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ON SIGNORINI PROBLEM FOR VON KÁRMÁN EQUATIONS

THE CASE OF ANGULAR DOMAIN

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INTRODUCTION

This paper is a continuation of the article [3]. It deals with the Signorini boundary value problem for the system of von Kármán equations.

The above mentioned problem (Section 1) is formulated in a more general form which contains the bilateral, the first and the second Signorini problems (problems R, S_{I}, S_{II} in [3]) and their combinations. The boundary conditions are generalized by introducing Γ_4 on which $w_n = 0$ is required. Together with the variational formulation (1.16)–(1.18) the corresponding classical formulation (1.19)–(1.29) is introduced.

The main result is the generalization of the existence theorem from the infinitely smooth domain to the case of the angular domain (the domain whose boundary is piecewise three times continuously differentiable) which is also studied in [1]. The used method of nonlinear pseudomonotone semicoercive operator inequality is the same as in the paper [3].

The contribution of the paper consists in overcoming the substantial technical difficulties connected with the non-smoothness of the boundary of the domain. The difficulties occur especially when the estimate of the term $B(w; \zeta F, w)$ is deduced, see Section 4. After the estimate (3.5) is obtained the proof of the existence theorem does not differ from that of Theorem 4.1 in [3], so we only refer to the latter.

1. NOTATION AND FORMULATION OF THE PROBLEM

Throughout the paper let Ω be a simply connected bounded domain in R^2 describing the shape of a plate. Let its boundary $\partial\Omega$ represented in the form $\partial\Omega = \{\omega(t) = (\omega_1(t), \omega_2(t)) \in R^2,\ t \in \langle O, l)\}$ be piecewise three times continuously differentiable; this means:

Let $\omega \in [C(\langle O, l\rangle)]^2$ be a continuous injective function and let a finite set $\boldsymbol{T} = \{\tau_j;\ O = \tau_0 < \tau_1 < \ldots < \tau_n = l\}$ exist such that

$$(1.1)\qquad \omega \in [C^3(\langle \tau_{j-1}, \tau_j \rangle)]^2 \quad \text{for} \quad j = 1, 2, \ldots, n\,, \qquad \omega(0) = \omega(l)\,.$$

(1.2) The parameter t is the length of arc hence $|\omega'(t)| = 1$; let the orientation be such that $(n_x, n_y) = (\omega_2', -\omega_1')$ is the unit vector of the outward normal to $\partial\Omega$ for $t \in (O, l) - \boldsymbol{T}$.

(1.3) The angles φ_j between the tangents $\omega'(\tau_j+)$, $-\omega'(\tau_j-)$ and φ_0 between $\omega'(O+)$, $-\omega(l-)$ fulfil $O < \varphi_j < 2\pi$ for $\tau_j \in \boldsymbol{T}$.

We denote by $f(t+), f(t-)$ the limits $\lim\limits_{s\to t+} f(s)$, $\lim\limits_{s\to t-} f(s)$.

As usual, we shall denote the partial derivatives by w_x, w_y, w_{xy}; the normal derivative by w_n, the tangential derivative by w_τ; the operators $\Delta^2 w$, $[w, f]$ are as in the paper [3]: $\Delta^2 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy}$, $[w, f] = w_{xx}f_{yy} + w_{yy}f_{xx} - 2w_{xy}f_{xy}$; and the form

$$(1.4)\qquad b^{xy}(u; v, w) = u_{xy}v_x w_y + u_{xy}v_y w_x - u_{xx}v_y w_y - u_{yy}v_x w_x\,.$$

Let us define the boundary operator Hw by

$$(1.5)\qquad Hw = (1 - \nu)\left(w_{xx}n_x n_y - w_{xy}(n_x^2 - n_y^2) - w_{yy}n_x n_y\right)$$

where ν is Poisson's constant $(0 \leqq \nu < \frac{1}{2})$, and operators Mw, Tw by

$$\begin{aligned} Mw &= \nu\Delta w + (1 - \nu)\left(w_{xx}n_x^2 + 2w_{xy}n_x n_y + w_{yy}n_y^2\right), \\ Tw &= -(\Delta w)_n + (Hw)_\tau\,. \end{aligned}$$

In order to specify the boundary conditions let $\partial\Omega$ be divided into four pairwise disjoint subsets $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ so that $\bigcup \Gamma_i = \partial\Omega$ and let $\Gamma' \subset \Gamma_3 \cup \Gamma_4$, $\Gamma'' \subset \Gamma_2 \cup \Gamma_3$. We suppose that each of the sets Γ_i, Γ', Γ'' is either empty or its interior with respect to $\partial\Omega$ is a union of homeomorphic images of open intervals.

The function k, l, m, r, P specifying the boundary problem are supposed to fulfil (with an arbitrary real number $p > 1$):

$$(1.6)\qquad \begin{aligned} &k \in L_p(\Gamma_2 \cup \Gamma_3)\,; && k \geqq 0 \quad \text{on} \quad \Gamma_2 \cup \Gamma_3\,, \\ &l \in L_1(\Gamma_3 \cup \Gamma_4)\,; && l \geqq 0 \quad \text{on} \quad \Gamma_3 \cup \Gamma_4\,, \\ &m \in L_p(\Gamma_2 \cup \Gamma_3)\,; && r \in L_1(\Gamma_3 \cup \Gamma_4)\,, \\ &P \in L_1(\Omega)\,. \end{aligned}$$

In the presence of corners $\omega(\tau_j)$ in the interior of $\Gamma_3 \cup \Gamma_4$ the function specifying the boundary problem must be completed by constants h_j. Denote by $\boldsymbol{T}^0$ the set of those corners τ_j and by $\boldsymbol{T}'$ the set of corners τ_j in the interior of Γ'. (Hence $\boldsymbol{T}' \subset \subset \boldsymbol{T}^0 \subset \boldsymbol{T}$).

