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WEAK PERIODIC SOLUTIONS OF THE BOUNDARY VALUE PROBLEM FOR NONLINEAR HEAT EQUATION

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1. INTRODUCTION

Let $\omega > 0$. Suppose that $f(t, x)$ is an ω-periodic function in t. Let $g : \mathbf{R}^1 \to \mathbf{R}^1$ be a continuous real valued function defined on the real line $\mathbf{R}^1$. In this paper we shall investigate the existence of a solution of the problem

$$
\begin{aligned}
(1) \qquad & u_t(t, x) - u_{xx}(t, x) - g(u(t, x)) = f(t, x)\,, \quad (t, x) \in Q = \mathbf{R}^1 \times (0, \pi)\,, \\
& u(t, 0) = u(t, \pi) = 0\,, \quad t \in \mathbf{R}^1\,, \\
& u(t + \omega, x) = u(t, x)\,, \quad (t, x) \in Q
\end{aligned}
$$

under the assumption that there exist finite limits

$$\mu = \lim_{\xi \to \infty} \frac{g(\xi)}{\xi}\,, \quad \nu = \lim_{\xi \to -\infty} \frac{g(\xi)}{\xi}\,.$$

We shall work with the concept of weak solution of (1). This notion is introduced together with the approrpiate function spaces in Section 2, where we shall also summarize the basic properties of the periodic solvability of the linear heat equation (i.e. of the problem (1) with $g(\xi) = \lambda\xi$).

In Sections 3 and 4 we consider the case

$$\mu = \nu = \lambda\,.$$

We distinguish two different cases. The regular case is defined as the when the problem

$$
\begin{aligned}
(2) \qquad & u_t(t, x) - u_{xx}(t, x) - \lambda\, u(t, x) = 0\,, \quad (t, x) \in Q \\
& u(t, 0) = u(t, \pi) = 0\,, \quad t \in \mathbf{R}^1
\end{aligned}
$$

$$u(t + \omega, x) = u(t, x), \quad (t, x) \in Q$$

has the trivial solution only. If (2) has also a nontrivial solution then we say that the singular case occurs.

In Section 5 we consider the case

$$\mu \neq \nu .$$

The proofs in the whole paper are based on the Schauder fixed point theorem and on the properties of the Leray-Schauder degree. The method is essentially the same as that used in the investigation of solvability of the Dirichlet problem for the second order ordinary differential equation

$$-u''(x) - g(u(x)) = f(x), \quad x \in (0, \pi),$$
$$u(0) = u(\pi) = 0$$

(see e.g. [4], [5], [6], [2], [7], [8]).

In writing this paper the authors were influenced by reading the manuscript of the prepared book by O. Vejvoda et al. [17]. The paper extends in a certain sense the results from [16]. Many assertions included below may be generalized (see Section 6). Some of these generalizations will be published later.

This paper is a part of the first author's thesis.

2. PRELIMINARIES

2.1 **Notations, basic definitions**

2.1.1. Denote by $\mathbf{N}$ the set of all positive integers.

2.1.2. All functions will be assumed real-valued. If u is a function defined on the set A then denote

$$u^+ : y \mapsto \max \{u(y), 0\}, \quad y \in A$$

(the positive part of the function u) and

$$u^- : y \mapsto \max \{-u(y), 0\}, \quad y \in A$$

(the negative part of the function u).

2.1.3. Integration will always be taken with respect to the Lebesgue measure.

2.1.4. Let X and Z be two vector spaces. If F is a mapping defined on X with values in the space Z (we write $F : X \to Z$), denote by Im $[F]$ the set of all values of the mapping F, i.e.

$$\operatorname{Im}[F] = F(X).$$

2.1.5. Let $L: X \to Z$ be a linear mapping. In this case we denote by Ker $[L]$ the null-space of the operator L, i.e.

$$\text{Ker}\,[L] = \{u \in X;\; Lu = \theta\}\,.$$

2.1.6. Suppose that X and Z are real Banach spaces. Let $F: X \to Z$. Then F is said to be completely continuous on X if for each bounded subset M of X, $F(M)$ is a relatively compact set in Z and F is continuous on X.

2.1.7. Let X and Z be two Banach spaces; then X is continuously imbedded in Z (we write $X \hookrightarrow Z$) if

(a) $X \subset Z$, and

(b) every convergent sequence in X is also convergent in Z.

Thus the imbedding operator $i: X \to Z$ defined by $i: u \mapsto u$ is a linear continuous mapping and hence there exists a positive number k such that

$$\|u\|_Z \leqq k\|u\|_X\,, \quad u \in X\,.$$

The space X is said to be compactly imbedded into Z (we shall write $X \hookrightarrow\hookrightarrow Z$) if the imbedding operator $i: X \to Z$ is completely continuous.

2.2. Function spaces

In the sequel Q denotes the set $\mathbf{R}^1 \times (0, \pi)$.

2.2.1. Denote by $H^0_\omega(Q)$ the space of all functions $u(t, x)$ defined almost everywhere on Q which are ω-periodic in the variable t, i.e.

$$u(t + \omega, x) = u(t, x) \quad \text{for almost all} \quad (t, x) \in Q\,,$$

and which are square integrable over $(0, \omega) \times (0, \pi)$. Introducing

$$\|u\|_0 = \left(\int_0^\omega \int_0^\pi u^2(t, x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2},$$

$H^0_\omega(Q)$ becomes a Banach space with the norm $\|u\|_0$.

2.2.2. Let the derivatives mean the derivatives in the sense of distributions. Put

$$H^1_\omega(Q) = \{u \in H^0_\omega(Q);\; u_t, u_x \in H^0_\omega(Q)\}\,,$$
$$H^{1,2}_\omega(Q) = \{u \in H^0_\omega(Q);\; u_t, u_x, u_{xx} \in H^0_\omega(Q)\}\,.$$

If we define

$$\|u\|_1 = \left(\int_0^\omega \int_0^\pi (u^2(t, x) + u_t^2(t, x) + u_x^2(t, x))\,\mathrm{d}x\,\mathrm{d}x\right)^{1/2}$$

and

$$\|u\|_{1,2} = \left(\int_0^\omega \int_0^\pi (u^2(t, x) + u_t^2(t, x) + u_x^2(t, x) + u_{xx}^2(t, x)) \, dx \, dt \right)^{1/2}$$

we obtain easily that $\|u\|_1$ and $\|u\|_{1,2}$ are norms on $H^1_\omega(Q)$ and $H^{1,2}_\omega(Q)$, respectively. Moreover, the spaces $H^1_\omega(Q)$ and $H^{1,2}_\omega(Q)$ are Banach spaces (with respect to the just defined norms).

