorname{grad} \beta)(L_1\sigma + \sigma^{(f)})] + L_2^* h = 0 . \quad \blacksquare$$

Theorem 7.1. *It holds:*

(i) *Problem I is solvable iff Problem IV is solvable.*

(ii) *Problem IV is solvable iff Problem IV* is solvable.*

Proof. Setting $\sigma = \tau + \sigma^{(f)}$, the boundary value problem (2.1)–(2.3) can be written in the equivalent form:

$$(+) \qquad \begin{cases} \textit{Find} \quad u \in \mathscr{V}, \quad \tau \in \boldsymbol{T}_0 \quad \textit{such that} \\ L_1\tau = (\operatorname{grad} \alpha)(K_1 u) - \sigma^{(f)} , \\ L_2\tau \in \partial\varphi(K_2 u) + g^{(f)} . \end{cases}$$

In virtue of Lemma 7.2 (i) the equivalence of formulation (+) to Problem IV is immediate.

The second statement is identical with [5; Theorem 2.1]. $\blacksquare$

2° Let us now consider once more the minimum problems III and III*.

To this end, we rewrite the functionals F and G as follows. First of all, using the generalized Green formula one gets

$$F(v) = \alpha(K_1 v) + \varphi(K_2 v) - (f, v) =$$
$$= \alpha(K_1 v) - (\sigma^{(f)}, K_1 v)_{\boldsymbol{S}} + \varphi(K_2 v) + \langle g^{(f)}, K_2 v\rangle_V$$

---

8) $D(\partial\varphi) = \{h \in V : \partial\varphi(h) \neq \emptyset\}$ (analogously for $\partial\varphi^*$).

for any $v \in \mathscr{V}$. Further, we obtain

$$G(\sigma) = \beta(\sigma) + \varphi^*(-\pi(\sigma)) = \\ = \beta(\tau + \sigma^{(f)}) + \varphi^*(-\pi(\tau) - g^{(f)})$$

where $\sigma \in D(I_f)$, $\sigma = \tau + \sigma^{(f)}$, $\tau \in \boldsymbol{T}_0$. Introducing the functional

$$G_1(\tau) = \beta(L_1\tau + \sigma^{(f)}) + \varphi^*(L_2\tau - g^{(f)}), \quad \tau \in \boldsymbol{T}_0,$$

and taking into account that the mapping $\tau \to \tau + \sigma^{(f)}$ is in fact bijective from $\boldsymbol{T}_0$ onto $D(I_f)$, it is easily seen that Problem III* is equivalent to the following one:

$$(++) \qquad \begin{cases} \textit{Find} \quad \sigma \in \boldsymbol{T}_0 \quad \textit{such that} \\ G_1(\tau) \geqq G_1(\sigma) \quad \forall \tau \in \boldsymbol{T}_0 . \end{cases}$$ ∎

Finally, we have

Lemma 7.3. *For all $v \in \mathscr{V}$ and all $\tau \in \boldsymbol{T}_0$ it holds:*

$$(7.2) \qquad \partial F(v) = K_1^*[(\operatorname{grad} \alpha)(K_1 v) - \sigma^{(f)}] + K_2^*[\partial\varphi(K_2 v) + g^{(f)}],$$

$$(7.3) \qquad \partial G_1(\tau) \supset L_1^*[(\operatorname{grad} \beta)(L_1\tau + \sigma^{(f)})] + L_2^*[\partial\varphi^*(L_2\tau - g^{(f)})].$$

Proof. The inclusion (7.3) and an analogous inclusion in (7.2) are obvious (cf. [9; § 4.2.2]). Now let $v^* \in \partial F(v)$, $v \in \mathscr{V}$, i.e.

$$(7.4) \qquad F(w) - F(v) \geqq \langle v^*, w - v\rangle_{\mathscr{V}} \quad \forall w \in \mathscr{V}.$$

K^* being surjective (cf. Lemma 7.1) there exists a $\{\tau, h^*\} \in \boldsymbol{S} \times V^*$ such that $v^* = K^*\{\tau, h^*\}$. Repeating the arguments of the proof of Theorem 4.1, one obtains from (7.4)

$$((\operatorname{grad} \alpha)(K_1 v), K_1(w - v))_{\boldsymbol{S}} + \varphi(K_2 w) - \varphi(K_2 v) \geqq \\ \geqq (\tau + \sigma^{(f)}, K_1(w - v))_{\boldsymbol{S}} + \langle h^* - g^{(f)}, K_2(w - v)\rangle_V \quad \forall w \in \mathscr{V}$$

and

$$K_1^*\tau = K_1^*[(\operatorname{grad} \alpha)(K_1 v) - \sigma^{(f)}],$$
$$K_2^* h^* \in K_2^*[\partial\varphi(K_2 v) + g^{(f)}].$$

Hence,

$$v^* = K_1^*\tau + K_2^* h^* \in K_1^*[(\operatorname{grad} \alpha)(K_1 v) - \varphi^{(f)}] + K_2^*[\partial\varphi(K_2 v) + g^{(f)}].$$ ∎

In virtue of the equivalence of Problem III* and $(++)$ the relations (7.2) and (7.3) lead to the following conclusions.