Boundary conditions of Airy stress function Φ are given by functions Φ_0, Φ_1 defined on $\partial\Omega$ which are supposed to fulfil

$$(1.7) \qquad \begin{aligned} \Phi_0 &\in W^{3/2,2}(\omega((\tau_{j-1}, \tau_j))) \\ \Phi_1 &\in W^{1/2,2}(\omega((\tau_{j-1}, \tau_j))) \end{aligned} \quad \text{for} \quad j = 1, 2, \dots, n\,,$$

$$(1.8) \qquad \Phi_0 \in W^{1,2}(\partial\Omega)\,,$$

$$(1.9) \qquad \Phi_{01} = -n_y \frac{\partial}{\partial t} \Phi_0 + n_x \Phi_1 \in W^{1/2,2}(\partial\Omega)\,,$$

$$\Phi_{10} = n_x \frac{\partial}{\partial t} \Phi_0 + n_y \Phi_1 \in W^{1/2,2}(\partial\Omega)\,.$$

Let us introduce the following linear form:

$$(1.10) \quad f(\varphi) = \int_{\Gamma_2 \cup \Gamma_3} m\varphi_n \,\mathrm{d}S + \int_{\Gamma_3 \cup \Gamma_4} r\varphi \,\mathrm{d}S + \sum_{\tau_j \in T^0} h_j \varphi(\omega(\tau_j)) + \int_\Omega P\varphi \,\mathrm{d}x \,\mathrm{d}y\,,$$

bilinear forms

$$(1.11) \qquad (u, v)_{W_0^{2,2}} = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy}) \,\mathrm{d}x \,\mathrm{d}y\,,$$

$$(1.12) \quad A(u, v) = (u, v)_{W_0^{2,2}} + \nu \int_\Omega [u, v] \,\mathrm{d}x \,\mathrm{d}y + \int_{\Gamma_2 \cup \Gamma_3} k u_n v_n \,\mathrm{d}S + \int_{\Gamma_3 \cup \Gamma_4} l u v \,\mathrm{d}S\,,$$

and the expression

$$(1.13) \qquad B(u; v, w) = \int_\Omega b^{xy}(u; v, w) \,\mathrm{d}x \,\mathrm{d}y\,.$$

Let us denote by V the closure of the set

$$(1.14) \qquad \mathscr{V} = \{u \in C^2(\bar{\Omega});\ u = 0 \text{ on } \Gamma_1 \cup \Gamma_2,\ u_n = 0 \text{ on } \Gamma_1 \cup \Gamma_4\}$$

in the norm of $W^{2,2}(\Omega)$ and by K the closure of the set

$$(1.15) \qquad \{u \in \mathscr{V};\ u \geqq 0 \text{ on } \Gamma',\ u_n \geqq 0 \text{ on } \Gamma''\}$$

in the norm of $W^{2,2}(\Omega)$.

Now we can introduce the notion of a variational solution of the problem:

Definition I. *The couple $|w, \Phi| \in K \times W^{2,2}(\Omega)$ is said to be a variational solution of the problem if*

$$(1.16) \quad A(w, v - w) \geqq B(w; \Phi, v - w) + f(v - w) \quad \textit{holds for each} \quad v \in K\,,$$

$$(1.17) \qquad (\Phi, \psi)_{W_0^{2,2}} = -B(w; w, \psi) \quad \textit{holds for each} \quad \psi \in W_0^{2,2}(\Omega),$$

$$(1.18) \qquad \Phi \textit{ satisfies } \Phi = \Phi_0, \; \Phi_n = \Phi_1 \textit{ on } \partial\Omega \textit{ in the sense of traces}.$$

Relation between classical and variational solutions

The sufficiently smooth variational solution defined above is the classical solution of the system of equations

$$(1.19) \qquad \left.\begin{aligned} \Delta^2 w &= [\Phi, w] + P \\ \Delta^2 \Phi &= -[w, w] \end{aligned}\right\} \quad \text{on} \quad \Omega$$

satisfying the boundary conditions

$$(1.20) \qquad \Phi = \Phi_0, \quad \Phi_n = \Phi_1 \qquad \text{on} \quad \partial\Omega,$$

$$(1.21) \qquad w = w_n = 0 \qquad \text{on} \quad \Gamma_1,$$

$$(1.22) \qquad w = 0 \qquad \text{on} \quad \Gamma_2,$$

$$(1.23) \qquad w_n = 0 \qquad \text{on} \quad \Gamma_4,$$

$$(1.24) \qquad Mw + kw_n = m \qquad \text{on} \quad \Gamma_2 \cup \Gamma_3 - \Gamma'',$$

$$(1.25) \qquad \left.\begin{aligned} &Mw + kw_n \geqq m, \quad w_n \geqq 0 \\ &(Mw + kw_n - m)\, w_n = 0 \end{aligned}\right\} \qquad \text{on} \quad \Gamma'',$$

$$(1.26) \qquad Tw + lw + \Phi_{y\tau} w_x - \Phi_{x\tau} w_y = r \qquad \text{on} \quad \Gamma_3 \cup \Gamma_4 - \Gamma';$$

$$(1.27) \qquad \left.\begin{aligned} &Tw + lw + \Phi_{y\tau} w_x - \Phi_{x\tau} w_y \geqq r, \quad w \geqq 0 \\ &(Tw + lw + \Phi_{y\tau} w_x - \Phi_{x\tau} w_y - r)\, w = 0 \end{aligned}\right\} \qquad \text{on} \quad \Gamma';$$

in the presence of corners in the interior of $\Gamma_3 \cup \Gamma_4$

$$(1.28) \qquad H\, w(\omega(\tau_j +)) - H\, w(\omega(\tau_j -)) = h_j \qquad \text{for} \quad \tau_j \in \boldsymbol{T}^0 - \boldsymbol{T}.$$

$$(1.29) \qquad \left.\begin{aligned} &H\, w(\omega(\tau_j +)) - H\, w(\omega(\tau_j -)) \geqq h_j, \quad w \geqq 0 \\ &(H\, w(\omega(\tau_j +)) - H\, w(\omega(\tau_j -)) - h_j)\, w = 0 \end{aligned}\right\} \qquad \text{for} \quad \tau_j \in \boldsymbol{T}'.$$

Mechanical interpretation of the boundary conditions

In the case $\Gamma' = \Gamma'' = \emptyset$ the bilateral problem is treated in the following ways:

In general, the plate is supported and clamped along Γ_1, supported and elastically clamped on Γ_2, elastically supported and clamped on Γ_4, and elastic supports and elastic clamping are prescribed on Γ_3. In particular, if $k = 0$ the elastic support converts to a transverse load and if $l = 0$ the elastic clamping converts to a loading with a moment distribution. If in addition $k = m = 0$ on Γ_2 the plate is simply supported, if $l = r = 0$ on Γ_4 it is simply clamped (allowing free vertical displacements) and if $k = l = m = r = 0$ on Γ_3 then Γ_3 is the free part of the boundary $\partial\Omega$.