2.2.3. Denote by ${}^\circ H^1_\omega(Q)$ the closure in $H^1_\omega(Q)$ of the set of all infinitely continuously differentiable ω-periodic functions $u(t, x)$ on Q satisfying

$$u(t, 0) = u(t, \pi) = 0 \,, \quad t \in \mathbf{R}^1 \,.$$

2.2.4. It is easy to see that the norms $\|u\|_1$, $\|u\|_{1,2}$, $\|u\|_0$ induce inner products $\langle u, v\rangle_1$, $\langle u, v\rangle_{1,2}$, $\langle u, v\rangle_0$ in the corresponding spaces and so the spaces $H^0_\omega(Q)$, $H^1_\omega(Q)$, $H^{1,2}_\omega(Q)$ (and also ${}^\circ H^1_\omega(Q)$ with the norm $\|u\|_1$) are Hilbert spaces.

2.2.5. It is (see [15]) $H^{1,2}_\omega(Q) \subset\subset \mathscr{C}_\omega(\bar{Q})$, where $\mathscr{C}_\omega(\bar{Q})$ is the space of all continuous functions $u(t, x)$ on $\bar{Q}$ which are ω-periodic in the variable t (the space $\mathscr{C}_\omega(\bar{Q})$ is equipped with the norm

$$\|u\|_{\mathscr{C}} = \max_{(t,x)\in[0,\omega]\times[0,\pi]} |u(t, x)|) \,.$$

Denote by m the norm of the imbedding operator from $H^{1,2}_\omega(Q)$ into $\mathscr{C}_\omega(\bar{Q})$.

2.2.6.

$$H^{1,2}_\omega(Q) \subset\subset H^0_\omega(Q) \,,$$
$$H^1_\omega(Q) \subset\subset H^0_\omega(Q)$$

(see e.g. [15]).

2.3. **Definition of the weak solution**

By a weak solution of the problem (1) we mean a function $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$ which satisfies the equation (1_1) almost everywhere on Q.

2.4. **The linear heat equation**

The following theorems are included e.g. in Chapter III of [17].

2.4.1. **Theorem.** *Let $\lambda \neq n^2$, $n \in \mathbf{N}$. Suppose that $f \in H^0_\omega(Q)$.*

Then the problem

(3)
$$u_t(t, x) - u_{xx}(t, x) - \lambda\, u(t, x) = f(t, x) \,, \quad (t, x) \in Q \,,$$
$$u(t, 0) = u(t, \pi) = 0 \,, \quad t \in \mathbf{R}^1 \,,$$
$$u(t + \omega, x) = u(t, x) \,, \quad (t, x) \in Q$$

has a unique weak solution $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$.

2.4.2. **Theorem.** *Let* $\lambda = n^2$ *for some* $n \in \mathbf{N}$. *Suppose that* $f \in H^0_\omega(Q)$. *Then the problem* (3) *has a weak solution* $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$ *if and only if*

$$\int_0^\omega \int_0^\pi f(t, x) \sin nx \, dx \, dt = 0 .$$

2.5. The operator L

2.5.1. For the sake of brevity, denote

$$X = H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q) ,$$
$$Z = H^0_\omega(Q)$$

and suppose that the space X is equipped with the norm $\|u\|_{1,2}$ (X with $\|u\|_{1,2}$ is a Hilbert space). For $\lambda \in \mathbf{R}^1$ introduce the operator

(4) $$L(\lambda) : u \mapsto u_t - u_{xx} - \lambda u .$$

Evidently, $L(\lambda)$ is a bounded linear mapping from X into Z.

Theorem 2.4.1 and 2.4.2 it immediately imply

2.5.2. If $\lambda \neq n^2$, $n \in \mathbf{N}$, then $L(\lambda) : X \to Z$ is a one-to-one mapping.

2.5.3. Im $[L(n^2)] = \{f \in Z; \int_0^\omega \int_0^\pi f(t, x) \sin nx \, dx \, dt = 0\}$.
It is easy to see that

2.5.4. Ker $[L(n^2)] = \{\alpha \sin nx; \alpha \in \mathbf{R}^1\}$.

2.6. The operator S

For fixed $f \in Z$ put

(5) $$S : u \mapsto f - \psi \circ u ,$$

where $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ is a given continuous and bounded function on $\mathbf{R}^1$. Obviously, $S : X \to Z$ and with respect to 2.2.6 and the main theorem on the Němyckij operators (see e.g. [11]) the mapping $S : X \to Z$ is completely continuous.

3. THE REGULAR CASE

3.1. **Theorem.** *Let* $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ *be a bounded and continuous function and let* $\lambda \neq n^2$, $n \in \mathbf{N}$. *Suppose that* $f \in H^0_\omega(Q)$.

Then the problem

$$(6)\qquad \begin{aligned}&u_t(t,x) - u_{xx}(t,x) - \lambda\, u(t,x) + \psi(u(t,x)) = f(t,x)\,, \quad (t,x) \in Q\,,\\ &u(t,0) = u(t,\pi) = 0\,, \quad t \in \mathbf{R}^1\,,\\ &u(t+\omega,x) = u(t,x)\,, \quad (t,x) \in Q\end{aligned}$$

has at least one weak solution $u \in H^{1,2}_{\omega}(Q) \cap {}^{\circ}H^{1}_{\omega}(Q)$.

Proof. With respect to 2.5.2 and the Banach open mapping theorem the operator $L(\lambda)$ defined by the relation (4) has a continuous inverse $L^{-1}(\lambda) : Z \to X$. As the function $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ is bounded there exists a constant $c > 0$ such that $\|Su\|_0 \leqq c$ for any $u \in X$ (the operator S is defined by (5)). The weak solvability of (6) is equivalent to the existence of a solution $u \in X$ of the equation

$$(7)\qquad u = L^{-1}(\lambda)\, Su\,.$$

In order to prove that the equation (7) has at least one solution we use the Schauder fixed point theorem (see e.g. [10]). Put

$$F : u \mapsto L^{-1}(\lambda)\, Su\,, \quad u \in X\,.$$

Then $F : X \to X$ is completely continuous (since $L^{-1}(\lambda)$ is continuous and S is completely continuous) and there exists $\tilde{c} > 0$ such that for arbitrary $u \in X$ we have

$$\|Fu\|_{1,2} \leqq \|L^{-1}(\lambda)\|\, \|Su\|_0 \leqq \|L^{-1}(\lambda)\|\, c = \tilde{c}\,.$$

Thus the operator F maps the closed ball $\{u \in X;\ \|u\|_{1,2} \leqq \tilde{c}\}$ into itself and the Schauder fixed point theorem together with the above argument implies the assertion.

