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Aplikace matematiky, Vol. 24 (1979), No. 5, 326–347

ASYMPTOTIC NORMALITY OF MULTIVARIATE LINEAR RANK STATISTICS UNDER GENERAL ALTERNATIVES

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(Received September 12, 1977)

1. INTRODUCTION

Let $X_j = (X_{1j}, \ldots, X_{pj})'$, $1 \leqq j \leqq N$, be independent random p-vectors with respective continuous cumulative distribution functions F_j, $1 \leqq j \leqq N$. Define the p-vectors $R_1, \ldots, R_N$ by setting R_{ij} equal to the rank of X_{ij} among $X_{i1}, \ldots, X_{iN}$, $1 \leqq i \leqq p$, $1 \leqq j \leqq N$. Denote by $a^{(N)}(\cdot)$ a multivariate score function taking values in $\mathbb{R}^p$, and given by $a^{(N)}(R) = (a_1^{(N)}(R_1), \ldots, a_p^{(N)}(R_p))'$; the $a_i^{(N)}$ are univariate score functions related to generating functions ϕ_i by either

$$(1.1) \qquad a_i^{(N)}(k) = \phi_i\left(\frac{k}{N+1}\right), \quad 1 \leqq k \leqq N$$

or

$$(1.2) \qquad a_i^{(N)}(k) = E\,\phi_i(U_{N,i}^{(k)}), \quad 1 \leqq k \leqq N,$$

where, for each i, $U_{N,i}^{(1)} < \ldots < U_{N,i}^{(N)}$ are the order statistics in a sample of size N from the uniform distribution on $[0, 1]$. Put

$$(1.3) \qquad S = \sum_{j=1}^{N} c_j\, a^{(N)}(R_j),$$

the c_j being arbitrary regression constants.

In this paper we investigate the asymptotic distribution of S under various sets of conditions on the constants, the generating functions, and the underlying distribution functions. Sen and Puri (1968) and Puri and Sen (1971) establish asymptotic normality of S using Chernoff-Savage (1958) techniques; Patel (1971) considers the distribution of S in the null case when $F_1 = \ldots = F_N$ and in the case of contiguous location shift alteratives. We herein establish asymptotic normality of S under extremely mild conditions on the underlying distribution functions. Our methodology devolves from Hájek (1968), who proves similar results in the univariate setting.

In Section 2 are found a multivariate version of Hájek's projection lemma and other preliminaries. Section 3 contains the main results concerning the approximation of the distributions of the multivariate rank statistics $\boldsymbol{S}$ of (1.3) by multivariate normal distribution. In Section 4 we extend certain results of Dupač (1970) and Hoeffding (1973) concerning simple centering values for $\boldsymbol{S}$.

2. PRELIMINARY RESULTS; HÁJEK'S PROJECTION METHOD

The classical central limit theorem is concerned with sequences of sums of independent random variables, so is not directly applicable to the linear rank statistics of interest (1.3). However, if we can show that our rank statistics are asymptotically equivalent to such sums, then the central limit theorem may be invoked to establish asymptotic normality. To approximate a linear rank statistic by a sum of independent random variables, we shall utilize the concept of orthogonal projection in the following manner: given independent random p-vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N$, the set of vector valued square integrable statistics $\boldsymbol{S}^{(p \times 1)} \equiv \boldsymbol{S}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N)$ together with the usual inner product $(\boldsymbol{S}_1, \boldsymbol{S}_2) = E(\boldsymbol{S}_1' \boldsymbol{S}_2)$ forms a Hilbert space $\mathscr{H}$. Any $\boldsymbol{S}$ in $\mathscr{H}$ can be approximated by a statistic $\hat{\boldsymbol{S}}$ belonging to the subspace $\mathscr{L}$ of $\mathscr{H}$ comprised of statistics $\boldsymbol{L}$, where

$$\boldsymbol{L} = \sum_{j=1}^{N} \boldsymbol{K}_j(\boldsymbol{X}_j), \quad \boldsymbol{K}_j : \mathbb{R}^p \to \mathbb{R}^p, \tag{2.1}$$

$$E[\boldsymbol{K}_j'(\boldsymbol{X}_j)\, \boldsymbol{K}_j(\boldsymbol{X}_j)] \text{ finite}, \quad 1 \leqq j \leqq N$$

and minimizing $E[(\boldsymbol{S} - \boldsymbol{S}_0)' (\boldsymbol{S} - \boldsymbol{S}_0)]$ for $\boldsymbol{S}_0 \in \mathscr{L}$. It is well known that $\hat{\boldsymbol{S}}$, a sum of independent random vectors, will be the orthogonal projection of $\boldsymbol{S}$ on $\mathscr{L}$; we state the following lemma, that the projection may be obtained explicitly in terms of conditional expectations.

Lemma 2.1. *Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N$ be independent random p-vectors, and* $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_N)$ *a $(p \times 1)$ statistic such that $E[\boldsymbol{S}'\boldsymbol{S}]$ is finite valued. Let*

$$\hat{\boldsymbol{S}} = \sum_{j=1}^{N} E(\boldsymbol{S} \mid \boldsymbol{X}_j) - (N - 1)\, E\boldsymbol{S}. \tag{2.2}$$

Then

$$E\hat{\boldsymbol{S}} = E\boldsymbol{S} \tag{2.3}$$

and

$$E[(\boldsymbol{S} - \hat{\boldsymbol{S}})(\boldsymbol{S} - \hat{\boldsymbol{S}})'] = \operatorname{cov} \boldsymbol{S} - \operatorname{cov} \hat{\boldsymbol{S}}. \tag{2.4}$$

Moreover, if L is given by (2.1) with $E[\boldsymbol{K}_j'(\boldsymbol{X}_j)\, \boldsymbol{K}_j(\boldsymbol{X}_j)] < \infty$, $1 \leqq j \leqq N$, then

$$E[(\boldsymbol{S} - \boldsymbol{L})(\boldsymbol{S} - \boldsymbol{L})'] = E[(\boldsymbol{S} - \hat{\boldsymbol{S}})(\boldsymbol{S} - \hat{\boldsymbol{S}})'] + E[(\hat{\boldsymbol{S}} - \boldsymbol{L})(\hat{\boldsymbol{S}} - \boldsymbol{L})']. \tag{2.5}$$

The projection lemma is an immediate generalization to vector-valued statistics of the univariate projection lemma of Hájek [(1968), Lemma 4.1]; hence the proof is omitted.

We now prove a lemma that relates conditional expectation of a vector-valued score function of a rank vector to a probability statement. The lemma will be utilized in the sequel, but is also of independent interest.

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ be independent random p-vectors, possessing arbitrary continuous distribution functions $F_1, \ldots, F_N$ respectively. Denote the marginal c.d.f. of the ith coordinate variate of $\boldsymbol{X}_j$, X_{ij}, by F_{ij}. Define $\boldsymbol{R}_j$, $1 \leqq j \leqq N$, as in Section 1. Let $a_i : \mathbb{R}^1 \to \mathbb{R}^1$, $i = 1, \ldots, p$, be arbitrary univariate score functions, and define $\boldsymbol{a} : \mathbb{R}^p \to \mathbb{R}^p$ by $\boldsymbol{a}(x_1, \ldots, x_p)' = (a_1(x_1), \ldots, a_p(x_p))'$. Let $u : \mathbb{R}^1 \to \{0, 1\}$ be defined by $u(x) = 1(0)$ if $x \geqq 0$ $(x < 0)$. Lastly, for $\boldsymbol{\alpha}, \boldsymbol{\beta}$ $p \times 1$ vectors, define $\boldsymbol{\alpha} * \boldsymbol{\beta}$, the Hadamard product of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, by $\boldsymbol{\alpha} * \boldsymbol{\beta} = (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_p\beta_p)'$. We now state and prove

Lemma 2.2.

$$(2.6) \qquad E[\boldsymbol{a}(\boldsymbol{R}_\alpha) \mid \boldsymbol{X}_\alpha = \boldsymbol{x}_\alpha, \boldsymbol{X}_\beta = \boldsymbol{x}_\beta] - E[\boldsymbol{a}(\boldsymbol{R}_\alpha) \mid \boldsymbol{X}_\alpha = \boldsymbol{x}_\alpha] =$$
$$= \sum_{k_1=2}^{N} \sum_{k_2=2}^{N} \cdots \sum_{k_p=2}^{N} \begin{bmatrix} u(x_{1\alpha} - x_{1\beta}) - F_{1\beta}(x_{1\alpha}) \\ u(x_{2\alpha} - x_{2\beta}) - F_{2\beta}(x_{2\alpha}) \\ \vdots \\ u(x_{p\alpha} - x_{p\beta}) - F_{p\beta}(x_{p\alpha}) \end{bmatrix} * \begin{bmatrix} a_1(k_1) - a_1(k_1 - 1) \\ a_2(k_2) - a_2(k_2 - 1) \\ \vdots \\ a_p(k_p) - a_p(k_p - 1) \end{bmatrix}$$
$$\times \, Pr(\boldsymbol{R}_\alpha = (k_1, \ldots, k_p)' \mid \boldsymbol{X}_\alpha = \boldsymbol{x}_\alpha, \boldsymbol{X}_\beta < \boldsymbol{X}_\alpha), \quad \text{for any} \quad \alpha \neq \beta,$$
$$\alpha, \beta = 1, \ldots, N,$$

where $\boldsymbol{X}_\beta < \boldsymbol{X}_\alpha$ connotes that all coordinates of $\boldsymbol{X}_\beta$ are less than the corresponding coordinates of $\boldsymbol{X}_\alpha$.

Proof. To simplify the cumbrous notation, we specify $\alpha = 1$ and $\beta = 2$. We shall prove the lemma only for $p = 2$; it will be clear that the method of proof for $p > 2$ does not involve any new notions.