The introduced formulation of the problem enables us to deal with unilateral problems as well. If $\Gamma' = \Gamma_3 \cup \Gamma_4$ the first Signorini problem is considered. The special case $l = r = 0$ of the condition (1.27) describes the situation of the edge of a plate which lies on a rigid base so that it can be deflected only upwards (see [3], Remark after Definition 2.2).

The second Signorini problem is described in the case $\Gamma'' = \Gamma_2 \cup \Gamma_3$.

2. REFORMULATION OF THE PROBLEM

In order to be able to use the abstract existence theorem for pseudomonotone semicoercive operators (see e.g. [3], Section 5) we shall reformulate the problem using Knightly's idea (see [5]) in the same way as in [3], Section 6.

Let $F \in W^{2,2}(\Omega)$ be a function such that

$$F = \Phi_0 \quad \text{and} \quad F_n = \Phi_1 \quad \text{on} \quad \partial\Omega \quad \text{in the sense of traces}\,. \tag{2.1}$$

The existence of such a function F follows from the assumptions (1.7)–(1.9) and Theorem 4 in [2].

Further, let $\zeta : \bar{\Omega} \to \langle 0, 1 \rangle$ be an arbitrary function such that

$$\zeta \in C^2(\bar{\Omega}) \quad \text{and} \quad \zeta = 1\,, \qquad \zeta_n = 0 \quad \text{on} \quad \partial\Omega\,. \tag{2.2}$$

Substituting $\Phi = g + \zeta F$ into (1.16) and (1.17) we obtain

$$A(w, v - w) - B(w; g, v - w) - B(w; \zeta F, v - w) \geqq f(v - w)\,, \tag{2.3}$$

$$(g, \psi)_{W_0^{2,2}} + (\zeta F, \psi)_{W_0^{2,2}} = -B(w; w, \psi)\,. \tag{2.4}$$

Let us introduce a real Hilbert space $H = V \times W_0^{2,2}(\Omega)$ with the norm generated by the scalar product

$$((U, Z)) = (w, v)_{W^{2,2}} + (g, \psi)_{W_0^{2,2}} \tag{2.5}$$

provided $U = |w, g|$, $Z = |v, \psi|$; $w, v \in V$ and $g, \psi \in W_0^{2,2}(\Omega)$. Let $\langle f, v \rangle$ denote the pairing between H' and H. Define a continuous functional $Q \in H'$ by

$$Q(Z) = f(v) \tag{2.6}$$

and a bounded operator $\mathcal{T}_\zeta : H \to H'$ by

$$(2.7) \qquad \langle \mathscr{T}_\zeta(U), Z\rangle = A(w, v) + (g, \psi)_{W_0^{2,2}} - B(w; g, v) - B(w; \zeta F, v) + \\ + B(w; w, \psi) + (\zeta F, \psi)_{W_0^{2,2}}.$$

Let us define the solution of the problem $\boldsymbol{K}_\zeta$ as follows:

Definition II. *The couple* $U = |w, g| \in K \times W_0^{2,2}(\Omega)$ *is said to be a solution of the problem* $\boldsymbol{K}_\zeta$ *if*

$$(2.8) \qquad \langle \mathscr{T}_\zeta(U), Z - U\rangle \geqq \langle Q, Z - U\rangle \quad \textit{holds for each} \quad Z \in K \times W_0^{2,2}(\Omega).$$

The problem from Definition I and the problem $\boldsymbol{K}_\zeta$ are equivalent, i.e. the solution $|w, \Phi|$ from Definition I exists if and only if the solution $|w, g|$ exists and $\Phi = g + \zeta F$.

3. MAIN RESULT

Define

$$(3.1) \qquad Y_V = \{v \in V;\ A(v, v) = 0\}.$$

Theorem. *Suppose that* Ω *is the domain described in Section 1* $(1.1)-(1.3)$. *Further let the following assumptions be satisfied:*

(3.2) Γ_2 *and* Γ_4 *are either empty or a union of finitely many segments of straight lines.*

(3.3) *The angles* φ_j (*see* (1.3)) *in the interior of* $\Gamma_2 \cup \Gamma_4$ *between segments range strictly between* 0 *and* π *and no two adjcent parts of* Γ_2 *and* Γ_4 *lie on the same straight line.*

$$(3.4) \qquad \Phi_0 = \Phi_1 = 0 \quad on \quad \Gamma_3.$$

Let the conditions $(1.6)-(1.9)$ *be satisfied.*

Then the following assertions hold:

(i) *If* $Y_V = \{0\}$ *then there exists a variational solution.*

(ii) *If* $Y_V \neq \{0\}$ *and simultaneously each* $z \in K \cap Y_V \setminus \{0\}$ *satisfies the inequality* $f(z) < 0$ (*see* (1.10)) *then there exists a variational solution.*

Sketch of the proof. We find such a function ζ that the operator $\mathscr{T}_\zeta$ defined by (2.7) satisfies the assumptions of the abstract existence theorem (it is pseudomonotone and semicoercive), see e.g. [3], Theorem 5.3. Then there exists a solution $|w, g|$ of the problem $\boldsymbol{K}_\zeta$. From the relation $\Phi = g + \zeta F$ we obtain a variational solution $|w, \Phi|$ introduced in Definition I.

To prove the semicoerciveness of $\mathscr{T}_\zeta$ (for some ζ) we find for each $\gamma > 0$ a function ζ satisfying (2.2) for which

$$|B(w; \zeta F, w)| \leqq \gamma \|w\|^2_{W^{2,2}} \quad \text{holds for each} \quad w \in V. \tag{3.5}$$

The estimate (3.5) is the crucial point in the proof of our theorem because the remainder of the proof repeats literally the corresponding parts of the proof of Theorem 4.1 in [3].