4. THE SINGULAR CASE

4.1. Results of the Landesman-Lazer type

4.1.1. **Theorem.** *Let* $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ *be a bounded and continuous function. Let* $n \in \mathbf{N}$. *Then the problem*

$$(8)\qquad \begin{aligned}&u_t(t,x) - u_{xx}(t,x) - n^2\, u(t,x) + \psi(u(t,x)) = f(t,x)\,, \quad (t,x) \in Q\,,\\ &u(t,0) = u(t,\pi) = 0\,, \quad t \in \mathbf{R}^1\,,\\ &u(t+\omega,x) = u(t,x)\,, \quad (t,x) \in Q\end{aligned}$$

has for $f \in H^0_{\omega}(Q)$ *at least one weak solution* $u \in H^{1,2}_{\omega}(Q) \cap {}^{\circ}H^1_{\omega}(Q)$ *provided*

$$(9)\qquad \limsup_{\xi \to \infty} \psi(\xi) \int_0^{\pi} (\sin nx)^+\, dx - \liminf_{\xi \to -\infty} \psi(\xi) \int_0^{\pi} (\sin nx)^-\, dx <$$

$$< \frac{1}{\omega} \int_0^\omega \int_0^\pi f(t, x) \sin nx \, dx \, dt < \liminf_{\xi \to -\infty} \psi(\xi) \int_0^\pi (\sin nx)^+ \, dx -$$

$$- \limsup_{\xi \to \infty} \psi(\xi) \int_0^\pi (\sin nx)^- \, dx .$$

(For example $\int_0^\pi (\sin nx)^- \, dx = 2/n \, . \, [n/2]$ for $n > 1$.)

Proof. (i) Let P be a mapping defined on the space Z by

$$(Pu)(x) = \frac{2}{\pi\omega} \left(\int_0^\omega \int_0^\pi u(t, y) \sin ny \, dy \, dt \right) \sin nx .$$

Obviously, P is the orthogonal projection from Z onto $\operatorname{Ker}[L(n^2)]$. Denote $P^c : u \mapsto u - Pu$, $u \in Z$. The mapping P^c is the orthogonal projection from Z onto the space $\operatorname{Im}[L(n^2)]$.

(ii) Let $K(n^2) : \operatorname{Im}[L(n^2)] \to P^c(X)$ be the right inverse of the operator $L(n^2)$, i.e.

$$K(n^2) z = u \quad \text{if and only if} \quad L(n^2) u = z, z \in \operatorname{Im}[L(n^2)], \; u \in P^c(X).$$

The mapping $K(n^2) : \operatorname{Im}[L(n^2)] \to X$ is continuous.

(iii) We shall investigate the solvability of the operator equation

$$L(n^2) u = Su . \tag{10}$$

The equation (10) can be rewritten to the equivalent system

$$\begin{aligned} P \, L(n^2) u &= PSu , \\ P^c \, L(n^2) u &= P^c Su . \end{aligned}$$

As $u = Pu + P^c u$ we denote $w = Pu$, $v = P^c u$ and thus we have a new system

$$\begin{aligned} P \, L(n^2) w + P \, L(n^2) v &= PS(w + v) , \\ P^c \, L(n^2) w + P^c \, L(n^2) v &= P^c S(w + v) . \end{aligned} \tag{11}$$

For $w \in \operatorname{Ker}[L(n^2)]$ we have $L(n^2) w = \theta$ and for $v \in X$ it is $L(n^2) v \in \operatorname{Im}[L(n^2)]$. Thus $P \, L(n^2) v = \theta$ and $P^c \, L(n^2) v = L(n^2) v$ and instead of (11) it is possible to write

$$\begin{aligned} PS(w + v) &= \theta , \\ v &= K(n^2) P^c S(w + v) . \end{aligned} \tag{12}$$

To be able to use the fixed point theorem, we rewrite (12) to a more convenient form. It is easy to see that (12) is solvable if and only if for some $\varepsilon > 0$ the system

$$(13) \qquad w - \varepsilon PS(w + K(n^2)\, P^c S(w + v)) = w\,,$$
$$v = K(n^2)\, P^c S(w + v)$$

is solvable.

Thus the operator equation (10) has a solution if and only if the system (13) has a solution. The system (13) is now prepared for using the Schauder fixed point theorem.

(iv) In $Y = \operatorname{Ker}[L(n^2)] \times P^c(X)$ we introduce the norm

$$(w, v) \mapsto \|v\|_0 + \|v\|_{1,2}\,.$$

For $\varepsilon > 0$, define an operator V_ε on the space Y by

$$V_\varepsilon : (w, v) \mapsto (w - \varepsilon PS(w + K(n^2)\, P^c S(w + v)),\ K(n^2)\, P^c S(w + v))\,.$$

It is easy to see that V_ε maps the space Y into Y.

To prove that the equation (10) has at least one solution it is sufficient to prove that for some $\varepsilon > 0$ the operator V_ε satisfies the assumptions of the Schauder fixed point theorem. As the operator $V_\varepsilon : Y \to Y$ is completely continuous, it is sufficient to show that there exists $\varepsilon_0 > 0$ and a nonempty convex closed bounded set $\mathscr{K} \subset Y$ such that $V_{\varepsilon_0}(\mathscr{K}) \subset \mathscr{K}$. Denote

$$\sup_{(w,v)\in Y} \|S(w + v)\|_0 = c < \infty\,,$$
$$\sup_{(w,v)\in Y} \|K(n^2)\, P^c S(w + v)\|_{1,2} = c_1 < \infty\,,$$
$$\sup_{(w,v)\in Y} \|PS(w + v)\|_0 = c_2 < \infty\,.$$