Denote by $B_N\left(\binom{k_1}{k_2} \middle| \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N\right)$ the probability of $\binom{k_1}{k_2}$ successes in N independent bivariate trials, where at trial j the outcome $\boldsymbol{O}_j$ is one of $\left\{\binom{1}{1}, \binom{1}{0}, \binom{0}{1}, \binom{0}{0}\right\}$, $\binom{k_1}{k_2} = \sum_{j=1}^{N} \boldsymbol{O}_j$, and $\boldsymbol{p}_j' = \left(Pr\left(\boldsymbol{O}_j = \binom{1}{1}\right), Pr\left(\boldsymbol{O}_j = \binom{1}{0}\right), Pr\left(\boldsymbol{O}_j = \binom{0}{1}\right), Pr\left(\boldsymbol{O}_j = \binom{0}{0}\right)\right)$. Since $R_{i1} = \sum_{j=1}^{N} u(X_{i1} - X_{ij})$, $i = 1, 2$, it follows that $Pr\left(\boldsymbol{R}_1 = \binom{k_1}{k_2} \middle| \boldsymbol{X}_1 = \boldsymbol{x}_1\right) = B_N\left(\binom{k_1}{k_2} \middle| \boldsymbol{p}_1' = (1, 0, 0, 0), \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N\right)$, where

$$(2.7)\quad \boldsymbol{p}_j' = (Pr(X_{1j} < x_{11} \text{ and } X_{2j} < x_{21}),\ Pr(X_{1j} < x_{11} \text{ and } X_{2j} > x_{21}),$$
$$Pr(X_{1j} > x_{11} \text{ and } X_{2j} < x_{21}),\ Pr(X_{1j} > x_{11} \text{ and } X_{2j} > x_{21}))$$
$$= (F_j(x_{11}, x_{21}),\ F_{1j}(x_{11}) - F_j(x_{11}, x_{21}),\ F_{2j}(x_{21}) - F_j(x_{11}, x_{21}),$$
$$1 - F_{1j}(x_{11}) - F_{2j}(x_{21}) + F_j(x_{11}, x_{21})),\quad j = 2, \ldots, N.$$

Similarly,

$$Pr\left(\boldsymbol{R}_1 = \binom{k_1}{k_2} \middle| \boldsymbol{X}_1 = \boldsymbol{x}_1,\ \boldsymbol{X}_2 = \boldsymbol{x}_2\right) =$$
$$= B_N\left(\binom{k_1}{k_2} \middle| \boldsymbol{p}_1' = (1, 0, 0, 0),\ \boldsymbol{p}_2 = \boldsymbol{p}_2^*, \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N\right),$$

where $\boldsymbol{p}_3, \ldots, \boldsymbol{p}_N$ are defined as in (2.7) and

$$\begin{aligned}
\boldsymbol{p}_2^{*\prime} &= (1, 0, 0, 0) \quad \text{if} \quad u_1 \equiv u(x_{11} - x_{12}) = 1 \quad \text{and} \quad u_2 \equiv u(x_{21} - x_{22}) = 1 \\
&= (0, 1, 0, 0) \quad \text{if} \quad u_1 = 1 \quad \text{and} \quad u_2 = 0 \\
&= (0, 0, 1, 0) \quad \text{if} \quad u_1 = 0 \quad \text{and} \quad u_2 = 1 \\
&= (0, 0, 0, 1) \quad \text{if} \quad u_1 = 0 \quad \text{and} \quad u_2 = 0.
\end{aligned}$$

From the definition of B_N, it is clear that

$$B_N\left(\binom{k_1}{k_2} \middle| \boldsymbol{p}_1' = (1, 0, 0, 0),\ \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N\right) = B_{N-1}\left(\binom{k_1 - 1}{k_2 - 1} \middle| \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N\right);$$

combining this with the definition of $\boldsymbol{p}_2$, we have

$$\begin{aligned}
(2.8)\quad Pr\left(\boldsymbol{R}_1 = \binom{k_1}{k_2} \middle| \boldsymbol{X}_1 = \boldsymbol{x}_1\right) &= F_2(x_{11}, x_{21})\, B_{N-1}\left(\binom{k_1 - 1}{k_2 - 1} \middle| \boldsymbol{p}_2' = \right. \\
&= \left. (1, 0, 0, 0),\ \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N\right) + \\
&+ [F_{12}(x_{11}) - F_2(x_{11}, x_{21})]\, B_{N-1}\left(\binom{k_1 - 1}{k_2 - 1} \middle| \boldsymbol{p}_2' = (0, 1, 0, 0),\ \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N\right) \\
&+ [F_{22}(x_{21}) - F_2(x_{11}, x_{21})]\, B_{N-1}\left(\binom{k_1 - 1}{k_2 - 1} \middle| \boldsymbol{p}_2' = (0, 0, 1, 0), \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N\right) \\
&+ [1 - F_{12}(x_{11}) - F_{22}(x_{21}) + F_2(x_{11}, x_{21})] \\
&\qquad B_{N-1}\left(\binom{k_1 - 1}{k_2 - 1} \middle| \boldsymbol{p}_2' = (0, 0, 0, 1),\ \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N\right) \\
&= F_2(x_{11}, x_{21})\, B_{N-2}\left(\binom{k_1 - 2}{k_2 - 2} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N\right) \\
&+ [F_{12}(x_{11}) - F_2(x_{11}, x_{21})]\, B_{N-2}\left(\binom{k_1 - 2}{k_2 - 1} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N\right)
\end{aligned}$$

$$+ [F_{22}(x_{21}) - F_2(x_{11}, x_{21})] B_{N-2}\left(\left(\begin{matrix} k_1 - 1 \\ k_2 - 2 \end{matrix}\right) \middle| p_3, \dots, p_N\right)$$

$$+ [1 - F_{12}(x_{11}) - F_{22}(x_{21}) + F_2(x_{11}, x_{21})] B_{N-2}\left(\left(\begin{matrix} k_1 - 1 \\ k_2 - 1 \end{matrix}\right) \middle| p_3, \dots, p_N\right).$$

In a like manner, we find

$$(2.9) \quad Pr\left(R_1 = \left(\begin{matrix} k_1 \\ k_2 \end{matrix}\right) \middle| X_1 = x_1, X_2 = x_2\right) = u_1 u_2 B_{N-2}\left(\left(\begin{matrix} k_1 - 2 \\ k_2 - 2 \end{matrix}\right) \middle| p_3, \dots, p_N\right)$$

$$+ u_1(1 - u_2) B_{N-2}\left(\left(\begin{matrix} k_1 - 2 \\ k_2 - 1 \end{matrix}\right) \middle| p_3, \dots, p_N\right)$$

$$+ (1 - u_1) u_2 B_{N-2}\left(\left(\begin{matrix} k_1 - 1 \\ k_2 - 2 \end{matrix}\right) \middle| p_3, \dots, p_N\right)$$

$$+ (1 - u_1)(1 - u_2) B_{N-2}\left(\left(\begin{matrix} k_1 - 1 \\ k_2 - 1 \end{matrix}\right) \middle| p_3, \dots, p_N\right).$$

Combining the last two results (2.8) and (2.9) yields

$$Pr\left(R_1 = \left(\begin{matrix} k_1 \\ k_2 \end{matrix}\right) \middle| X_1 = x_1, X_2 = x_2\right) - Pr\left(R_1 = \left(\begin{matrix} k_1 \\ k_2 \end{matrix}\right) \middle| X_1 = x_1\right)$$

$$= [u_1 u_2 - F_2(x_{11}, x_{21})]\left[B_{N-2}\left(\left(\begin{matrix} k_1 - 2 \\ k_2 - 2 \end{matrix}\right) \middle| p_3, \dots, p_N\right)\right) -$$

$$- B_{N-2}\left(\left(\begin{matrix} k_1 - 2 \\ k_2 - 1 \end{matrix}\right) \middle| p_3, \dots, p_N\right)\right)$$

$$- B_{N-2}\left(\left(\begin{matrix} k_1 - 2 \\ k_2 - 2 \end{matrix}\right) \middle| p_3, \dots, p_N\right) + B_{N-2}\left(\left(\begin{matrix} k_1 - 1 \\ k_2 - 1 \end{matrix}\right) \middle| p_3, \dots, p_N\right)\Bigg]$$

$$+ [u_1 - F_{12}(x_{11})]\left[B_{N-2}\left(\left(\begin{matrix} k_1 - 2 \\ k_2 - 1 \end{matrix}\right) \middle| p_3, \dots, p_N\right) -\right.$$

$$\left.- B_{N-2}\left(\left(\begin{matrix} k_1 - 1 \\ k_2 - 1 \end{matrix}\right) \middle| p_3, \dots, p_N\right)\right]$$

$$+ [u_2 - F_{22}(x_{21})]\left[B_{N-2}\left(\left(\begin{matrix} k_1 - 1 \\ k_2 - 2 \end{matrix}\right) \middle| p_3, \dots, p_N\right) -\right.$$

$$\left.- B_{N-2}\left(\left(\begin{matrix} k_1 - 1 \\ k_2 - 1 \end{matrix}\right) \middle| p_3, \dots, p_N\right)\right]$$

$= S_1 + S_2 + S_3$, say, where it is implicit that the S_i are functions of k_1 and k_2. Then

$$(2.10)\quad E[a(\boldsymbol{R}_1) \mid \boldsymbol{X}_1 = \boldsymbol{x}_1, \boldsymbol{X}_2 = x_2] - E[a(\boldsymbol{R}_1) \mid \boldsymbol{X}_1 = \boldsymbol{x}_1]$$

$$= \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \begin{pmatrix} a_1(k_1) \\ a_2(k_2) \end{pmatrix} \cdot \left[Pr\left(\boldsymbol{R}_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \middle| \boldsymbol{X}_1 = \boldsymbol{x}_1, \boldsymbol{X}_2 = \boldsymbol{x}_2 \right) - \right.$$

$$\left. - Pr\left(\boldsymbol{R}_1 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \middle| \boldsymbol{X}_1 = \boldsymbol{x}_1 \right) \right] = \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \begin{pmatrix} a_1(k_1) \\ a_2(k_2) \end{pmatrix} (S_1 + S_2 + S_3) .$$

To evaluate this expression, we first consider

$$\sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \begin{pmatrix} a_1(k_1) \\ a_2(k_2) \end{pmatrix} \left[B_{N-2}\left(\begin{pmatrix} k_1 - 2 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) - B_{N-2}\left(\begin{pmatrix} k_1 - 1 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) \right] .$$