4. AUXILIARY FUNCTION ζ AND ESTIMATE OF TERM $B(w; \zeta F, w)$

We obtain the desired function ζ satisfying the estimate (3.5) immediately from the following lemma by a proper choice of ε, δ.

Lemma. *There exist two positive constants $\bar{c}$, $\bar{d}$ with the following property: For each ε, $\delta > 0$ there exists a function ζ satisfying (2.2) such that*

$$|B(w; \zeta F, w)| \leqq (\bar{c}\varepsilon + \bar{d}\delta^{1/4}) \|F\|_{W^{2,2}} \|w\|^2_{W^{2,2}} \quad \textit{holds for each} \quad w \in V. \tag{4.1}$$

Before proving the lemma in detail we sketch the idea of the proof and clear up some technical details.

Idea of the proof.

I The boundary strip Ω_Δ of a sufficiently small width $\Delta > 0$ will be divided into a finite number of "oblongs" Ω_i, $\bigcup \Omega_i = \Omega_\Delta$ (see Fig. 1).

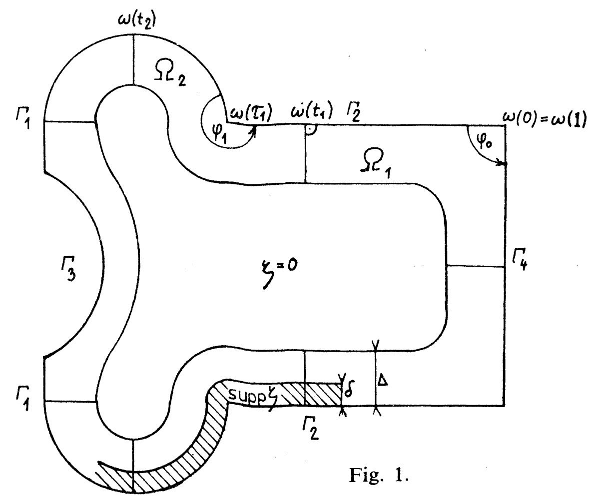

Fig. 1.

II The auxiliary function ζ will be constructed on each oblong Ω_i, while $\zeta = 0$ inside of $\Omega - \Omega_\Delta$. To obtain the function ζ on Ω_i we use special functions of distance from the boundary and a special function z with parameters ε, δ. The parameter δ determines the width of support of the function ζ and thus its measure (denote $\operatorname{supp} \zeta = \Omega_\delta \subset \Omega_\Delta$) is

$$\mu(\Omega_\delta) \leqq \text{const}\, \delta\,. \tag{4.2}$$

The parameter ε determines the estimate of derivatives of the function ζ, see (4.22).

III It is

$$B(w; \zeta F, w) = \int_{\Omega_\delta} \zeta\, b^{xy}(w; F, w) + \int_{\Omega_\delta} F\, b^{xy}(w; \zeta, w)\,. \tag{4.3}$$

The first integral in (4.3) will be estimated by using (4.2) and the inequality of imbedding in Sobolev spaces (see [6]):

$$\|f_x\|_{L^4(\Omega_\delta)} \leqq [\mu(\Omega_\delta)]^{1/8}\, \|f_x\|_{L^8(\Omega_\delta)} \leqq \text{const}\, \delta^{1/8} \|f\|_{W^{2,2}(\Omega)}\,. \tag{4.4}$$

The second integral in (4.3) will be estimated in each oblong separately. In order to estimate the terms with the "unpleasant" derivatives ζ_z (4.22) we transform the differential form b^{xy} into a differential form with derivatives in the other directions and introduce local Cartesian (or oblique) co-ordinates (x^*, y^*). After that we are ready to use Hardy's inequality (4.11).

Let us introduce some technical details:

Auxiliary function h_a^b. For $a < b$ let $h_a^b : R^1 \to \langle 0, 1\rangle$ be the function with the following properties:

$$\begin{aligned} &h_a^b \in C^2(R^1)\,, \\ &h_a^b(t) = 0 \quad \text{for} \quad t \leqq a\,, \\ &h_a^b(t) = 1 \quad \text{for} \quad t \geqq b\,, \\ &|(h_a^b)'| \leqq \frac{2}{b-a}\,. \end{aligned} \tag{4.5}$$

The function $z \in C^2(\langle 0, \infty))$ is introduced in [4], (4.16)–(4.18). Let us recall its properties:

$$\begin{aligned} &z(t) = 1\,, \quad z'(t) = 0 \quad \text{for} \quad t \in \langle 0, h)\,, \\ &\operatorname{supp} z \subset \langle 0, \delta)\,, \end{aligned} \tag{4.6}$$

$$|z'(t)| \leqq \frac{\varepsilon}{t} \quad \text{for} \quad t > 0\,. \tag{4.7}$$

Regularized distance σ. There exist a function σ and positive constants c_1, c_2, c such that

$$\sigma \in C^2(\Omega) \cap C(\bar{\Omega}) , \tag{4.8}$$

$$c_1 \operatorname{dist}((x, y), \partial\Omega) \leqq \sigma(x, y) \leqq c_2 \operatorname{dist}((x, y), \partial\Omega) \quad \text{for} \quad (x, y) \in \Omega , \tag{4.9}$$

$$|\sigma_x| \leqq c , \quad |\sigma_y| \leqq c . \tag{4.10}$$

For this assertion see [6], Chapitre 5, Lemma 3.1.

Hardy's inequality. Let $p > 1$, $a < b$, $f \in C^1(\langle a, b\rangle)$, $f(a) = 0$. Then

$$\int_a^b \left| \frac{f(x)}{x - a} \right|^p \mathrm{d}x \leqq \left(\frac{p}{p - 1} \right)^p \int_a^b |f'(x)|^p \, \mathrm{d}x . \tag{4.11}$$

For this assertion see [6], Chapitre 2, Lemma 5.1.