Using the Fatou lemma and the assumption (9) we have

$$\liminf_{\xi\to\infty} \langle S(\xi \sin nx + v(t, x)), \sin nx\rangle_0 = \int_0^\omega \int_0^\pi f(t, x) \sin nx \,dx\,dt -$$
$$- \limsup_{\xi\to\infty} \int_0^\omega \int_0^\pi \psi(\xi \sin nx + v(t, x)) \sin nx \,dt \geqq$$
$$\geqq \int_0^\omega \int_0^\pi f(t, x) \sin nx \,dx\,dt - \limsup_{\xi\to\infty} \int_0^\omega \int_0^\pi \psi(\xi \sin nx + v(t, x))\,.$$
$$. (\sin nx)^+ \,dx\,dt + \limsup_{\xi\to\infty} \int_0^\omega \int_0^\pi \psi(\xi \sin nx + v(t, x))\,(\sin nx)^- \,dx\,dt \geqq$$
$$\geqq \int_0^\omega \int_0^\pi f(t, x) \sin nx \,dx\,dt - \limsup_{\xi\to\infty} \psi(\xi) \int_0^\omega \int_0^\pi (\sin nx)^+ \,dx\,dt +$$
$$+ \liminf_{\xi\to-\infty} \psi(\xi) \int_0^\omega \int_0^\pi (\sin nx)^- \,dx\,dt > 0$$

and

$$\limsup_{\xi \to -\infty} \langle S(\xi \sin nx + v(t, x)), \sin nx \rangle_0 < 0$$

uniformly for $v \in P^c(X)$, $\|v\|_{1,2} \leqq c_1$.

We can find a sufficiently large $t_0 > 0$ and then a sufficiently small $\delta > 0$, such that if $v \in P^c(X)$, $\|v\|_{1,2} \leqq c_1$ and $w \in \text{Ker}\,[L(n^2)]$, $\|w\|_0 \geqq t_0/2$ then $\langle S(w + v), w \rangle_0 \geqq \delta t_0/2$. This implies that for an arbitrary $w \in \text{Ker}\,[L(n^2)]$ with $t_0/2 \leqq \|w\|_0 \leqq t_0$ and for $\varepsilon > 0$ such that $\varepsilon \leqq \delta t_0/c_2^2$ it is

$$\|w - \varepsilon PS(w + K(n^2)\, P^c S(w + v))\|_0^2 = \|w\|_0^2 - 2\varepsilon \langle S(w + K(n^2)\, P^c S(w + v)), w \rangle_0 + \\ + \varepsilon^2 \|PS(w + K(n^2)\, P^c S(w + v))\|_0^2 \leqq t_0^2 - \varepsilon \delta t_0 + \varepsilon^2 c_2^2 \leqq t_0^2 .$$

For $v \in P^c(X)$ and $w \in \text{Ker}\,[L(n^2)]$ with $\|w\|_0 \leqq t_0/2$ and for an arbitrary $\varepsilon > 0$ such that $\varepsilon \leqq t_0/2c_2$ it is

$$\|w - \varepsilon PS(w + K(n^2)\, P^c S(w + v))\|_0 \leqq \|w\|_0 + \\ + \varepsilon \|PS(w + K(n^2)\, P^c S(w + v))\|_0 \leqq t_0/2 + \varepsilon c_2 \leqq t_0 .$$

Put

$$\varepsilon_0 = \min \{\delta t_0/c_2^2,\, t_0/2c_2\}$$

and

$$\mathscr{K} = \{(w, v) \in Y;\ \|w\|_0 \leqq t_0,\ \|v\|_{1,2} \leqq c_1\} .$$

Then the above calculation yields $V_{\varepsilon_0}(\mathscr{K}) \subset \mathscr{K}$. (Obviously $\mathscr{K}$ is a nonempty convex bounded and closed subset of Y.)

4.1.2. **Theorem.** *Let $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ be a bounded continuous function. Let $n \in \mathbf{N}$. Then the problem (8) has for $f \in H_\omega^0(Q)$ at least one weak solution $u \in H_\omega^{1,2}(Q) \cap {}^\circ H_\omega^1(Q)$ provided*

(14)
$$\limsup_{\xi \to -\infty} \psi(\xi) \int_0^\pi (\sin nx)^+ \,\mathrm{d}x - \liminf_{\xi \to \infty} \psi(\xi) \int_0^\pi (\sin nx)^- \,\mathrm{d}x < \\ < \frac{1}{\omega} \int_0^\omega \int_0^\pi f(t, x) \sin nx \,\mathrm{d}x\,\mathrm{d}t < \\ < \limsup_{\xi \to \infty} \psi(\xi) \int_0^\pi (\sin nx)^+ \,\mathrm{d}x - \liminf_{\xi \to -\infty} \psi(\xi) \int_0^\pi (\sin nx)^- \,\mathrm{d}x .$$

Proof. The proof follows from Theorem 4.1.1 by considering the equation

$$-L(n^2)\, u = -Su$$

instead of (10).

4.1.3. **Corollary** (see [1]). *Let $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ be a bounded and continuous function such that*

(15) $$\lim_{\xi \to \infty} \psi(\xi) = \psi(\infty) \in \mathbf{R}^1 ,$$

(16) $$\lim_{\xi \to -\infty} \psi(\xi) = \psi(-\infty) \in \mathbf{R}^1 .$$

Then the problem (8) *has at least one weak solution $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$ for $f \in H^0_\omega(Q)$ provided*

(17) $$\psi(\infty) \int_0^\pi (\sin nx)^+ \,\mathrm{d}x - \psi(-\infty) \int_0^\pi (\sin nx)^- \,\mathrm{d}x <$$
$$< \frac{1}{\omega} \int_0^\omega \int_0^\pi f(t, x) \sin nx \,\mathrm{d}x \,\mathrm{d}t < \psi(-\infty) \int_0^\pi (\sin nx)^+ \,\mathrm{d}x - \psi(\infty) \int_0^\pi (\sin nx)^- \,\mathrm{d}x$$

or

(18) $$\psi(-\infty) \int_0^\pi (\sin nx)^+ \,\mathrm{d}x - \psi(\infty) \int_0^\pi (\sin nx)^- \,\mathrm{d}x <$$
$$< \frac{1}{\omega} \int_0^\omega \int_0^\pi f(t, x) \sin nx \,\mathrm{d}x \,\mathrm{d}t < \psi(\infty) \int_0^\pi (\sin nx)^+ \,\mathrm{d}x - \psi(-\infty) \int_0^\pi (\sin nx)^- \,\mathrm{d}x .$$

4.1.4. **Theorem.** *Let $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ be a bounded continuous function satisfying* (15), (16) *and*

(19) $$\psi(\infty) < \psi(\xi) < \psi(-\infty) , \quad \xi \in \mathbf{R}^1 .$$

Let $f \in H^0_\omega(Q)$. Then the conditions (17) *are necessary and sufficient for the weak solvability of the problem* (8).

If we suppose

(20) $$\psi(-\infty) < \psi(\xi) < \psi(\infty) , \quad \xi \in \mathbf{R}^1$$

instead of (19) then the conditions (18) are necessary and sufficient for the existence of a weak solution of (8) with $f \in H^0_\omega(Q)$.