(i) The solvability of Problem IV or IV* is sufficient for both Problems III and III* to have a solution.

(ii) Let Problem III be solvable, i.e. there exists an element $u \in \mathscr{V}$ such that $0 \in \partial F(u)$. Then, by virtue of (7.2), Problem IV is solvable⁹).

(iii) Let Problem III* be solvable. If φ^* is continuous at some point of $\operatorname{Im} L_2$ the inclusion in (7.3) becomes an equality (cf. [9; § 4.2.2]). In this case the solvability of Problem III* yields the solvability of Problem IV*.

In general, the inclusion in (7.3) cannot be sharpened to an equality within the framework of the above developed Hilbert space theory. *If there exists a "statically admissible stress field" $\sigma \in \mathbf{T}_{ad}$ at which the functional of "complementary energy" attains its minimum on $\mathbf{T}_{ad}$* ¹⁰) *and if the proper inclusion in* (7.3) *holds then there need not exist a displacement field $u \in \mathscr{V}$ which fulfils the constitutive law* (2.2) *and the boundary conditions* (2.3).

The authors are indebted to Dr. K. Gröger for a number of helpful discussions when preparing the present part of the paper.

References

[1] *Duvaut, G.* et *Lions, J. L.*: Les inéquations en mécanique et en physique. Dunod, Paris 1972.

[2] *Ekeland, I.* et *Temam, R.*: Analyse convexe et problèmes variationnels. Dunod, Gauthier-Villars, Paris 1974.

[3] *Fichera, G.*: Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei, Memorie (Cl. Sci. fis., mat. e nat.), serie 8, vol. 7 (1964), 91—140.

[4] *Fichera, G.*: Boundary value problems of elasticity with unilateral constraints. In: Handbuch der Physik (Herausg.: S. Flügge), Band VI a/2, Springer, 1972.

[5] *Gajewski, H.* und *Gröger, K.*: Konjugierte Operatoren und a-posteriori-Fehlerabschätzungen. Math. Nachr. 73 (1976), 315—333.

[6] *Hess, P.*: On semi-coercive nonlinear problems. Indiana Univ. Math. J., 23 (1974), 645—654.

[7] *Hlaváček, I.*: Variational principles in the linear theory of elasticity for general boundary conditions. Apl. Matem., 12 (1967), 425—447.

[8] *Hlaváček, I.* and *Nečas, J.*: On inequalities of Korn's type. II: Applications to linear elasticity. Arch. Rat. Mech. Anal., 36 (1970), 312—334.

[9] *Ioffe, A. D.* and *Tikhomirov, V. M.*: The theory of extremum problems (Russian). Moscow, 1974.

[10] *Krein, S. G.*: Linear equations in a Banach space (Russian). Moscow, 1971.

[11] *Lions, J. L.* and *Stampacchia, G.*: Variational inequalities. Comm. Pure Appl. Math., 20 (1967), 493—519.

[12] *Nayroles, B.*: Duality and convexity in solid equilibrium problems. Laboratoire Méc. et d'Accoustique, C.N.R.S., Marseille 1974.

[13] *Rockafellar, R. T.*: Duality and stability in extremum problems involving convex functions. Pacific J. Math., 21 (1967), 167—187.

[14] *Schatzman, M.*: Problèmes aux limites non linéaires, non coercifs. Ann. Scuola Norm. Sup. Pisa, 27, serie III (1973), 640—686.

⁹) Let us note that assertions (i), (ii) are seen without making use of (7.2), (7.3). To show this it suffices to combine Theorems 4.1 and 6.2 with Theorem 7.1.

¹⁰) Note that the minimizing element σ exists under the only assumption $\mathbf{T}_{ad} \neq \emptyset$ (cf. Theorem 6.3).

Souhrn

OBECNÉ OKRAJOVÉ ÚLOHY A DUALITA V LINEÁRNÍ TEORII PRUŽNOSTI, I

ROLF HÜNLICH, JOACHIM NAUMANN

Tato část článku doplňuje diskusi, která byla obsahem první části, ve dvou směrech. Předně, v kapitole 5. se dokazuje řada existenčních vět pro řešení problému III (princip minima potenciální energie). Za druhé, kapitoly 6 a 7 jsou věnovány jednak diskusi klasického i abstraktního přístupu k teorii duality, jednak vztahu mezi řešitelností problému III a jeho duálního problému.

Authors' addresses: Dr. *Rolf Hünlich,* Akademie der Wissenschaften der DDR, Zentralinstitut für Mathematik und Mechanik, Mohrenstrasse 39, DDR 108 Berlin; Dr. *Joachim Naumann,* Humboldt-Universität zu Berlin, Sektion Mathematik, Unter den Linden 6, DDR 108 Berlin.