Transformation of the differential form b^{xy}. In addition to the partial derivatives f_x, f_y we shall use the direction derivatives. Let $z_i = (x_i, y_i) \in R^2$ be a unit vector and let us denote

$$f_{z_i}(x, y) = \lim_{h \to 0} \frac{1}{h} (f(x + hx_i, \ y + hy_i) - f(x, y)) . \tag{4.12}$$

Obviously

$$|f_{z_i}| \leqq |f_x| + |f_y| . \tag{4.13}$$

Let z_1, z_2 and z_3, z_4 be two pairs of unit vectors $z_i = (x_i, y_i)$ and let $J_{12} = x_1 y_2 - x_2 y_1$, $J_{34} = x_3 y_4 - x_4 y_3$, both $J_{12}, J_{34} \neq 0$. Then by virtue of $f_x = 1/J_{12} \cdot (f_{z_1} y_2 - f_{z_2} y_1)$ etc. we obtain

$$b^{xy}(u; v, w) = \frac{1}{J_{12} J_{34}} (u_{z_1 z_4} v_{z_2} w_{z_3} + u_{z_2 z_3} v_{z_1} w_{z_4} - u_{z_1 z_3} v_{z_2} w_{z_4} - u_{z_2 z_4} v_{z_1} w_{z_3}) . \tag{4.14}$$

Local Cartesian (oblique) co-ordinates

$$(x^*, y^*) : \Omega_i \to \langle t_{i-1}, t_i \rangle \times \langle -\Delta, \Delta \rangle \tag{4.15}$$

have the following properties:

$$\frac{\partial(x^*, y^*)}{\partial(x, y)} = 1 \quad (\text{resp.} = \text{const.} > 0) \tag{4.16}$$

$$|f_{x^*}|, |f_{y^*}| \leqq |f_x| + |f_y| .$$

We shall denote the transformed sets and functions by Ω^*, f^*.

Proof of Lemma

I Let us consider the "special" points of $\partial\Omega$ – the "corners" in $\Gamma_2 \cup \Gamma_4$ and the points at which Γ_2 or Γ_4 neighbour with Γ_1 or Γ_3. The boundary strip Ω_Δ is divided with respect to these "special" points by inner normals $n(t_i)$ at the points $\omega(t_i)$, $t_i \notin T$ which are not "special". It is done in such a way that each arc $S_i = \omega(\langle t_{i-1}, t_i \rangle)$ – the side of the oblong Ω_i – contains inside at most one "special" point, see Fig. 1. Further, we shall suppose that the width of Ω_Δ is sufficiently small and the arcs S_i are sufficiently short.

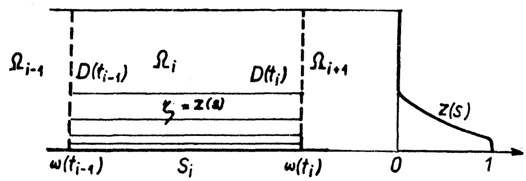

Fig. 2.

II We shall consider four types of oblongs Ω_i according to the boundary conditions prescribed on their arcs S_i.

a Let $S_i \subset \Gamma_2$ or Γ_4 and let S_i be a segment of a straight line, see Fig. 2. We put

(4.17) $$\zeta = z(s) \quad \text{on} \quad \Omega_i$$

where s is the perpendicular distance from the straight line containing the segment S and z is the function with parameters ε, δ satisfying (4.6), (4.7).

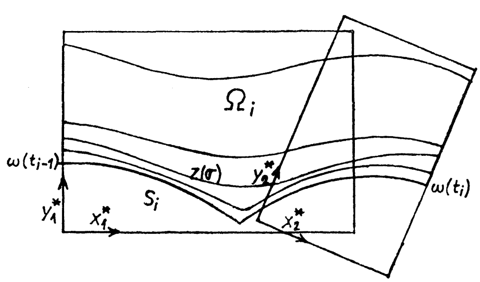

Fig. 3.

b Let $S_i \subset \Gamma_1 \cup \Gamma_3$, Fig. 3. In this case we put

(4.18) $$\zeta = z(\sigma) \quad \text{on} \quad \Omega_i$$

where σ is the regularized distance (4.8)–(4.10).

c Let Γ_2 or Γ_4 neighbour in S_i with $\Gamma_1 \cup \Gamma_3$. Let e.g. $S^- = \omega(\langle t_{i-1}, \tau_j \rangle) \subset \Gamma_1$ or Γ_3, $S^+ = \omega(\langle \tau_j, t_i \rangle) \subset \Gamma_2$ or Γ_4 be a segment of a straight line and the angle φ_j, see (1.3), ranges between 0 and 2π, Fig. 4. Let us denote by $\varphi(x, y)$ the function of the

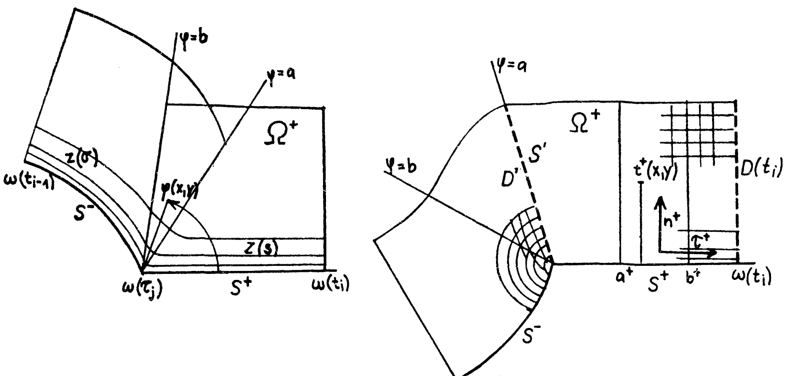

Fig. 4.

angle between the vectors $\omega'(\tau_j+)$ and $(x, y) - \omega(\tau_j)$ for $(x, y) \in \Omega_i$. Then we can put

$$\zeta = z(s) + h_a^b(\varphi)\,(z(\sigma) - z(s)) \quad \text{on} \quad \Omega_i \tag{4.19}$$

where s is the perpendicular distance from the straight line containing S^+, σ is the regularized distance and

$$\begin{aligned} &\text{either} \qquad 0 < a < b < \varphi_j \quad \text{if} \quad 0 < \varphi_j \leqq \pi \\ &\text{or} \qquad \varphi_j - \pi < a < b < \pi \quad \text{if} \quad \pi < \varphi_j < 2\pi\,. \end{aligned} \tag{4.20}$$

d Let $S_i \subset \Gamma_2 \cup \Gamma_4$ but let S_i consist of two straight segments, $S^- = \omega(\langle t_{i-1}, \tau_j \rangle)$, $S^+ = \omega(\langle \tau_j, t_i \rangle)$, Fig. 5. According to (3.3) the angle φ_j between the segments S^+ and S^- is $0 < \varphi_j < \pi$. In this case we put

$$\zeta = z(s^-) + z(s^+) - z(s^-)\,z(s^+) \quad \text{on} \quad \Omega_i \tag{4.21}$$

where s^k is the perpendicular distance from the straight line containing S^k, $k = -, +$.