Decompose $\begin{pmatrix} a_1(k_1) \\ a_2(k_2) \end{pmatrix} = \begin{pmatrix} a_1(k_1) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2(k_2) \end{pmatrix}$, and note that

$$(2.11)\quad \sum_{k_1=1}^{N} \begin{pmatrix} 0 \\ a_2(k_2) \end{pmatrix} \left[B_{N-2}\left(\begin{pmatrix} k_1 - 2 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) - \right.$$

$$\left. - B_{N-2}\left(\begin{pmatrix} k_1 - 1 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) \right]$$

$$= \begin{pmatrix} 0 \\ a_2(k_2) \end{pmatrix} \left\{ \left[0 - B_{N-2}\left(\begin{pmatrix} 0 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) \right] + \right.$$

$$+ \left[B_{N-2}\left(\begin{pmatrix} 0 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) - B_{N-2}\left(\begin{pmatrix} 1 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) \right] + \ldots$$

$$+ \left[B_{N-2}\left(\begin{pmatrix} N - 3 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) - B_{N-2}\left(\begin{pmatrix} N - 2 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) \right]$$

$$\left. + \left[B_{N-2}\left(\begin{pmatrix} N - 2 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) - 0 \right] \right\} = \begin{pmatrix} 0 \\ a_2(k_2) \end{pmatrix} . 0 = 0 .$$

Also,

$$(2.12)\quad \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} \begin{pmatrix} a_1(k_1) \\ 0 \end{pmatrix} \left[B_{N-2}\left(\begin{pmatrix} k_1 - 2 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) - \right.$$

$$- B_{N-2}\left(\begin{pmatrix} k_1 - 1 \\ k_2 - 1 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right)$$

$$= \sum_{k_1=2}^{N} \sum_{k_1=2}^{N} \begin{pmatrix} a_1(k_1) - a_1(k_1 - 1) \\ 0 \end{pmatrix} . B_{N-2}\left(\begin{pmatrix} k_1 - 2 \\ k_2 - 2 \end{pmatrix} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N \right) \Bigg] .$$

From (2.11) and (2.12), we have

(2.13)

$$\sum_{k_1=1}^{N}\sum_{k_2=1}^{N}\begin{pmatrix}a_1(k_1)\\a_2(k_2)\end{pmatrix}S_2=\sum_{k_1=2}^{N}\sum_{k_2=2}^{N}[u_1-F_{12}(x_{11})]\begin{pmatrix}a_1(k_1)-a_1(k_1-1)\\0\end{pmatrix}$$

$$\times B_{N-2}\left(\begin{pmatrix}k_1-2\\k_2-2\end{pmatrix}\middle|p_3,\ldots,p_N\right)$$

$$=\sum_{k_1=2}^{N}\sum_{k_2=2}^{N}\begin{pmatrix}u_1-F_{12}(x_{11})\\0\end{pmatrix}*\begin{pmatrix}a_1(k_1)-a_1(k_1-1)\\0\end{pmatrix}$$

$$\times B_{N-2}\left(\begin{pmatrix}k_1-2\\k_2-2\end{pmatrix}\middle|p_3,\ldots,p_N\right).$$

A completely analogous argument yields

(2.14)
$$\sum_{k_1=1}^{N}\sum_{k_2=2}^{N}\begin{pmatrix}a_1(k_1)\\a_2(k_2)\end{pmatrix}S_3=$$

$$=\sum_{k_1=2}^{N}\sum_{k_2=2}^{N}\begin{pmatrix}0\\u_2-F_{22}(x_{21})\end{pmatrix}*\begin{pmatrix}0\\a_2(k_2)-a_2(k_2-1)\end{pmatrix}B_{N-2}\left(\begin{pmatrix}k_1-2\\k_2-2\end{pmatrix}\middle|p_3,\ldots,p_N\right).$$

Finally,

(2.15)
$$\sum_{k_1=1}^{N}\sum_{k_2=1}^{N}\begin{pmatrix}a_1(k_1)\\a_2(k_2)\end{pmatrix}$$

$$\times\left[B_{N-2}\left(\begin{pmatrix}k_1-2\\k_2-2\end{pmatrix}\middle|p_3,\ldots,p_N\right)-B_{N-2}\left(\begin{pmatrix}k_1-2\\k_2-1\end{pmatrix}\middle|p_3,\ldots,p_N\right)\right.$$

$$\left.-B_{N-2}\left(\begin{pmatrix}k_1-1\\k_2-2\end{pmatrix}\middle|p_3,\ldots,p_N\right)+B_{N-2}\left(\begin{pmatrix}k_1-1\\k_2-1\end{pmatrix}\middle|p_3,\ldots,p_N\right)\right]$$

$$=\sum_{k_1=2}^{N}\sum_{k_2=2}^{N}B_{N-2}\left(\begin{pmatrix}k_1-2\\k_2-2\end{pmatrix}\middle|p_3,\ldots,p_N\right)$$

$$\times\left[\begin{pmatrix}a_1(k_1)\\a_2(k_2)\end{pmatrix}-\begin{pmatrix}a_1(k_1)\\a_2(k_2-1)\end{pmatrix}-\begin{pmatrix}a_1(k_1-1)\\a_2(k_2)\end{pmatrix}+\begin{pmatrix}a_1(k_1-1)\\a_2(k_2-1)\end{pmatrix}\right]=0\,.$$

Hence $\sum_{k_1=1}^{N}\sum_{k_2=1}^{N}\begin{pmatrix}a_1(k_1)\\a_2(k_2)\end{pmatrix}S_1=0$, and (2.11), in light of (2.12) and (2.13), becomes

(2.16)
$$E[a(\boldsymbol{R}_1)\mid \boldsymbol{X}_1=x_1,\ X_2=x_2]-E[a(\boldsymbol{R}_1)\mid \boldsymbol{X}_1=\boldsymbol{x}]=$$

$$=\sum_{k_1=2}^{N}\sum_{k_2=2}^{N}\begin{pmatrix}u_1-F_{12}(x_{11})\\u_2-F_{22}(x_{21})\end{pmatrix}*\begin{pmatrix}a_1(k_1)-a_1(k_1-1)\\a_2(k_2)-a_2(k_2-1)\end{pmatrix}B_{N-2}\left(\begin{pmatrix}k_1-2\\k_2-2\end{pmatrix}\middle|p_3,\ldots,p_N\right).$$

But by definition of B_{N-2}, $Pr\left(\boldsymbol{R}_1 = \binom{k_1}{k_2} \middle| \boldsymbol{X}_1 = \boldsymbol{x}_1, \boldsymbol{X}_2 < \boldsymbol{x}_1\right) = B_{N-2}\left(\binom{k_1-2}{k_2-2} \middle| \boldsymbol{p}_3, \ldots, \boldsymbol{p}_N\right)$. Hence (2.16) reduces to (2.6), and we are done.

Theorem 2.3. *Suppose that the conditions of Lemma 2.2 hold. Consider the statistic* $\boldsymbol{S} = \sum_{j=1}^{N} c_j\, \boldsymbol{a}^{(N)}(\boldsymbol{R}_j)$, *where the marginal univariate scores* $a_i^{(N)}(\cdot)$ *are related to generating functions* $\phi_i : \mathbb{R}^1 \to \mathbb{R}^1$ *by* $a_i^{(N)}(R) = \phi_i(R/N+1)$, $1 \leqq i \leqq p$. *Suppose in addition, that each* ϕ_i *has bounded second derivative. Then there exists a constant* M *depending only on* $\phi_1, \ldots, \phi_p$ *such that for any* N, $(c_1, \ldots, c_N)$, *and continuous* $F_1, \ldots, F_N$,

$$(2.17)\qquad E[(\boldsymbol{S} - E\boldsymbol{S} - \sum_{j=1}^{N} \boldsymbol{Z}_j)(\boldsymbol{S} - E\boldsymbol{S} - \sum_{j=1}^{N} \boldsymbol{Z}_j)'] \leqq MN^{-1} \sum_{j=1}^{N} (c_j - \bar{c})^2 \boldsymbol{I},$$

where $\boldsymbol{I}$ *is the* $p \times p$ *identity matrix,*

$$(2.18)\qquad \boldsymbol{Z}_j = \boldsymbol{Z}_j(\boldsymbol{X}_j) = (N+1)^{-1} \sum_{k=1}^{N} (c_k - c_j)$$
$$\int \begin{pmatrix} u(x_1 - X_{1j}) - F_{1j}(x_1) \\ u(x_2 - X_{2j}) - F_{2j}(x_2) \\ \vdots \\ u(x_p - X_{pj}) - F_{pj}(x_p) \end{pmatrix} * \begin{pmatrix} \phi_1'(H_1(x_1)) \\ \phi_2'(H_2(x_2)) \\ \vdots \\ \phi_p'(H_p(x_p)) \end{pmatrix} \mathrm{d}F_k(x_1, \ldots, x_p)\,, \quad 1 \leqq j \leqq N\,,$$

and $H_i(x) = 1/N \sum_{j=1}^{N} F_{ij}(x)$.