Proof. The sufficiency follows from Corollary 4.1.3. If we suppose that for $f \in H^0_\omega(Q)$ the problem (8) has a weak solution $u_0 \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$ then $Su_0 \in \operatorname{Im} [L(n^2)]$. This means (according to 2.5.3) that $\langle Su_0, \sin nx \rangle_0 = 0$, i.e.

$$\int_0^\omega \int_0^\pi f(t, x) \sin nx \,\mathrm{d}x \,\mathrm{d}t = \int_0^\omega \int_0^\pi \psi(u_0(t, x)) \sin nx \,\mathrm{d}x \,\mathrm{d}t .$$

The last relation together with the assumption (19) imply

$$\int_0^\omega \int_0^\pi \psi(u_0(t, x)) \sin nx \,\mathrm{d}x \,\mathrm{d}t < \psi(-\infty)\, \omega \int_0^\pi (\sin nx)^+ \,\mathrm{d}x \,\mathrm{d}t -$$

$$- \psi(\infty)\, \omega \int_0^\pi (\sin nx)^- \, dx \,,$$

$$\int_0^\omega \int_0^\pi \psi(u_0(t, x)) \sin nx \, dx \, dt > \psi(\infty)\, \omega \int_0^\pi (\sin nx)^+ \, dx - \psi(-\infty)\, \omega \int_0^\pi (\sin nx)^- \, dx$$

and hence also the desired assertion.

4.2. **Vanishing nonlinearities**

The set of the right-hand sides $f \in H^0_\omega(Q)$ satisfying one from the relations (17), (18) may be empty e.g. in the case of

$$\psi(-\infty) = \psi(\infty) = 0 \tag{21}$$

and thus the results from 4.1 have no sense in this case.

The idea how to prove the existence of a weak solution of (8) with ψ satisfying (21) is based on the so-called method of truncated equations. We change the function ψ outside a sufficiently large interval $(-a, a)$ so that we obtain a function $\tilde{\psi}$ for which one of the conditions (17), (18) has sense. According to the assertions of Theorems 4.1.1 and 4.1.2 the problem

$$\begin{aligned} &u_t(t, x) - u_{xx}(t, x) - n^2\, u(t, x) + \tilde{\psi}(u(t, x)) = f(t, x)\,, \quad (t, x) \in Q \\ &u(t, 0) = u(t, \pi) = 0\,, \quad t \in \mathbf{R}^1 \\ &u(t + \omega, x) = u(t, x)\,, \quad (t, x) \in Q \end{aligned} \tag{22}$$

will be weakly solvable. We shall prove that an arbitrary weak solution $u \in H^{1,2}_\omega(Q) \cap$ $\cap\, {}^\circ H^1_\omega(Q)$ of (22) satisfies

$$\|u\|_{\mathscr{C}} < a\,. \tag{23}$$

Thus any weak solution of (22) is also a weak solution of (8). The main part of the proof will be to establish the a priori estimate (23).

4.2.1. The first a priori estimate. Let $\tilde{\psi} : \mathbf{R}^1 \to \mathbf{R}^1$ be a bounded continuous function on $\mathbf{R}^1$ and let $f \in H^0_\omega(Q)$. Then an arbitrary weak solution $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$ of (22) satisfies

$$\left| u(t, x) - \frac{2}{\pi\omega} \left(\int_0^\omega \int_0^\pi u(t, y) \sin ny \, dy \, dt \right) \sin nx \right| \leqq \tag{24}$$

$$\leqq m \|K(n^2)\| \left(\|f\|_0 + \pi^{1/2} \omega^{1/2} \sup_{\xi \in \mathbf{R}^1} |\tilde{\psi}(\xi)| \right) = c_1(\tilde{\psi}, f, n) = c_1\,.$$

Proof. Let the notation introduced in the proof of Theorem 4.1.1 be observed. If $u \in H^{1,2}_{\omega}(Q) \cap {}^{\circ}H^1_{\omega}(Q)$ is a solution of the equation $L(n^2)\, u = Su$ then $P^c u = K(n^2)\, P^c Su$ and thus

$$\|P^c u\|_{1,2} \leqq \|K(n^2)\|\ \|Su\|_0 \leqq \|K(n^2)\| \cdot \left(\|f\|_0 + \pi^{1/2}\omega^{1/2} \sup_{\xi \in \mathbf{R}^1} |\tilde{\psi}(\xi)|\right).$$

The last inequalities together with 2.2.5 yield (24).

4.2.2. The second a priori estimate. Let $\tilde{\psi} : \mathbf{R}^1 \to \mathbf{R}^1$ be a continuous and bounded function, $a > \xi_0 \geqq 0$, $f \in H^0_{\omega}(Q)$. Suppose

$$\inf_{\xi \geqq a} \tilde{\psi}(\xi) > 0\,,$$

$$\tilde{\psi}(\xi) = -\tilde{\psi}(-\xi)\,, \quad |\xi| \geqq \xi_0\,,$$

and denote

$$\Gamma = \Gamma(a, \tilde{\psi}) = 2\omega \inf_{\xi \geqq a} \tilde{\psi}(\xi)\,.$$

Let

$$\left| \int_0^{\omega} \int_0^{\pi} f(t, x) \sin nx \,\mathrm{d}x \,\mathrm{d}t \right| < \Gamma\,.$$

Then an arbitrary weak solution $u \in H^{1,2}_{\omega}(Q) \cap {}^{\circ}H^1_{\omega}(Q)$ of (22) satisfies

$$\left| \frac{2}{\pi\omega} \int_0^{\omega} \int_0^{\pi} u(t, x) \sin nx \,\mathrm{d}x \,\mathrm{d}t \right| \leqq c_2(a, \tilde{\psi}, f, n) = c_2\,.$$

where

$$c_2 = (a + c_1)\, c_3\,,$$

$$c_3 = \left(\Gamma + \pi\omega \sup_{\xi \in \mathbf{R}^1} |\tilde{\psi}(\xi)|\right)^{1/2} \left(\Gamma - \left| \int_0^{\omega} \int_0^{\pi} f(t, x) \sin nx \,\mathrm{d}x \,\mathrm{d}t \right|\right)^{-1/2}.$$

Proof. Suppose that there exists a weak solution $u \in H^{1,2}_{\omega}(Q) \cap {}^{\circ}H^1_{\omega}(Q)$ of (22) such that

$$\zeta = \frac{2}{\pi\omega} \int_0^{\omega} \int_0^{\pi} u(t, x) \sin nx \,\mathrm{d}x \,\mathrm{d}t > (a + c_1)\, c_3\,.$$