The function ζ constructed above satisfies (2.2). According to (4.6) and (4.9) the condition (4.2) is satisfied. In addition, we require the derivatives of the function ζ to satisfy the estimate (with a positive constant c_0)

$$|\zeta_z(x, y)| \leqq \varepsilon c_0 \frac{1}{\operatorname{dist}((x, y), \partial\Omega)} \quad \text{for} \quad (x, y) \in \Omega_\Delta \tag{4.22}$$

for each unit vector z. This condition is satisfied, too, as can be easily seen in all these cases from the construction of the function ζ and from the properties of the functions z, σ, s.

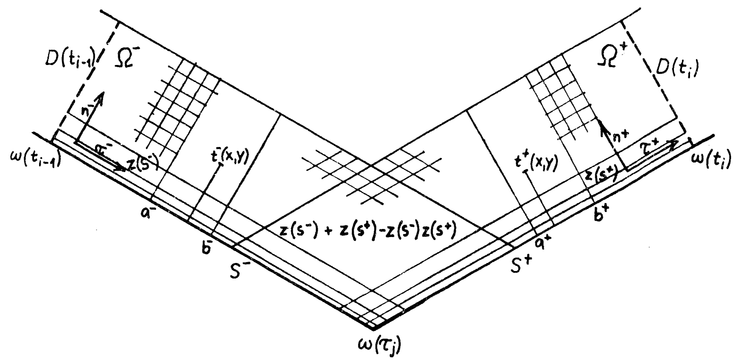

Fig. 5.

III All constants are supposed to be independent of ε, δ. The members in the first integral in (4.3) can be simply estimated by means of Hölder's inequality and (4.4) as follows:

$$(4.23) \quad \left|\int_{\Omega_\delta} \zeta w_{xy} F_y w_x\right| \leqq \|w_{xy}\|_{L^2(\Omega)} \|F_y\|_{L^4(\Omega_\delta)} \|w_x\|_{L^4(\Omega_\delta)} \leqq \text{const}\, \delta^{1/4} \|F\|_{W^{2,2}} \|w\|^2_{W^{2,2}}\,.$$

The remaining terms can be estimated similarly, so we obtain

$$(4.24) \qquad \left|\int_{\Omega_\delta} \zeta b^{xy}(w; F, w)\right| \leqq \bar{d}\delta^{1/4} \|F\|_{W^{2,2}} \|w\|^2_{W^{2,2}}\,.$$

Let us consider the second integral in (4.3). We restrict ourselves to smoother functions $w \in \mathscr{V}$, see (1.14). We shall establish the estimate in detail in one case, in the other possible cases we only mention the different technical details.

a1 Let $S_i \subset \Gamma_2$ and let S_i be a segment of a straight line, Fig. 2. The function ζ is given by (4.17). We transform the form b^{xy} into derivatives in the directions of the normal n and the tangent τ to S_i. In virtue of (4.14) we have $b^{xy} = b^{n\tau}$. We introduce local Cartesian co-ordinates such that S_i in the new co-ordinates become $S_i^* = \{(x^*, y^*);\ y^* = 0,\ x^* \in \langle t_{i-1}, t_i \rangle\}$. The terms containing ζ_τ equal zero because $\zeta_\tau = 0$ on Ω_i. The term containing w_τ can be estimated by means of (4.22) and Hardy's inequality (because $w_\tau^*(x^*, 0) = w^*(x^*, 0) = 0$, see (1.22) as follows:

$$(4.25) \quad \left|\int_{\Omega_i} F w_{\tau n} \zeta_n w_\tau \,dx\,dy\right| \leqq \varepsilon c_0 \|F\|_{L^\infty} \|w_{\tau n}\|_{L^2} \left[\int_{t_{i-1}}^{t_i} \left(\int_0^\Delta \left|\frac{w_\tau^*}{y^*}\right|^2 dy^*\right) dx^*\right]^{1/2} \leqq$$

$$\leqq \varepsilon \text{ const } \|F\|_{L^\infty} \|w_{\tau n}\|_{L^2} \left[\int_{t_{i-1}}^{t_i} \left(\int_0^\Delta |w_{\tau y^*}^*|^2 \, dy^*\right) dx^*\right]^{1/2} \leqq \varepsilon \text{ const } \|F\|_{W^{2,2}} \|w\|_{W^{2,2}}^2 .$$

The remaining term $Fw_{\tau\tau}\zeta_n w_n$ can be transformed by integrating it by parts with respect to τ:

$$(4.26) \qquad -\int_{\Omega_i^*} F^* w_{\tau\tau}^* \zeta_n^* w_n^* \, dy^* \, dx^* = -\left[\int_0^\Delta (F^* w_\tau^* \zeta_n^* w_n^*)(x^*, y^*) \, dy^*\right]_{x^* = t_{i-1}}^{t_i} + $$
$$+ \int_{\Omega i^*} w_\tau^* [F^* \zeta_n^* w_n^*]_\tau \, dy^* \, dx^* .$$

The latter integral contains w_τ, so it can be estimated in the same way as in (4.25). Let us denote

$$(4.27) \qquad D(t) = \int_0^\Delta F^* w_\tau^* w_n^* \zeta_n^*(t, y^*) \, dy^* .$$

Thus we obtain

$$(4.28) \qquad \left|\int_{\Omega_i} F b^{xy}(w; \zeta, w)\right| \leqq D(t_{i-1}) - D(t_i) + c_i \varepsilon \|F\|_{W^{2,2}} \|w\|_{W^{2,2}}^2 .$$

The terms $-D(t_i)$, $D(t_{i-1})$ will be cancelled by the terms with opposite signs from the estimates of the neighbouring oblongs Ω_{i+1}, Ω_{i-1}.