Proof. Define $\boldsymbol{\phi}, \boldsymbol{\phi}', \boldsymbol{\phi}' : \mathbb{R}^p \to \mathbb{R}^p$ by $\boldsymbol{\phi}(\boldsymbol{x}) = (\phi_1(x_1), \phi_2(x_2), \ldots, \phi_p(x_p))'$, and similarly for $\boldsymbol{\phi}'$ and $\boldsymbol{\phi}''$. Also, let $\boldsymbol{H} : \mathbb{R}^p \to \mathbb{R}^p$ be given by $\boldsymbol{H}(\boldsymbol{x}) = (H_1(x_1), H_2(x_2), \ldots, H_p(x_p))'$. Then, $\boldsymbol{S} = \sum_{j=1}^{N} c_j \boldsymbol{\phi}(1/(N+1)\,\boldsymbol{R}_j)$, and from Lemma 2.1,

$$(2.19)\qquad \hat{\boldsymbol{S}} - E\hat{\boldsymbol{S}} = \sum_{k=1}^{N} E\left[\sum_{j=1}^{N} c_j \boldsymbol{\phi}\left(\frac{1}{N+1}\boldsymbol{R}_j\right) \middle| \boldsymbol{X}_k\right] - NE\boldsymbol{S}$$
$$= \sum_{k=1}^{N} \sum_{j=1}^{N} c_j \left\{E\left[\boldsymbol{\phi}\left(\frac{1}{N+1}\boldsymbol{R}_j\right) \middle| \boldsymbol{X}_k\right] - E\boldsymbol{\phi}\left(\frac{1}{N+1}\boldsymbol{R}_j\right)\right\}.$$

Observe that $\sum_{j=1}^{N} \boldsymbol{\phi}(1/(N+1)\,\boldsymbol{R}_j)$ is a constant vector $(= \sum_{j=1}^{N} \boldsymbol{\phi}(j/(N+1)\,\boldsymbol{e})$, where $\boldsymbol{e} = (1, 1, \ldots, 1)')$; hence $\sum_{j=1}^{N} \boldsymbol{\phi}(1/(N+1)\,\boldsymbol{R}_j) = E[\sum_{j=1}^{N} \boldsymbol{\phi}(1/(N+1)\,\boldsymbol{R}_j) \mid \boldsymbol{X}_k] = \sum_{j=1}^{N} E\boldsymbol{\phi}(1/(N+1)\,\boldsymbol{R}_j) \mid \boldsymbol{X}_k] = \sum_{j=1}^{N} E\boldsymbol{\phi}(1/(N+1)\,\boldsymbol{R}_j)$. So, from (2.19) we have also

$$(2.20)\quad \hat{S} - E\hat{S} = \sum_{k=1}^{N}\sum_{j=1}^{N} -(c_k - c_j)\left\{E\left[\phi\left(\frac{1}{N+1}R_j\right)\middle| X_k\right] - E\phi\left(\frac{1}{N+1}R_j\right)\right\}.$$

By Lemma 2.2,

$$E\left[\phi\left(\frac{1}{N+1}R_j\right)\middle| X_k\right] - E\phi\left(\frac{1}{N+1}R_j\right)$$

$$= E\left\{E\left[\phi\left(\frac{1}{N+1}R_j\right)\middle| X_j, X_k\right] - E\left[\phi\left(\frac{1}{N+1}R_j\right)\middle| X_j\right]\right\}$$

$$= \int_{\mathbb{R}^p} \sum_{k_1=2}^{N}\cdots\sum_{k_p=2}^{N} \begin{pmatrix} u(x_1 - X_{1k}) - F_{1k}(x_1) \\ \vdots \\ u(x_p - X_{pk}) - F_{pk}(x_p) \end{pmatrix} * \begin{pmatrix} \phi_1\left(\frac{k_1}{N+1}\right) - \phi_1\left(\frac{k_1-1}{N+1}\right) \\ \vdots \\ \phi_p\left(\frac{k_p}{N+1}\right) - \phi_p\left(\frac{k_p-1}{N+1}\right) \end{pmatrix}$$

$$\times\ Pr(R_j = (k_1, \ldots, k_p)' \mid X_j = x,\ X_k < X_j)\, \mathrm{d}F_j(x_1, \ldots, x_p)\,.$$

But

$$\begin{pmatrix} \phi_1\left(\frac{k_1}{N+1}\right) - \phi_1\left(\frac{k_1-1}{N+1}\right) \\ \vdots \\ \phi_p\left(\frac{k_p}{N+1}\right) - \phi_p\left(\frac{k_p-1}{N+1}\right) \end{pmatrix} = \frac{1}{N+1}\begin{pmatrix} \phi_1'\left(\frac{k_1}{N+1}\right) \\ \vdots \\ \phi_p'\left(\frac{k_p}{N+1}\right) \end{pmatrix} + \left(\frac{1}{N+1}\right)^2 C_2\boldsymbol{\alpha},$$

where

$C_2 = \max_{1 \leq i \leq p} \sup_{t \in (0,1)} |\phi_i''(t)|$, and each $|\alpha_i| \leq 1$. Further,

$$\sum_{k_1=2}^{N}\cdots\sum_{k_p=2}^{N} \phi'\left(\frac{1}{N+1}\begin{pmatrix} k_1 \\ \vdots \\ k_p \end{pmatrix}\right) Pr\left(R_j = \begin{pmatrix} k_1 \\ \vdots \\ k_p \end{pmatrix}\middle| X_j = x,\ X_k < X_j\right) =$$

$$= E\left[\phi'\left(\frac{1}{N+1}R_j\right)\middle| X_j = x,\ X_k < X_j\right].$$

So,

$$(2.21)\qquad E\left[\phi\left(\frac{1}{N+1}R_j\right)\middle| X_k\right] - E\phi\left(\frac{1}{N+1}R_j\right)$$

$$= (N+1)^{-1}\int \begin{pmatrix} u(x_1 - X_{1k}) - F_{1k}(x_1) \\ \vdots \\ u(x_p - X_{pk}) - F_{pk}(x_p) \end{pmatrix} * E\left[\phi'\left(\frac{1}{N+1}R_j\right)\middle| X_j = x,\ X_k < X_j\right]$$

$$\times\ \mathrm{d}F_j(x_1, \ldots, x_p) + (N+1)^{-2}\, C_2\boldsymbol{\alpha}\,.$$

The remainder of the proof is quite analogous to the proof of Theorem 4.2 of Hájek (1968), so is deferred.

3. APPROXIMATE NORMALITY OF A CLASS OF MULTIVARIATE RANK STATISTICS

Let the assumptions concerning $\mathbf{X}_1, \ldots, \mathbf{X}_N$, independent random p-vectors, be as in Section 1. In this section, we shall prove that the distributions of statistics of the class (1.3) can, under various joint restrictions on the constants, the distribution functions, and the generating functions, be approximated by certain multivariate normal distributions. We start with a theorem that imposes rather stringent restraints on the generating functions ϕ_i, counterbalanced by rather relaxed conditions on $F_1, F_2, \ldots, F_N$ and choice of $c_1, c_2, \ldots, c_N$. (We shall preserve the notational conventions introduced in the previous section. In particular, we remind the reader that if A, B are positive definite matrices, $A > B$ if $(A - B)$ is positive definite, and that if $\boldsymbol{x}, \boldsymbol{y}$ are $p \times 1$ vectors, $\boldsymbol{x} > \boldsymbol{y}$ if $x_i > y_i$, $1 \leqq i \leqq p$.)

Theorem 3.1. *Consider the statistic* $\boldsymbol{S}$ *of* (1.3) *where the scores are given either by* (1.1) *or by* (1.2). *Assume that each* ϕ_i *has bounded second derivative. Then for every* $\varepsilon > 0$, *there exists a constant* $K = K(\varepsilon) > 0$ *such that if*

$$\text{(3.1)} \qquad \operatorname{cov} \boldsymbol{S} > K \max_{i \leqq j \leqq N} (c_j - \bar{c})^2 \boldsymbol{I}^{(p \times p)}$$

then

$$\text{(3.2)} \qquad \sup_{-\infty < x < \infty} \left| Pr[\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S}) < x(\boldsymbol{d}'(\operatorname{cov} \boldsymbol{S})\, \boldsymbol{d})^{1/2}] - \Phi(x) \right| < \varepsilon\,,$$

where $\boldsymbol{d}^{(p \times 1)}$ *is an arbitrary non-null vector and* Φ *denotes the cumulative normal distribution function. In other words, under the hypothesis* (3.1), *the distribution of* $\boldsymbol{S}$ *can be approximated by that of a multivariate normal distribution with natural parameters* ($E\boldsymbol{S}$, $\operatorname{cov} \boldsymbol{S}$). *Furthermore, the conclusion* (3.2) *remains true if we replace* $\operatorname{cov} \boldsymbol{S}$ *in* (3.1) *and* (3.2) *by*

$$\text{(3.3)} \qquad \Sigma = \sum_{j=1}^{N} E[\boldsymbol{Z}_j \boldsymbol{Z}_j']\,,$$

where $\boldsymbol{Z}_j$, $1 \leqq j \leqq N$, *are given as in* (2.18).

Remark. Note that $E\boldsymbol{Z}_j = 0$, $1 \leqq j \leqq N$ since

$$\int \begin{pmatrix} u(x_1 - y_1) - F_{1j}(x_1) \\ \vdots \\ u(x_p - y_p) - F_{pj}(x_p) \end{pmatrix} \mathrm{d}F_j(y_1, \ldots, y_p) = \boldsymbol{O}\,; \quad \text{hence } \textstyle\sum \text{ in (3.3)}$$

is equal to $\sum_{j=1}^{N} \operatorname{cov} \boldsymbol{Z}_j = \operatorname{cov}\left(\sum_{j=1}^{N} \boldsymbol{Z}_j\right)$.

Proof. Choose an $\varepsilon > 0$, and let $\boldsymbol{d}^{(p \times 1)}$ be an arbitrary non-null vector. Then by the Lindeberg-Feller theorem, there exists $\lambda = \lambda(\varepsilon) > 0$ such that

(3.4) $$(d'\Sigma d)^{-1}\sum_{j=1}^{N}\int_{\{|x|>\lambda(d'\Sigma d)^{1/2}\}} x^2\,dPr(d'Z_j \leqq x) < \lambda$$

implies

(3.5) $$\sup_{-\infty<x<\infty}\left|Pr\left(\sum_{j=1}^{N} d'Z_j < x(d'\Sigma d)^{1/2}\right) - \Phi(x)\right| < \tfrac{1}{4}\varepsilon\,.$$

Let $\eta > 0$ be such that $\sup_{-\infty<x<\infty}|\Phi(x) - \Phi(x \pm \eta)| < \frac{1}{4}\varepsilon$; then (3.5) in turn implies

(3.6) $$\sup_{-\infty<x<\infty}\left|Pr\left(\sum_{j=1}^{N} d'Z_j < x(d'\Sigma d)^{1/2} \pm \eta(d'\Sigma d)^{1/2}\right) - \Phi(x)\right| < \tfrac{1}{2}\varepsilon\,.$$

Suppose cov S is such that

(3.7) $$\operatorname{cov} S \geqq \{[2p^{1/2}\lambda^{-1}\max_i\sup_t|\phi_i'(t)| + (2\varepsilon^{-1/2}\eta^{-1} + 1)\,M^{1/2}]^2 \max_{1\leqq k\leqq N}|c_k - \bar{c}|^2\}\,I, \quad M \text{ as in (2.17)}\,.$$