Note that $c_3 \geqq 1$. Choose $\alpha > 0$ sufficiently small and such that $\zeta > (a + c_1) \cdot (c_3^{-1} - \alpha)^{-1}$. Put

$$\varepsilon = \frac{1}{n} \arcsin\left(c_3^{-1} - \alpha\right).$$

Then

$$\left| \int_0^{\omega} \int_0^{\pi} f(t, x) \sin nx \,\mathrm{d}x \,\mathrm{d}t \right| < \Gamma \cos^2 n\varepsilon - \pi\omega \sup_{\xi \in \mathbf{R}^1} |\tilde{\psi}(\xi)| \sin^2 n\varepsilon \leqq$$

$$\leqq \Gamma \cos n\varepsilon - 2n\varepsilon\omega \sup_{\xi \in \mathbf{R}^1} |\tilde{\psi}(\xi)| \sin n\varepsilon =$$

$$= \inf_{\xi \geq a} \tilde{\psi}(\xi)\, \omega \sum_{k=0}^{n-1} \int_{(\pi/n)k+\varepsilon}^{(\pi/n)(k+1)-\varepsilon} |\sin ny|\, \mathrm{d}y - 2n\varepsilon\omega \sup_{\xi \in \mathbf{R}^1} |\tilde{\psi}(\xi)| \sin n\varepsilon \leq$$

$$\leq \int_0^\omega \int_0^\pi \psi(\zeta \sin nx + P^c u(t, x)) \sin nx \, \mathrm{d}x \, \mathrm{d}t =$$

$$= \int_0^\omega \int_0^\pi \psi(u(t, x)) \sin nx \, \mathrm{d}x \, \mathrm{d}t = \int_0^\omega \int_0^\pi f(t, x) \sin nx \, \mathrm{d}x \, \mathrm{d}t$$

which is a contradiction.

Analogously it is possible to prove that $\zeta > -(a + c_1)\, c_3$.

4.2.3. **Theorem.** *Let $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ be a bounded continuous and odd function, $0 < a < b$, $b - a > 2c_1$, $f \in H^0_\omega(Q)$.*

Then the problem (8) has at least one weak solution $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$ provided

(25)
$$\inf_{\xi \in [a,b]} \psi(\xi) > \frac{1}{2\omega} \left(\pi\omega \sup_{\xi \in \mathbf{R}^1} |\psi(\xi)| + \right.$$
$$\left. + \left(\frac{b - c_1}{a + c_1}\right)^2 \left| \int_0^\omega \int_0^\pi f(t, x) \sin nx \, \mathrm{d}x \, \mathrm{d}t \right| \right) \left(\left(\frac{b - c_1}{a + c_1}\right)^2 - 1 \right)^{-1}.$$

Proof. Choose $a < b_1 < b$ such that the function

$$\tilde{\psi} : \xi \mapsto \begin{cases} \psi(\xi), & |\xi| \leq b_1 \\ \psi(b_1), & \xi > b_1 \\ \psi(-b_1), & \xi < -b_1 \end{cases}$$

satisfies the following conditions:

(26)
$$\left| \int_0^\omega \int_0^\pi f(t, x) \sin nx \, \mathrm{d}x \, \mathrm{d}t \right| < 2\omega \inf_{\xi \geq a} \tilde{\psi}(\xi) \leq 2\omega\, \psi(b_1),$$
$$b_1 > c_1(\tilde{\psi}, f, n) + c_2(a, \tilde{\psi}, f, n).$$

Since $\tilde{\psi}(\infty) = \tilde{\psi}(b_1)$ and $\tilde{\psi}$ is odd, then according to the inequalities (26) and with respect to Corollary 4.1.3 there exists at least one weak solution $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$ of the problem (22). From 4.2.1 and 4.2.2 we conclude

$$|u(t, x)| \leq b_1, \quad (t, x) \in Q.$$

Thus the function u is also a weak solution of the problem (8).

In the following theorem and in Section 4.3 we shall give sufficient conditions for the validity of the undecipherable assumption (25).

4.2.4. **Theorem.** *Let* $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ *be a continuous bounded and odd function. Suppose* $a > 0$ *and*

$$\lim_{\xi \to \infty} \xi^2 \min_{\tau \in [a,\xi]} \psi(\tau) = \infty . \tag{27}$$

Then the problem (8) *has at least one weak solution for any* $f \in H^0_\omega(Q)$ *with*

$$\int_0^\omega \int_0^\pi f(t, x) \sin nx \, dx \, dt = 0 .$$

Proof. If we multiply (25) by b^2 we obtain that the limit of the left hand side is infinite and the limit superior of the right hand side is finite. Thus for sufficiently large b the inequality (25) holds.

4.3. Expansive nonlinearities

4.3.1. **Definition.** *A bounded odd continuous and nontrivial function* $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ *is said to be expansive if for each p,*

$$0 \leqq p < \sup_{\xi \in \mathbf{R}^1} \psi(\xi) ,$$

there exist sequences $0 < a_k < b_k$, $\lim_{k \to \infty} b_k a_k^{-1} = \infty$, *such that*

$$\lim_{k \to \infty} \min_{\xi \in [a_k, b_k]} \psi(\xi) > p .$$

The following theorem extends the assertion of Theorem 4.1.4 to the case of expansive functions which can have no limits $\psi(\infty)$, $\psi(-\infty)$.

4.3.2. **Theorem.** *Let* $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ *be an expansive function. Then*

$$f \in H^0_\omega(Q) , \quad \left| \int_0^\omega \int_0^\pi f(t, x) \sin nx \, dx \, dt \right| < 2\omega \sup_{\xi \in \mathbf{R}^1} \psi(\xi)$$

is a necessary and sufficient condition for the weak solvability of the problem (8).

Proof. The limit $ba^{-1} \to \infty$ of the right hand side in (25) is

$$\frac{1}{2\omega} \left| \int_0^\omega \int_0^\pi f(t, x) \sin nx \, dx \, dt \right| .$$

Thus we can choose k sufficiently large so that the assumption (25) holds with $[a, b] = [a_k, b_k]$. The sufficiency follows from Theorem 4.2.3. The necessity is obtained analogously as in the proof of Theorem 4.1.4.