a2 Let $S_i \subset \Gamma_4$ and let S_i be a segment of a straight line. In this case we shall proceed similarly: we transform the form b^{xy} into derivatives in the directions of n, τ. We can apply Hardy's inequality to the term containing w_n because $w_n = 0$ on Γ_4 (1.23). The remaining term $Fw_{n\tau}\zeta_n w_\tau$ is integrated by parts with respect to τ and we obtain

$$(4.29) \qquad \left|\int_{\Omega_i} F b^{xy}(w; \zeta, w)\right| \leqq D(t_i) - D(t_{i-1}) + c_i \varepsilon \|F\|_{W^{2,2}} \|w\|_{W^{2,2}}^2 .$$

Remark. In this case the terms $D(t)$ in (4.29) have the signs opposite to those in (4.28). This is the reason why we suppose in (3.3) that no two adjacent parts of Γ_2 and Γ_4 lie on the same straight line.

b Let $S_i \subset \Gamma_1 \cup \Gamma_3$. The function ζ is given by (4.18). In this case we need not transform the form b^{xy}. We introduce such a system (or two systems) of local Cartesian co-ordinates that we can estimate (integrating with respect to y^*) each point of Ω_i by a boundary point from S_i, see Fig. 3. Let S_i in the new co-ordinates become $S_i^* = \{(x^*, \alpha(x^*)), x^* \in I\}$, where α is a proper function.

Let us consider the case $S_i \subset \Gamma_3$. We can use Hardy's inequality because $F_x = F_y = 0$ on Γ_3 (3.4); e.g. let us estimate the term $Fw_{xx}\zeta_y w_y$:

$$(4.30) \qquad \left| \int_{\Omega_i} F w_{xx} \zeta_y w_y \, \mathrm{d}x \, \mathrm{d}y \right| \leqq$$

$$\leqq \varepsilon \operatorname{const} \left[\int_I \left(\int_{\alpha(x^*)}^{\Delta} \left| \frac{F^*}{y^* - \alpha(x^*)} \right|^4 \mathrm{d}y^* \right) \mathrm{d}x^* \right]^{1/4} \|w_{xx}\|_{L^2} \|w_y\|_{L^4} \leqq$$

$$\leqq \varepsilon \operatorname{const} \left[\int_I \left(\int_{\alpha(x^*)}^{\Delta} |F^*_{y^*}|^4 \, \mathrm{d}y^* \right) \mathrm{d}x^* \right]^{1/4} \|w_{xx}\|_{L^2} \|w_y\|_{L^4} \leqq \varepsilon \operatorname{const} \|F\|_{W^{2,2}} \|w\|^2_{W^{2,2}}.$$

If S_i contains a piece of Γ_1 then it can be estimated by Hardy's inequality as in (4.25) because $w_x = w_y = 0$ on Γ_1 (1.21). Finally, we obtain

$$(4.31) \qquad \left| \int_{\Omega_i} F\, b^{xy}(w; \zeta, w) \right| \leqq \varepsilon\, c_i \|F\|_{W^{2,2}} \|w\|^2_{W^{2,2}}.$$

c Let Γ_2 or Γ_4 neighbour in S_i with $\Gamma_1 \cup \Gamma_3$. The function ζ is given by (4.19). Let us consider e.g. the case $S^+ \subset \Gamma_2$ or Γ_4 and $S^- \subset \Gamma_1 \cup \Gamma_3$ – for notation see paragraph **IIc** of the proof and Fig. 4. Let $\tau_j < a^+ < b^+ < t_i$ and let us denote

$$(4.32) \qquad \Omega^+ = \{(x, y) \in \Omega_i;\ 0 < \varphi(x, y) < a\}, \qquad S' = \{(x, y) \in \Omega_i;\ \varphi(x, y) = a\}.$$

We split the integral into three:

(4.33)

$$\int_{\Omega_i} F b^{xy}(w; \zeta, w) = \int_{\Omega^+} (1 - h^{b^+}_{a^+}(t^+))\, F b^{xy} + \int_{\Omega^+} h^{b^+}_{a^+}(t^+)\, F b^{xy} + \int_{\Omega_i - \Omega^+} F b^{xy}$$

where t^+ is a function on Ω^+ given by the relation

$$(4.34) \qquad t^+(x, y) = t \quad \text{if} \quad (x, y) - \omega(t) \quad \text{is a normal vector to } S^+.$$

We transform the form b^{xy} into derivatives in the directions of the normal n^+ and the tangent τ^+ to S^+ in the first and the second integral. They can be estimated as in the case **a1** or **a2**, only we use oblique co-ordinates instead of Cartesian ones in the first integral. Integration by parts yields the terms $D(t_i)$ in the second integral and D' in the first one (the other equals zero):

$$(4.35) \qquad D' = \int_{S'} F w_\tau w_n \zeta_n \frac{\mathrm{d}S}{\sin a}.$$

The term $D(t_i)$ will be cancelled by the same term with the opposite sign from the neighbouring oblong Ω_{i+1}; D' will be estimated later. The third integral can be estimated in the same way as in the case **b** because $S^- \subset \Gamma_1 \cup \Gamma_3$.

Let us consider the remaining term D'. We shall integrate with respect to the local co-ordinates r, ψ given by $r = |(x, y) - \omega(\tau_j)|$, $\psi = r\,\varphi(x, y)$. We can write

$$(4.36)\qquad \int_{(S^-)^*} F^* w_\tau^* w_n^*\, \mathrm{d}r - \int_{(S')^*} F^* w_\tau^* w_n^*\, \mathrm{d}r = \int_{(\Omega_i - \Omega^+)^*} (F^* w_\tau^* w_n^*)_\psi\, \mathrm{d}\psi\, \mathrm{d}r\,.$$

The first integral on the left hand side equals zero because $w_\tau = w_n = 0$ or $F_\tau = F_n = 0$, (1.14) or (3.4). In virtue of $\partial(r, \psi)/\partial(x, y) = 1$ and (4.13) we can estimate the integral on the right hand side (integrating with respect to the oblique co-ordinates (x^*, y^*)) as in the case **b**. The second integral on the left hand side estimates the term D'.