We shall show that (3.2) will follow if we choose $K = [2p^{1/2}\lambda^{-1}\max_{1\leqq i\leqq p}\sup_{0<t<1}|\phi_i(t)| + (2\varepsilon^{-1/2}\eta^{-1} + 1)\,M^{1/2}]^2$. Suppose first that the scores are generated from the ϕ_i by (1.1). From the definition of Z_j, we have

$$|Z_{ij}| = \frac{1}{N+1}\left|\sum_{k=1}^{N}(c_k - c_j)\int[u(x - X_{ij}) - F_{ij}(x)]\,\phi_i'(H_i(x))\,dF_{ij}(x)\right|$$
$$\leqq 2\max_{1\leqq k\leqq N}|c_k - \bar{c}|\sup_{t\in(0,1)}|\phi_i'(t)|\,.$$

So,

(3.8) $$d'Z_j \leqq 2\max_k|c_k - \bar{c}|\max_{1\leqq i\leqq p}\sup_{t\in(0,1)}|\phi_i'(t)|\sum_{i=1}^{p}|d_i|\,.$$

From (2.17) and Minkowski's inequality,

(3.9) $$|(d'\Sigma d)^{1/2} - (d'(\operatorname{cov} S)\,d)^{1/2}| \leqq M^{1/2}\max_k|c_k - \bar{c}|\,[\sum_{i=1}^{p} d_i^2]^{1/2}\,.$$

Combining (3.9) and (3.7),

$$(d'\Sigma d)^{1/2} + M^{1/2}\max_{1\leqq k\leqq N}|c_k - \bar{c}|\,[\sum_{i=1}^{p} d_i^2]^{1/2} \geqq (d'(\operatorname{cov} S)\,d)^{1/2}$$
$$\geqq \max_k|c_k - \bar{c}|\,[\sum_{i=1}^{p} d_i^2]^{1/2}\,[2p^{1/2}\lambda^{-1}\max_i\sup_t|\phi_i'(t)| + M^{1/2}]\,.$$

Or,

(3.10) $$\lambda(d'\Sigma d)^{1/2} \geqq 2p^{1/2}\max_k|c_k - \bar{c}|\,[\sum_{i=1}^{p} d_i^2]^{1/2}\max_i\sup_t|\phi_i'(t)|\,.$$

From the Cauchy-Schwarz inequality, $p \sum_{i=1}^{p} d_i^2 \geqq [\sum_{i=1}^{p} |d_i|]^2$; hence, comparing (3.8) and (3.10) we conclude that the left hand side of (3.4) is 0 for cov $\boldsymbol{S}$ satisfying (3.7). But, since $\boldsymbol{d}$ was arbitrary, this implies that the distribution of $\sum_{j=1}^{N} \boldsymbol{Z}_j$ can be approximated by that of a multivariate normal distribution with parameters $(\boldsymbol{O}, \Sigma)$.

We now show that $\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S})$ is "near" $\boldsymbol{d}' \sum_{j=1}^{N} \boldsymbol{Z}_j$.

$$Pr[\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S}) \leqq x(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2}] \leqq Pr[\boldsymbol{d}' \sum_{j=1}^{N} \boldsymbol{Z}_j \leqq x(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2} + \eta(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2}]$$

$$+ Pr[|\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S} - \sum_{j=1}^{N} \boldsymbol{Z}_j)| > \eta(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2}]$$

$$\leqq \Phi(x) + \tfrac{1}{2}\varepsilon + Pr[|\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S} - \sum_{j=1}^{N} \boldsymbol{Z}_j)| > \eta(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2}] \quad \text{from (3.9)}$$

$$\leqq \Phi(x) + \tfrac{1}{2}\varepsilon + E[\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S} - \sum_{j=1}^{N} \boldsymbol{Z}_j)]^2/\eta^2(\boldsymbol{d}'\Sigma\boldsymbol{d})$$

$$\leqq \Phi(x) + \tfrac{1}{2}\varepsilon + M \max_k |c_k - \bar{c}|^2 \sum_{i=1}^{p} d_i^2/\eta^2(\boldsymbol{d}'\Sigma\boldsymbol{d}) \quad \text{from (2.17).}$$

From (3.7) and (3.9),

$$(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2} + M^{1/2} \max_k |c_k - \bar{c}| \, [\sum_{i=1}^{p} d_i^2]^{1/2}$$

$$\leqq [\boldsymbol{d}'(\text{cov}\, \boldsymbol{S})\, \boldsymbol{d}]^{1/2} \leqq (2\varepsilon^{-1/2}\eta^{-1} + 1)\, M^{1/2} \max_k |c_k - \bar{c}| \, [\sum_{i=1}^{p} d_i^2]^{1/2} .$$

Hence,

$$\eta^2(\boldsymbol{d}'\Sigma\boldsymbol{d}) \leqq 4\varepsilon^{-1} M \max_k |c_k - \bar{c}|^2 \sum_{i=1}^{p} d_i^2$$

and so

$$Pr[\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S}) \leqq x(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2}] \leqq \Phi(x) + \tfrac{3}{4}\varepsilon .$$

We could similarly prove the other inequality $Pr[\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S}) \leqq x(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2}] \geqq \geqq \Phi(x) - \tfrac{3}{4}\varepsilon$, so that

$$\sup_x |Pr[\boldsymbol{d}'(\boldsymbol{S} - E\boldsymbol{S}) \leqq x(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2}] - \Phi(x)| \leqq \tfrac{3}{4}\varepsilon .$$

Next, assuming $\varepsilon < 1$, we again argue from (3.9) and (3.7):

$$\eta(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2} > M^{1/2} \max_k |c_k - \bar{c}| \, [\sum_{i=1}^{p} d_i^2]^{1/2} ,$$

so

$$|(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2} - (\boldsymbol{d}'(\text{cov}\, \boldsymbol{S})\, \boldsymbol{d})^{1/2}| \leqq M^{1/2} \max_k |c_k - \bar{c}| \, [\sum_{i=1}^{p} d_i^2]^{1/2} \leqq \eta(\boldsymbol{d}'\Sigma\boldsymbol{d})^{1/2} .$$

Then, $\sup_x |Pr[d'(S - ES) < x(d'(\text{cov } S)\, d)^{1/2}] - \Phi(x)| < \varepsilon$, as was to be shown. Note that in the course of the proof we have also proved that (3.2) remains valid if cov S is replaced by $\sum$ in (3.7).

Finally, suppose that the scores are generated from the ϕ_i by (1.2). But then

$$E\phi_i(U_{N,i}^{(k)}) = \phi_i\left(\frac{k}{N + 1}\right) + \xi_{i,N},$$

where $|\xi_{i,N}| \leqq CN^{-1}$, and C is a constant that does not depend on i or N. It follows that

$$E\{[(S_1 - ES_1) - (S_2 - ES_2)]\,[(S_1 - ES_1) - (S_2 - ES_2)]'\} \leqq CpN^{-1} \sum_{j=1}^{N} (c_j - \bar{c})^2\, I^{(p \times p)},$$

where $S_1(S_2)$ connotes the statistic S of (1.3) with scores generated by (1.1) ((1.2)). Thus $S_1 - ES_1$ is equivalent to $S_2 - ES_2$ in asymptotic considerations. This completes the proof.

We now prove a version of Theorem 3.1 that is useful in situations in which the distribution functions of the random vectors are "almost identical", in the sense of (3.12). In particular, the following theorem will yield the approximate null distribution (that is, under the hypothesis $F_1 = F_2 = \ldots = F_N$) of the statistic S of (1.3) as an immediate consequence.

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *hold. Define*

$$\begin{aligned} W_j &= \phi(F_{1j}(X_{1j}), F_{2j}(X_{2j}), \ldots, F_{pj}(X_{pj}))', \quad j = 1, 2, \ldots, N \\ V_j &= (c_j - \bar{c})\, W_j \\ \Psi &= \Psi^{(N)} = \sum_{j=1}^{N} E[(V_j - EV_j)(V_j - EV_j)']. \end{aligned}$$

Then for every $\varepsilon > 0$ *and* $\zeta > 0$ *there exists a* $\delta = \delta(\varepsilon, \zeta) > 0$ *such that if*

(3.11) $$\sum_{j=1}^{N} (c_j - \bar{c})^2 > \delta^{-1} \max_{1 \leqq j \leqq N} (c_j - \bar{c})^2$$

(3.12) $$\max_{\substack{1 \leqq j,k \leqq N \\ x \in \mathbb{R}^p}} (F_j(x) - F_k(x)) < \delta$$

and

(3.13) $$\text{cov } W_1 > \zeta I$$

then

(3.14) $$\sup_x |Pr[d'(S - ES) < x(d'\Psi d)^{1/2}] - \Phi(x)| < \varepsilon, \quad d^{(p \times 1)} \neq O.$$

That is, the distribution of $\mathbf{S}$ can be approximated by that of a multinormal random variable with parameters $(E\mathbf{S}, \boldsymbol{\Psi})$.

Proof. We first relate (3.13) to the positive definiteness of $\boldsymbol{\Psi}$. Note that $E\mathbf{W}_1 = E\mathbf{W}_j, j = 2, 3, \ldots, N$, and that

$$\begin{aligned}
E(W_{i1}W_{l1}) - E(W_{ij}W_{lj}) &= \int \phi_i[F_{i1}(X_{i1})]\, \phi_l[F_{l1}(X_{l1})]\, \mathrm{d}F_{il,1}(X_{i1}, X_{l1}) \\
&\quad - \int \phi_i[F_{ij}(X_{ij})]\, \phi_l[F_{lj}(X_{lj})]\, \mathrm{d}F_{il,j}(X_{ij}, X_{lj}) \\
&= \int \phi_i[F_{i1}(X_{i1})]\, \phi_l[F_{l1}(X_{l1})]\, \mathrm{d}[F_{il,1}(X_{i1}, X_{l1}) - F_{il,j}(X_{i1}, X_{l1})] \\
&\quad + \int \{\phi_i[F_{i1}(X_{ij})]\, \phi_l[F_{l1}(X_{lj})] - \phi_i[F_{ij}(X_{ij})]\, \phi_l[F_{lj}(X_{lj})]\} \times \mathrm{d}F_{il,j}(X_{ij}, X_{lj}) \\
&\leqq C_0^2 \max_{\substack{2 \leqq j \leqq N \\ x \in \mathbb{R}^p}} |F_1(x) - F_j(x)| + \int \{\phi_i[F_{i1}(X_{ij})] - \phi_i[F_{ij}(X_{ij})]\} \\
&\quad \times \phi_j[F_{lj}(X_{lj})]\, \mathrm{d}F_{il,j}(X_{ij}, X_{lj}) + \int \phi_i[F_{ij}(X_{ij})]\, \{\phi_l[F_{l1}(X_{lj})] \\
&\quad - \phi_l[F_{lj}(X_{lj})]\}\, \mathrm{d}F_{il,j}(X_{ij}, X_{lj}) \\
&\leqq C_0^2 \max_{\substack{1 \leqq j \leqq N \\ x \in \mathbb{R}^p}} |F_1(x) - F_j(x)| + 2C_0C_1 \max_{\substack{1 \leqq j \leqq N \\ x \in \mathbb{R}^p}} |F_1(x) - F_j(x)| \\
&\leqq \max_{\substack{1 \leqq j,k \leqq N \\ x \in \mathbb{R}^p}} |F_k(x) - F_j(x)|\, [C_0^2 + 2C_0C_1]
\end{aligned}$$

where

$$C_0 = \max_{1 \leqq i \leqq p} \sup_{t \in (0,1)} |\phi_i(t)|, \quad C_1 = \max_{1 \leqq i \leqq p} \sup_{t \in (0,1)} |\phi_i'(t)|.$$