5.1. **Theorem.** *Let $\mu, \nu \in \mathbf{R}^1$ and suppose that*

$$g : \xi \mapsto \mu\xi^+ - \nu\xi^- - \psi(\xi), \quad \xi \in \mathbf{R}^1 \tag{28}$$

where $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ is a bounded and continuous function. Put

$$\mathfrak{M} = \{(\mu, \nu) \in \mathbf{R}^2;\ \mu < 1,\ \nu < 1\} \cup \bigcup_{k=0}^{\infty} \{(\mu, \nu) \in \mathbf{R}^2;\ \mu^{1/2} > k + 1,\ \omega_k(\mu^{1/2}) <$$

$$< \nu^{1/2} < \zeta_{k+1}(\mu^{1/2})\} \cup \bigcup_{k=1}^{\infty} \{(\mu, \nu) \in \mathbf{R}^2;\ \mu^{1/2} > k,\ \zeta_k(\mu^{1/2}) < \nu^{1/2} < \vartheta_k(\mu^{1/2})\},$$

where

$$\vartheta_k(\tau) = \begin{cases} \dfrac{(k+1)\tau}{\tau - k}, & \tau \in (k, 2k+1\rangle \\[2ex] \dfrac{k\tau}{\tau - (k+1)}, & \tau \in (2k+1, \infty) \end{cases},$$

$$\omega_k(\tau) = \begin{cases} \dfrac{k\tau}{\tau - (k+1)}, & \tau \in (k+1,\ 2k+1\rangle \\[2ex] \dfrac{(k+1)\tau}{\tau - k}, & \tau \in (2k+1, \infty) \end{cases},$$

$$\zeta k(\tau) = \frac{k\tau}{\tau - k}, \quad \tau \in (k, \infty).$$

(i) *If $(\mu, \nu) \in \mathfrak{M}$ and $f \in H^0_\omega(Q)$ then the problem* (1) *with g given by* (28) *has at least one weak solution $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$.*

(ii) *If $(\mu, \nu) \in \mathbf{R}^2 - \overline{\mathfrak{M}}$ then there exists $f \in H^0_\omega(Q)$ such that the problem* (1) *with g given by* (28) *has no weak solution in $H(^{1,2}_\omega Q) \cap {}^\circ H^1_\omega(Q)$.*

5.2. Proof of Theorem 5.1

First of all we shall prove one simple and very useful lemma.

(i) **Lemma.** *Let the function $f \in H^0_\omega(Q)$ be independent of t. Then any weak solution $u \in H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$ of the problem*

$$\begin{aligned} &u_t(t, x) - u_{xx}(t, x) - \mu\, u^+(t, x) + \nu\, u^-(t, x) = f(x), \quad (t, x) \in Q \\ &u(t, 0) = u(t, \pi) = 0, \quad t \in \mathbf{R}^1 \\ &u(t + \omega, x) = u(t, x), \quad (t, x) \in Q \end{aligned} \tag{29}$$

is independent of t.

Proof. Multiplying (29_1) by $u_t(t, x)$ and integrating over $(0, \omega) \times (0, \pi)$ we obtain

$$\int_0^\omega \int_0^\pi u_t^2(t, x)\, dx\, dt = 0$$

and thus

$$u_t(t, x) = 0 \quad \text{for almost all} \quad (t, x) \in Q .$$

So $u(t, x) = u(0, x)$ for all $(t, x) \in Q$.

(ii) According to the previous lemma the problem

$$\begin{aligned} (30) \qquad & u_t(t, x) - u_{xx}(t, x) - \mu\, u^+(t, x) + \nu\, u^-(t, x) = 0 , \quad (t, x) \in Q \\ & u(t, 0) = u(t, \pi) = 0 , \quad t \in \mathbf{R}^1 \\ & u(t + \omega, x) = u(t, x) , \quad (t, x) \in Q \end{aligned}$$

has a nontrivial weak solution if and only if the Dirichlet problem

$$\begin{aligned} (31) \qquad & -u''(x) - \mu\, u^+(x) + \nu\, u^-(x) = 0 , \quad x \in (0, \pi) \\ & u(0) = u(\pi) = 0 \end{aligned}$$

has a nontrivial solution. The problem (31) has a nontrivial solution if and only if $(\mu, \nu) \in \partial\mathfrak{M}$ (see e.g. [5]).

(iii) Denote $J = L(0)$, $T_{(\mu,\nu)} : u \mapsto \mu u^+ - \nu u^-$. Obviously $T_{(\mu,\nu)} : X \to Z$ is completely continuous. To solve

$$(32) \qquad Ju - T_{(\mu,\nu)}(u) + Uu = f$$

where $U : u(t, x) \mapsto \psi(u(t, x))$ is nothing else than to seek the weak solution of (1) with g given by (28). Recall that $J : X \to Z$ is an isomorphism.

(iv) If $(\mu, \nu) \notin \partial\mathfrak{M}$ then the equation $Ju - T_{(\mu,\nu)}(u) = \theta$ has only the trivial solution. Thus for $(\mu, \nu) \notin \partial\mathfrak{M}$ there exists $c_1(\mu, \nu) > 0$ such that

$$(33) \qquad \|Ju - T_{(\mu,\nu)}(u)\|_0 \geqq c_1(\mu, \nu)\, \|u\|_{1,2} , \quad u \in X ,$$

(this follows from the complete continuity and the homogeneity of $T_{(\mu,\nu)}$) and

$$\|z - T_{(\mu,\nu)}(J^{-1}z)\|_0 \geqq c_1(\mu, \nu) \| J^{-1}z\|_{1,2} , \quad z \in Z .$$

(v) To prove the assertion of Theorem 5.1(i) we use the Leray-Schauder degree theory (see e.g. [10]). It is sufficient to show that the mapping

$$z \mapsto z - T_{(\mu,\nu)}(J^{-1}z) + U(J^{-1}z) , \quad z \in Z$$

is onto Z.