d Let $S_i \subset \Gamma_2 \cup \Gamma_4$ but let S_i consist of two segments S^- and S^+, Fig. 5. The function ζ is given by (4.21). Let us denote $\Omega^+ = \operatorname{supp} z(s^+)$. Let t^+ be the function on Ω^+ defined as in (4.34) and let Ω^- and t^- have the analogous meaning, see Fig. 5. Further let $t_{i-1} < a^- < b^- < \tau_j < a^+ < b^+ < t_i$. The integral splits into five parts:

(4.37)

$$\int_{\Omega_i} F b^{xy} = \int_{\Omega^-} (1 - h_{a^-}^{b^-}(t^-))\, F b^{xy}(w; z(s^-), w) + \int_{\Omega^-} h_{a^-}^{b^-}(t^-)\, F b^{xy}(w; z(s^-), w) +$$
$$+ \int_{\Omega^+} (1 - h_{a^+}^{b^+}(t^+))\, F b^{xy}(w; z(s^+), w) + \int_{\Omega^+} h_{a^+}^{b^+}(t^+)\, F b^{xy}(w; z(s^+), w) -$$
$$- \int_{\Omega^- \cap \Omega^+} F b^{xy}(w; z(s^-)\, z(s^+), w)\,.$$

The first and the fourth integral can be estimated in the same way as in the case **a1** and **a2**; integration by parts yields the terms $D(t_{i-1})$, $D(t_i)$ which will be cancelled by the same terms with opposite signs from the neighbouring oblongs.

As for the second, the third and the fifth integral in (4.37), we shall integrate with respect to the oblique co-ordinates with the co-ordinate axes in the directions of the tangents τ^- to S^- and τ^+ to S^+.

We shall consider three cases:

d1 Let $S_i \subset \Gamma_4$. We transform the form b^{xy} into derivatives in the directions of the normals and the tangents to the segments S^- and S^+ : n^-, n^+, τ^-, τ^+. In virtue of (4.14) we obtain

$$(4.38)\qquad b^{xy}(w; \zeta, w) =$$
$$= \mathrm{const}\,(w_{\tau^- n^-} \zeta_{\tau^+} w_{n^+} + w_{\tau^+ n^+} \zeta_{\tau^-} w_{n^-} - w_{\tau^- n^+} \zeta_{\tau^+} w_{n^-} - w_{\tau^+ n^-} \zeta_{\tau^-} w_{n^+})\,.$$

Let us estimate the second integral in (4.37). The terms with $z(s^-)_{\tau^-}$ equal zero, the term with $w_{n^-} z(s^-)_{\tau^+}$ can be estimated as in (4.25) because $w_{n^-} = 0$ on S^-. The remaining term $Fw_{\tau^-n^-}z(s^-)_{\tau^+}w_{n^+}h^{b^-}_{a^-}(t^-)$ is integrated by parts with respect to τ^-. The traces equal zero because $w_{n^+} = 0$ on S^+ and $h^{b^-}_{a^-}(t^-) = 0$ for $t^- \leqq a^-$. The third integral can be estimated similarly.

In the fifth integral the terms with $w_{n^-}z(s^-)_{\tau^+}$, $w_{n^+}z(s^+)_{\tau^-}$ can be estimated as in (4.25). The remaining two terms are integrated by parts (e.g. $Fw_{\tau^-n^-}z(s^-)\, z(s^+)\, w_{n^+}$ in τ^-, the traces equal zero because $w_{n^+} = 0$ on S^+ and $z(s^+) = 0$ for $s^+ \geqq \delta$) and estimated in the usual way.

d2 Let $S_i \subset \Gamma_2$. We transform the form b^{xy} into $b^{\tau^-\tau^+}$ and we can proceed similarly as in the case **d1** because $z(s^-)_{\tau^-} = 0$ on Ω^-, $z(s^+)_{\tau^+} = 0$ on Ω^+, $w_{\tau^-} = 0$ on S^-, and $w_{\tau^+} = 0$ on S^+.

d3 Let one of the segments be a subset of Γ_2, the other of Γ_4, e.g., $S^- \subset \Gamma_2$ and $S^+ \subset \Gamma_4$. In this case we can use $z(s^-)_{\tau^-} = 0$ on Ω^-, $z(s^+)_{\tau^+} = 0$ on Ω^+, $w_{\tau^-} = 0$ on S^- and $w_{n^+} = 0$ on S^+. Let the angle φ_j be $\varphi_j \neq \frac{1}{2}\pi$. Then we can transform the form b^{xy} into derivatives in the directions $\tau^-, \tau^+, \tau^-, n^+$:

$$(4.39) \qquad b^{xy}(w; \zeta, w) =$$
$$= \text{const}\,(w_{\tau^-\tau^-}\zeta_{\tau^+}w_{n^+} + w_{\tau^+n^+}\zeta_{\tau^-}w_{\tau^-} - w_{\tau^-n^+}\zeta_{\tau^+}w_{\tau^-} - w_{\tau^+\tau^-}\zeta_{\tau^-}w_{n^+})$$

and we proceed like in the case **d1**.

If $\varphi_j = \frac{1}{2}\pi$ the transformation yields $b^{xy} = b^{\tau^-\tau^+}$ because $\tau^- = -n^+$ and $\tau^+ = n^-$, and we proceed like **d1**.

We complete the proof by putting $\bar{c} = \sum c_i$.

Remark. The boundary strip Ω_Δ can be divided so that there is no oblong of type **a**. We consider it only as the simplest case.

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Souhrn

O SIGNORINIOVĚ PROBLÉMU PRO VON KÁRMÁNOVY ROVNICE

PŘÍPAD OBLASTI S „ROHY"

JAN FRANCŮ

Článek se zabývá existencí řešení zobecněného Signoriniova problému. Použitá metoda, která spočívá v převedení příslušné okrajové úlohy na nerovnici s pseudomonotónním semikoercitivním operátorem je uvedena v [3]. Existenční výsledek pro oblasti s hladkou hranicí z [3] je zobecněn na technicky důležité oblasti s „rohy". Rozhodujícím krokem důkazu je odhad nelineárního členu, který se objevuje v operátorové formulaci problému. Podstatné technické obtíže, které jsou spojeny s nehladkostí hranice jsou překonány speciální volbou pomocné funkce.

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