Then,

$$\begin{aligned}
(3.15) \qquad \boldsymbol{\Psi} &= \sum_{j=1}^{N} (c_j - \bar{c})^2 \operatorname{cov} \mathbf{W}_j \\
&= \sum_{j=1}^{N} (c_j - \bar{c})^2 \operatorname{cov} \mathbf{W}_1 + \sum_{j=1}^{N} (c_j - \bar{c})^2 (\operatorname{cov} \mathbf{W}_j - \operatorname{cov} \mathbf{W}_1) \\
&\geqq \sum_{j=1}^{N} (c_j - \bar{c})^2 [\zeta - \max |F_j(x) - F_k(x)| \,.\, (C_0^2 + 2C_0C_1)\, p]\, \mathbf{I},
\end{aligned}$$

where the matrix on the right hand side is positive definite for $\max |F_j(x) - F_k(x)|$ sufficiently small.

The remainder of the proof consists of showing that $\boldsymbol{\Psi}$ is "near" $\sum$ of Theorem 3.1, so that the proof may follow from the preceding theorem. The methods are straightforward, so are omitted.

Remark. Suppose the assumptions of Theorem 3.2 are satisfied, and that $F_1 = F_2 = \ldots = F_N$. Then the diagonal elements of Ψ, the approximate covariance matrix of $\boldsymbol{S}$, are given by $\varphi_{ii} = \sum_{j=1}^{N} (c_j - \bar{c})^2 \int_0^1 [\phi_i(t) - \bar{\Phi}_i]^2 \, dt$, where $\bar{\Phi}_i = \int_0^1 \phi_i(t) \, dt$. However, the off-diagonals are generally dependent upon the underlying distribution functions $F_1, \ldots, F_N$: for example, assuming continuous, non-vanishing densities, we have

$$E[(W_{ij} - EW_{ij})(W_{lj} - EW_{lj})] = \int_{w=0}^{1} \int_{z=0}^{1} \phi_i(w) \, \phi_l(z) \, [f_{ij} F_{ij}^{-1}(w)]^{-1} [f_{lj} F_{lj}^{-1}(z)]^{-1} f_{il,j}(F_{ij}^{-1}(w), F_{lj}^{-1}(z)) \, dw \, dz - \bar{\Phi}_i \bar{\Phi}_l \, ,$$

where $f_{ij}(f_{lj})$ is the ith (lth) marginal p.d.f. of $\boldsymbol{X}_j$, and $f_{il,j}$ is the joint p.d.f. of X_{ij} and X_{lj}. In orther words, the distribution of our multivariate rank statistic $\boldsymbol{S}$ is generally not independent of the underlying distributions of the observations. Thus in circumstances in which the F_j are postulated equal but unknown, only the diagonals of Ψ – the variances of the marginal univariate rank statistics, the individual components of $\boldsymbol{S}$ – can be calculated; the additional information – specifically, the covariance structure of $\boldsymbol{S}$ – that Theorem 3.2 provides over the univariate asymptotic results cannot be utilized readily.

In the next theorem, we shall relax our restrictions on the generating functions ϕ_i to the situation in which they are absolutely continuous and square integrable. Recall (cf. Natanson (1961), p. 242) that a real valued function ϕ is absolutely continuous inside $(0, 1)$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $[\sum_{k=1}^{N} |\phi(b_k) - \phi(a_k)|] < \varepsilon$ for all numbers $a_1, b_1, \ldots, a_N, b_N$ where $0 < a_1 < b_1 \leqq a_2 < b_2 \leqq \ldots \leqq a_N < b_N < 1$, and $\sum_{k=1}^{N} (b_k - a_k) < \delta$. It s a well known fact (Natanson (1961), p. 246) that if ϕ is absolutely continuous in $(0, 1)$, then ϕ is of bounded variation; hence ϕ' exists, is finite almost everywhere, and is integrable on $(0, 1)$. However, the limit of $\phi(t)$ as $t \to 0$ or $t \to 1$ may not exist or may exist, but be infinite.

Theorem 3.3. *Consider the statistic* $\boldsymbol{S}$ *of* (1.3), *where the scores are given either by* (1.1) *or by* (1.2). *Assume that each* ϕ_i *satisfies* $\phi_i(t) = \phi_i^{(1)}(t) - \phi_i^{(2)}(t), 0 < t < 1$, *where the* $\phi_i^{(m)}$, $1 \leqq i \leqq p$, $m = 1, 2$, *are non-decreasing, square integrable, and absolutely continuous inside* $(0, 1)$. *Then for every* $\varepsilon > 0$ *and* $\zeta > 0$ *there exists* $N_0 = N_0(\varepsilon, \zeta)$ *such that if*

(3.16) $$N > N_0(\varepsilon, \zeta)$$

and

(3.17) $$\operatorname{cov} \boldsymbol{S} > \zeta N \max_{1 \leqq j \leqq N} (c_j - \bar{c})^2 \, \boldsymbol{I}$$

hold, then (3.2) *obtains. That is,* $\boldsymbol{S}$ *can be approximated by a multivariate normal distribution with parameters* ($E\boldsymbol{S}$, cov $\boldsymbol{S}$). *In addition, as in Thoerem* 3.1. cov $\boldsymbol{S}$ *in* (3.17) *and* (3.2) *may be replaced by* Σ *of* (3.3).

Proof. Let $\varepsilon > 0$, $\zeta > 0$ be given. Choose $\beta > 0$ and $\eta > 0$ such that

$$\sup_x \left|\Phi(x) - \Phi[(x \pm \beta)(1 \pm \eta)^{-1}]\right| < \tfrac{1}{4}\varepsilon \tag{3.18}$$

and let $\alpha > 0$ be such that

$$\alpha < \zeta p^{-1} \min(\eta^2, \tfrac{1}{4}\beta^2)/84 \,. \tag{3.19}$$

From Lemma 5.1 of Hájek (1968), we can decompose $\phi_i(t) = \chi_i(t) + \tau_i^{(1)}(t) - \tau_i^{(2)}(t)$, $0 < t < 1$, where χ_i is a polynomial, $\tau_i^{(1)}$ and $\tau_i^{(2)}$ are non-decreasing, and $\int_0^1 [\tau_i^{(1)}(t)]^2 \,. \,. \mathrm{d}t + \int_0^1 [\tau_i^{(2)}(t)]^2 \,\mathrm{d}t < \frac{1}{2}\alpha$. Denote

$$\boldsymbol{\chi}(\boldsymbol{x}) = (\chi_1(x_1), \ldots, \chi_p(x_p))'$$

$$\boldsymbol{\tau}^{(k)}(\boldsymbol{x}) = (\tau_1^{(k)}(x_1), \ldots, \tau_p^{(k)}(x_p))' \quad k = 1, 2$$

$$\boldsymbol{S}_x = \sum_{j=1}^{N} c_j \boldsymbol{\chi}\left(\frac{1}{N+1}\boldsymbol{R}_j\right)$$

$$\boldsymbol{S}_k = \sum_{j=1}^{N} c_j \boldsymbol{\tau}^{(k)}\left(\frac{1}{N+1}\boldsymbol{R}_j\right) \quad k = 1, 2 \,.$$

Then $\boldsymbol{S} = \boldsymbol{S}_x + \boldsymbol{S}_1 - \boldsymbol{S}_2$. Now, let $\boldsymbol{d}^{(p\times 1)}$ be a non-zero vector. Then

$$\begin{aligned} \left|[\operatorname{var} \boldsymbol{d}'\boldsymbol{S}]^{1/2} - [\operatorname{var} \boldsymbol{d}'\boldsymbol{S}_x]^{1/2}\right| &\leqq [\operatorname{var} \boldsymbol{d}'(\boldsymbol{S} - \boldsymbol{S}_x)]^{1/2} \\ &\leqq [\operatorname{var} \boldsymbol{d}'\boldsymbol{S}_1]^{1/2} + [\operatorname{var} \boldsymbol{d}'\boldsymbol{S}_2]^{1/2} \,. \end{aligned} \tag{3.20}$$

Denote by $T^{(k)} = \{t_{ij}^{(k)}\}$ the covariance matrix of $\boldsymbol{S}_k$, $k = 1, 2$. Then by Hájek's variance inequality [(1968), Theorem 3.1]

$$t_{ii}^{(k)} \leqq 21 \max_{1\leqq j\leqq N} (c_j - \bar{c})^2 \sum_{j=1}^{N} \left[\tau_i^{(k)}\left(\frac{1}{N+1}\right)\right]^2 .$$

Hence,

$$\operatorname{var} \boldsymbol{d}'\boldsymbol{S}_k \leqq p(\boldsymbol{d}'\boldsymbol{d})\, 21 \max_{1\leqq j\leqq N} (c_j - \bar{c})^2 \max_{1\leqq i\leqq p} \left\{\sum_{j=1}^{N} \left[\tau_i^{(k)}\left(\frac{j}{N+1}\right)\right]^2\right\} .$$

Consequently,

$$\begin{aligned} &[\operatorname{var} \boldsymbol{d}'\boldsymbol{S}_1]^{1/2} + [\operatorname{var} \boldsymbol{d}'\boldsymbol{S}_2]^{1/2} \\ &\leqq p^{1/2}(\boldsymbol{d}'\boldsymbol{d})^{1/2} (21)^{1/2} \max_{1\leqq j\leqq N} |c_j - \bar{c}| \left\{\left[\max_{1\leqq i\leqq p} \sum_{j=1}^{N} \left[\tau_i^{(1)}\left(\frac{j}{N+1}\right)\right]^2\right]^{1/2} \right. \\ &\quad \left. + \left[\max_{1\leqq i\leqq p} \sum_{j=1}^{N} \left[\tau_i^{(2)}\left(\frac{j}{N+1}\right)\right]^2\right]^{1/2}\right\} \end{aligned} \tag{3.21}$$

$$\leqq p^{1/2}(\boldsymbol{d}'\boldsymbol{d})^{1/2}(21)^{1/2}\max_{1\leqq j\leqq N}|c_j-\bar{c}|\,2^{1/2}\left\{\max_{1\leqq i_1,i_2\leqq p}\left[\sum_{j=1}^{N}\left[\tau_{i_1}^{(1)}\left(\frac{j}{N+1}\right)\right]^2+\sum_{j=1}^{N}\left[\tau_{i_2}^{(2)}\left(\frac{j}{N+1}\right)\right]^2\right]\right\}^{1/2}.$$

But for non-decreasing τ_i,

$$\sum_{j=1}^{N}\left[\tau_i\left(\frac{j}{N+1}\right)\right]^2\leqq 2N\int_0^1[\tau_i(t)]^2\,dt\,, \tag{3.22}$$

and by construction of $\tau_i^{(1)}$ and $\tau_i^{(2)}$,

$$\int_0^1[\tau_{i_1}^{(2)}(t)]^2\,dt+\int_0^1[\tau_{i_2}^{(2)}(t)]^2\,dt<\alpha\quad\text{for any choice of } i_1 \text{ and } i_2\,. \tag{3.23}$$

Combining (3.21), (3.22), (3.23), and (3.19), we have

$$\begin{aligned}&[\operatorname{var}\boldsymbol{d}'\boldsymbol{S}_1]^{1/2}+[\operatorname{var}\boldsymbol{d}'\boldsymbol{S}_2]^{1/2}\\&\leqq p^{1/2}(\boldsymbol{d}'\boldsymbol{d})^{1/2}(42)^{1/2}\max_{1\leqq j\leqq N}|c_j-\bar{c}|\,(2N\alpha)^{1/2}\\&\leqq(\boldsymbol{d}'\boldsymbol{d})^{1/2}\max_{1\leqq j\leqq N}|c_j-\bar{c}|\,N^{1/2}\zeta^{1/2}\min(\eta,\tfrac{1}{2}\beta\varepsilon^{1/2})\,.\end{aligned}\tag{3.24}$$

But if (3.17) is satisfied, then

$$\begin{aligned}|(\operatorname{var}\boldsymbol{d}'\boldsymbol{S})^{1/2}-(\operatorname{var}\boldsymbol{d}'\boldsymbol{S}_x)^{1/2}|&\leqq[\operatorname{var}\boldsymbol{d}'(\boldsymbol{S}-\boldsymbol{S}_x)]^{1/2}\\&\leqq[\boldsymbol{d}'(\operatorname{cov}\boldsymbol{S})\,\boldsymbol{d}]^{1/2}\min(\eta,\tfrac{1}{2}\beta\varepsilon^{1/2})\,.\end{aligned}\tag{3.25}$$

Let $K_{1/2\varepsilon}=K_{1/2\varepsilon}(\chi)$ be the constant, the existence of which is established in Theorem 3.1. (The χ_i, being polynomials, have bounded second derivatives; thus Theorem 3.1 can be applied to $\boldsymbol{S}_x$.) Put $N_0(\varepsilon,\zeta)=(1-\eta)^{-2}\zeta^{-1}K_{1/2\varepsilon}$; then from (3.25),

$$\operatorname{var}\boldsymbol{d}'\boldsymbol{S}_x\geqq\operatorname{var}\boldsymbol{d}'\boldsymbol{S}[1-\min(\eta,\tfrac{1}{2}\beta\varepsilon^{1/2})]^2$$

(this follows from the fact that if $|a-b|<ca$, then $b>(1-c)\,a$). So $\operatorname{cov}\boldsymbol{S}>\zeta N(\xi,\zeta)\max_{1\leqq j\leqq N}(c_j-\bar{c})^2\,\boldsymbol{I}$ implies $\operatorname{cov}\boldsymbol{S}_x>K_{1/2\varepsilon}\max_{1\leqq j\leqq N}(c_j-\bar{c})^2\,\boldsymbol{I}$.

Arguing as in the proof of Theorem 3.1, we have

$$\begin{aligned}&Pr[\boldsymbol{d}'(\boldsymbol{S}-E\boldsymbol{S})<x[\boldsymbol{d}'(\operatorname{cov}\boldsymbol{S})\,\boldsymbol{d}]^{1/2}]\\&\leqq Pr[\boldsymbol{d}'(\boldsymbol{S}_x-E\boldsymbol{S}_x)\leqq(x+\beta)\,[\boldsymbol{d}'(\operatorname{cov}\boldsymbol{S})\,\boldsymbol{d}]^{1/2}]\\&+Pr[|\boldsymbol{d}'(\boldsymbol{S}-E\boldsymbol{S}-\boldsymbol{S}_x+E\boldsymbol{S}_x)|>\beta[\boldsymbol{d}'(\operatorname{cov}\boldsymbol{S})\,\boldsymbol{d}]^{1/2}]\\&\leqq Pr[\boldsymbol{d}'(\boldsymbol{S}_x-E\boldsymbol{S}_x)\leqq(x+\beta)(1-\eta)^{-1}[\operatorname{var}\boldsymbol{d}'\boldsymbol{S}_x]^{1/2}]\\&+[\operatorname{var}\boldsymbol{d}'(\boldsymbol{S}-\boldsymbol{S}_x)]\,[\beta^2\boldsymbol{d}'(\operatorname{cov}\boldsymbol{S})\,\boldsymbol{d}]^{-1}\\&\leqq\Phi((x+\beta)(1-\eta)^{-1})+\tfrac{1}{2}\varepsilon+\tfrac{1}{4}\varepsilon\leqq\Phi(x)+\varepsilon\,.\end{aligned}$$

Similarly, we could prove the opposite inequality
$Pr[d'(S - ES) < x(d'(\text{cov } S)\, d)^{1/2}] \geqq \Phi(x) - \varepsilon$; this establishes (3.2).

The version of the theorem in which cov S is replaced by Σ may be shown by means of a decomposition of Σ corresponding to the decomposition $\phi = \chi + \tau^{(1)} - \tau^{(2)}$. Define Σ_x, Σ_1, and Σ_2 by (2.18) and (3.3) wherein ϕ is replaced by $\chi, \tau^{(1)}$, and $\tau^{(2)}$ respectively. Then

$$|(d'\Sigma d)^{1/2} - (d'\Sigma_x d)^{1/2}| \leqq (d'\Sigma_1 d)^{1/2} + (d'\Sigma_2 d)^{1/2}.$$

Utilizing a univariate variance inequality of Hájek [(1968), eqn. (5.37)],

$$[(d'\Sigma d)^{1/2} - (d'\Sigma_x d)^{1/2}]^2 \leqq 8pN \max_{1 \leqq j \leqq N} (c_j - \bar{c})^2 \, \alpha(d'd).$$

Then for α sufficiently small, $|(\mathbf{d}'\Sigma \mathbf{d})^{1/2} - (\mathbf{d}'\Sigma_x \mathbf{d})^{1/2}|$ can be made negligible in comparison with $\zeta N \max_{1 \leqq j \leqq N} (c_j - \bar{c})^2 (\mathbf{d}'\mathbf{d})$. As seen in (3.9) and (3.25) both $|[\text{var } \mathbf{d}'S_x]^{1/2} - [d'\Sigma_x d]^{1/2}|$ and $|[\text{var } d'S]^{1/2} - (\text{var } \mathbf{d}'S_x)^{1/2}|$ are suitably bounded. The difference $|(d'\Sigma d)^{1/2} - (\mathbf{d}'(\text{cov } S)\, \mathbf{d})^{1/2}|$ is thus negligible with respect to $(\mathbf{d}'\Sigma\mathbf{d})^{1/2}$ if $\Sigma > \zeta N \max_{1 \leqq j \leqq N} (c_j - \bar{c})^2 I$ and N is sufficiently large.

We have implicitly assumed in the above derivation that the scores are given as in (1.1). But if (1.2) obtains, we could argue as in the proof of Theorem 3.1, with the aid of the inequality $\sum_{j=1}^{N} [E\phi_i(U_{N,j}^{(i)})]^2 \leqq N \int_0^1 [\phi_i(t)]^2 \, dt$, valid for all i.

Remark. In anticipation of section 4, we note that if the scores are indeed given by (1.2) where the ϕ_i are square integrable and absolutely continuous, then $\sum_{j=1}^{N} |a_i^{(N)}(j) - \phi_i(j)| = 0(1)$, for all N and i. This observation, together with Theorem 4.2, yields a different proof of the assertion that Theorem 3.3 remains valid when the scores are related to the generating functions by (1.2). Indeed, from the results contained in section 4, it is sufficient to prove the theorems of this section merely under the assumption that the scores are related to the generating functions by (1.1); as we shall see, the theorems all remain valid if (1.2) instead obtains.

As we had done with Theorem 3.1, we now specialize Theorem 3.3 to the situation in which the distribution functions of the random observations $X_1, X_2, \ldots, X_N$ are "nearly identical". The following theorem is especially useful (as is Theorem 3.2) in determining the null distribution of S.

Theorem 3.4. *Let the assumptions of Theorem* 3.3 *hold. Define* V_j, W_j, $1 \leqq j \leqq N$ *and* Ψ *as in Theorem* 3.2. *Then given any* $\varepsilon > 0$, $\zeta_1 > 0$, *and* $\zeta_2 > 0$ *there exists a* $\delta = \delta(\varepsilon, \zeta_1, \zeta_2) > 0$ *and* $N_0 = N_0(\varepsilon, \zeta_1, \zeta_2) > 0$ *such that if*

$$N > N_0, \quad \sum_{j=1}^{N} (c_j - \bar{c})^2 > \zeta_1 N \max_{1 \leqq j \leqq N} (c_j - \bar{c})^2 \tag{3.26}$$

$$(3.27) \qquad \max_{\substack{1 \le j,k \le N \\ x \in \mathbb{R}^p}} (F_j(x) - F_k(x)) < \delta$$

and

$$(3.28) \qquad \operatorname{cov} \boldsymbol{W}_1 > \zeta_2 \boldsymbol{I}$$

are jointly satisfied, then (3.14) *holds.*

Proof. The proof is an amalgam of techniques used in the proofs of Theorems 3.2 and 3.3, and that of Theorem 2.4 of Hájek (1968), hence is omitted.

4. CENTERING VALUES FOR CLASSES OF MULTIVARIATE RANK STATISTICS

In section 3 we have shown, under various joint restrictions on the constants $c_1, c_2, \ldots, c_N$. the distribution functions $F_1, F_2, \ldots, F_N$, and the generating functions $\phi_1, \phi_2, \ldots, \phi_p$, that the distribution of the linear rank statistic $\boldsymbol{S}$ can be approximated by a multivariate normal distribution with natural parameters $(E\boldsymbol{S}, \operatorname{cov} \boldsymbol{S})$. Furthermore, we had indicated that the covariance matrix $\operatorname{cov} \boldsymbol{S}$ could itself be approximated by another covariance matrix, the latter being characterized by its ready expressibility and calculability in terms of known parameters. We have deferred until this section, however, the analogous problem with regard to the centering values of $\boldsymbol{S}$: namely, under what conditions can we replace $E\boldsymbol{S}$ in the conclusions of the theorems of section 3 by relatively simpler expressions? The question is of paramount importance in applications, because $E\boldsymbol{S}$ generally is not easily computable.

We remark that in the case of univariate rank statistics $S = \sum_{j=1}^{N} c_j \, a(R_j)$, the problem of finding centering constants that have simpler structure and are easier to evaluate than ES was left unanswered by Hájek, but was subsequently investigated by Dupač (1970) and by Hoeffding (1973). Dupač successfully found centering constants for S under the hypothesis that the generating functions had bounded second derivatives; Hoeffding, too, succeeded, upon imposing on the generating functions a condition slightly stronger than square integrability, but still weaker than existence of second derivatives. We shall provide in this section the multivariate analogs of these results.

With the following theorem, we show that Theorem 3.1 can be generalized with regard to the choice of scores $a_i^{(N)}(\cdot)$ – that is, the scores need not be specified exactly by (1.1) or (1.2) but may instead satisfy a broader relation to the generating functions ϕ_i. We also provide relatively simple expressions that can be substituted for $E\boldsymbol{S}$ in the conclusion of Theorem 3.1. The theorem is based upon Theorem H, 2.1 of Dupač (1970), in which are proved similar assertions, but for the univariate case.

To simplify the exposition, we introduce here the following definitions:

$$\bar{a}_i^{(N)} = N^{-1} \sum_{j=1}^{N} a_i^{(N)}(j)$$
$$\bar{\boldsymbol{a}} = (\bar{a}_1^{(N)}, \bar{a}_2^{(N)}, \ldots, \bar{a}_p^{(N)})'$$
$$\bar{\boldsymbol{\phi}} = (\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_p)' .$$

Also, we shall adhere to the notational conventions adopted in earlier sections.

Theorem 4.1. *Under the assumptions of Theorem* 3.1, (3.2) *remains valid*:

(i) *with* $\boldsymbol{ES}$ *replaced by* $\boldsymbol{\mu} \equiv \sum_{j=1}^{N} c_j \, E\boldsymbol{\phi}(\boldsymbol{H}(\boldsymbol{X}_j))$, *if the scores are related to the generating functions by* (1.2); *or*

(ii) *with* $\boldsymbol{ES}$ *replaced by* $\boldsymbol{\mu}^* \equiv \boldsymbol{\mu} + N\bar{c}(\bar{\boldsymbol{a}} - \bar{\boldsymbol{\phi}})$, *if the scores are given by* (1.1), *or more generally, if the scores are related to the generating functions by*

$$\sum_{j=1}^{N} \left| a_i^{(N)}(j) - \phi_i\left(\frac{j}{N+1}\right) \right| = 0(1) , \quad 1 \leqq i \leqq p ; \tag{4.1}$$

or

(iii) *with* $\boldsymbol{ES}$ *replaced by* $\boldsymbol{\mu}$ *if* (4.1) *holds and if*

$$\bar{c} = 0(\max_{1 \leqq j \leqq N} |c_j - \bar{c}|) . \tag{4.2}$$

The next theorem stands in relation to Theorem 3.3 as the previous theorem does to Theorem 3.1: we prove that Theorem 3.3 remains valid if the scores are related to the generating functions by (4.1), and we provide simple centering values for $\boldsymbol{S}$. The theorem is a multivariate generalization of Theorem 1 of Hoeffding (1973).

Theorem 4.2. *Let the assumptions of Theorem* 3.3 *be satisfied, but with the square integrability condition on the* $\phi_i^{(k)}$ *be replaced by*

$$J(\phi_i^{(k)}) = \int_0^1 t^{1/2}(1-t)^{1/2} \, \mathrm{d}\phi_i^{(k)}(t) < \infty , \quad k = 1, 2 , \quad 1 \leqq i \leqq p . \tag{4.3}$$

Then the conclusion of Theorem 3.3 *follows*:

(i) *with* $\boldsymbol{ES}$ *replaced by* $\boldsymbol{\mu}$ *in* (3.2), *if the scores are given by* (1.2); *or*

(ii) *with* $\boldsymbol{ES}$ *replaced by* $\boldsymbol{\mu}^*$, *if the scores are given by* (1.1), *or more generally, if the scores are related to the generating functions by* (4.1); *or*

(iii) *with* $\boldsymbol{ES}$ *replaced by* $\boldsymbol{\mu}$ *if both* (4.1) *and* (4.2) *obtain.*

Remark. Integrating by parts in (4.3) we obtain

$$J(\phi_i) = \int_0^1 \phi_i(t) \, (t - \tfrac{1}{2}) \, t^{-1/2}(1-t)^{-1/2} \, \mathrm{d}t .$$

Hence condition (4.3) is equivalent to

$$\int_0^1 |\phi_i^{(k)}(t)|\, t^{-1/2}(1-t)^{-1/2}\, dt < \infty\,, \quad k = 1, 2\,, \quad 1 \leqq i \leqq p\,.$$

Hoeffding's Proposition 2 implies that, in the univariate case,

$$(ES - \mu)^2 = 0\left[N \max\,(c_j - \bar{c})^2 \left(\int_0^1 |\phi(t)|\, t^{-1/2}(1-t)^{-1/2}\, dt\right)^2\right],$$

from whence arises the assumption $\phi = \phi^{(1)} - \phi^{(2)}$, with $\phi^{(k)}(t)\, t^{-1/2}(1-t)^{-1/2}$ integrable. Hoeffding shows that if ϕ is non-decreasing, then the condition $J(\phi) < \infty$ implies square integrability of ϕ and is implied by $\int_0^1 \phi^2(t)\, [\log(1 + |\phi(t)|)]^{1+\delta}\, dt < < \infty$ for some $\delta > 0$. In this sense, the condition (4.17) is not much more restrictive than square integrability.

The proofs of Theorems 4.1 and 4.2 follow directly from the aforementioned univariate results of Dupač (1970) and Hoeffding (1973), and the observation that if

$$|S_i - \mu_i|\, (\mathrm{var}\, S_i)^{1/2} < \tau\,, \quad i = 1, \dots, p\,,$$

then

$$\begin{aligned} &|Pr[(S_i - \mu_i)(\mathrm{var}\, S_i)^{1/2} < x_i\,, \quad i = 1, \dots, p] \\ &- Pr[(S_i - ES_i)(\mathrm{var}\, S_i)^{1/2} < x_i\,, \quad i = 1, \dots, p]| \\ &\leqq \max\{\sum_{i=1}^{p} Pr[x_i \leqq (S_i - ES_i)(\mathrm{var}\, S_i)^{1/2} < x_i + \tau]\,, \\ &\sum_{i=1}^{p} Pr[x_i - \tau \leqq (S_i - ES_i)(\mathrm{var}\, S_i)^{1/2} < x_i]\}\,. \end{aligned}$$

Acknowledgments. This paper is based on the author's Ph.D. dissertation submitted to the Department of Statistics of the University of Chicago. Financial assistance under the National Science Foundation Graduate Fellowship Program is gratefully acknowledged. The author is deeply indebted to Professor Michael D. Perlman of the University of Chicago for his interest and encouragement, comments and criticisms during the course of this research. The author also thanks the referee, whose perceptive suggestions were most helpful.

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Souhrn

ASYMPTOTICKÁ NORMALITA MNOHOROZMĚRNÝCH LINEÁRNÍCH POŘADOVÝCH STATISTIK PŘI OBECNÝCH ALTERNATIVÁCH

JAMES A. KOZIOL

Nechť X_j, $1 \leqq j \leqq N$, jsou nezávislé náhodné p-vektory se spojitými distribučními funkcemi F_j, $1 \leqq j \leqq N$. Definujme p-vektory $\boldsymbol{R}_j$ tak, že položíme R_{ij} rovno pořadí X_{ij} mezi hodnotami $X_{i1}, \ldots, X_{iN}$, $1 \leqq i \leqq p$, $1 \leqq j \leqq N$. Budiž $\boldsymbol{a}^{(N)}(\cdot)$ mnohorozměrná skórová funkce v R^p a položme $\boldsymbol{S} = \sum_{j=1}^{N} c_j\, \boldsymbol{a}^{(N)}(\boldsymbol{R}_j)$, kde c_j jsou libovolné regresní konstanty. V článku se vyšetřuje asymptotické rozložení $\boldsymbol{S}$ při různých podmínkách na konstanty, na skórovou funkci a na základní distribuční funkce. Speciálně je dokázána asymptotická normalita $\boldsymbol{S}$ pouze za předpokladu, že F_j jsou spojité. Dále jsou za určitých slabých předpokladů nalezeny centrující vektory pro $\boldsymbol{S}$.

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