Let $\zeta \in Z$ be arbitrary but fixed. If $(\mu, \nu) \in \mathfrak{M}$ then denote by C the component of $\mathfrak{M}$ such that $(\mu, \nu) \in C$. There exists $(\lambda, \lambda) \in C$. Thus

$$(34) \qquad \mathrm{d}[z - T_{(\lambda,\lambda)}(J^{-1}z);\ K_Z(R), \theta] = \mathrm{d}[z - T_{(\mu,\nu)}(J^{-1}z);\ K_Z(R), \theta]$$

for any $R > 0$, where $K_Z(R)$ is the open ball in Z centered at origin and with the radius R and $\mathrm{d}[\cdot\,; K_Z(R), \theta]$ is the Leray-Schauder degree of the given mapping with respect to the set $K_Z(R)$ and the point θ. The mapping $z \mapsto T_{(\lambda,\lambda)}(J^{-1}z)$ is linear and thus

$$(35) \qquad \mathrm{d}[z - T_{(\lambda,\lambda)}(J^{-1}z);\ K_Z(R), \theta] = \pm 1\,.$$

As

$$\|z - T_{(\mu,\nu)}(J^{-1}z) + \tau(UJ^{-1}z - \zeta)\|_0 \geqq c_1(\mu, \nu)\,\|J^{-1}z\|_{1,2} - \sup_{u\in X} \|Uu\|_0 - \|\zeta\|_0$$

we obtain that there exists $R > 0$ sufficiently large such that for an arbitrary $z \in Z$, $\|z\|_0 = R$ and any $\tau \in [0, 1]$ it is

$$z - T_{(\mu,\nu)}(J^{-1}z) + \tau(UJ^{-1}z - \zeta) \neq \theta\,.$$

From the homotopy property of the Leray-Schauder degree we have

$$(36) \qquad \begin{aligned} &\mathrm{d}[z - T_{(\mu,\nu)}(J^{-1}z) + UJ^{-1}z - \zeta;\ K_Z(R), \theta] = \\ &= \mathrm{d}[z - T_{(\mu,\nu)}(J^{-1}z);\ K_Z(R), \theta]\,. \end{aligned}$$

From (34)–(36) we have

$$\mathrm{d}[z - T_{(\mu,\nu)}(J^{-1}z) + UJ^{-1}z - \zeta;\ K_Z(R), \theta] = \pm 1$$

and thus (with respect to the main property of the Leray-Schauder degree) we have that the equation

$$z - T_{(\mu,\nu)}(J^{-1}z) + UJ^{-1}z = \zeta$$

is solvable in Z. Part (i) of Theorem 5.1 is proved.

(vi) If $(\mu, \nu) \in \mathbf{R}^2 - \overline{\mathfrak{M}}$ then there exists $f \in \mathscr{C}^\infty([0, \pi])$ such that the Dirichlet problem

$$\begin{gathered} -u''(x) - \mu\, u^+(x) + \nu\, u^-(x) = f(x)\,, \quad x \in (0, \pi) \\ u(0) = u(\pi) = 0 \end{gathered}$$

has no solution (see [2], [3]). Thus according to Lemma (i) the problem (29) has no weak solution in $H^{1,2}_\omega(Q) \cap {}^\circ H^1_\omega(Q)$. In other words, $\mathrm{Im}\,[J - T_{(\mu,\nu)}] \neq Z$. According to (33) the set $\mathrm{Im}\,[J - T_{(\mu,\nu)}]$ is a closed subset of Z. As the mappings J and $T_{(\mu,\nu)}$ are homogeneous, there exists an open cone $V \subset Z$ such that $Z - \mathrm{Im}\,[J - T_{(\mu,\nu)}] \supset V$. As any point from $\mathrm{Im}\,[J - T_{(\mu,\nu)} + U]$ has a distance from $\mathrm{Im}\,[J - T_{(\mu,\nu)}]$ not bigger than $\sup_{u\in X} \|Uu\|_0$ we have $\mathrm{Im}\,[J - T_{(\mu,\nu)} + U] \neq Z$ and thus part (ii) of Theorem 5.1 is proved.

6. REMARKS

6.1. Using the same method as in Sections 3 and 4 we can extend the results to the periodic solvability of abstract ordinary differential equations in Hilbert spaces and, particularly, to the periodic solvability of boundary value problems for higher dimensional nonlinear partial differential equations of the parabolic type

$$(37) \qquad u_t - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - g(u(t, x)) = f(t, x)\,, \quad (t, x) \in \mathbf{R}^1 \times \Omega$$

$$u(t, x) = 0\,, \quad x \in \partial\Omega\,, \quad t \in \mathbf{R}^1$$

$$u(t + \omega, x) = u(t, x)\,, \quad (t\ x) \in \mathbf{R}^1 \times \Omega\,.$$

Also the nonlinearity g can depend on t and x.

6.2. The function ψ in Section 3 (and also in the case of the generalizations sketched in Section 6.1) can also depend on u_x.

6.3. Instead of the boundedness of the function ψ in Section 3 we can suppose that ψ satisfies the growth condition

$$(38) \qquad |\psi(\xi)| \leqq \alpha + \beta|\xi|^\delta\,, \quad \xi \in \mathbf{R}^1$$

where $\alpha \geqq 0$, $\beta \geqq 0$ and $\delta \in (0, 1)$, and we can prove (rewriting literally the proof of Theorem 3.1) the same assertion as in Theorem 3.1. It is also possible to investigate by the same method the case of $\delta = 1$ in (38) under the assumption that the constant $\beta > 0$ is sufficiently small (the smallness depends on the distance of λ from $\{n^2;\ n \in \mathbf{N}\}$).

6.4. As an analogous assertion to that of Theorem 4.1.1 for boundary value problems for partial differential equations of elliptic type was first proved by E. M. Landesman and A. C. Lazer (see [14]) the results of such kind are usually called "results of the Landesman-Lazer type".

6.5. The method of the proof of Theorem 4.1.1 is essentially a special case of the abstract method of proving solvability of operator equations $Lu = Su$ with a non-invertible linear L and a completely continuous S which is explained in [4]. It is also possible to use Mawhin's coincidence degree theory (see e.g. [12]).

6.6. Also in Section 4 it is possible to suppose (38) (of course with β sufficiently small if $\delta = 1$) and $\psi(\infty) = \infty$, $\psi(-\infty) = -\infty$ and we obtain the solvability of (8) for an arbitrary $f \in H^0_\omega(Q)$.

6.7. The results analogous to those from Sections 4.2 and 4.3 for boundary value problems are proved in [7], [9]. Theorem 4.2.4 can be extended to the case of the problem (37) if f is sufficiently smooth and the condition (27) is replaced by

$$\lim_{\xi \to \infty} \xi \min_{\tau \in [a, \xi]} \psi(\tau) = \infty\,.$$

6.8. The notion of the expansive function is introduced in [7] and further invetigated in [9], [13]. Typical example of expansive function is

$$\psi : \xi \mapsto \begin{cases} \sin (\log \xi)^{1/2}, & \xi \geqq 1 \\ 0, & 0 \leqq \xi < 1 \\ -\psi(-\xi), & \xi \leqq 0 . \end{cases}$$

Also the generalization of Theorem 4.3.2 to the problem (37) if f is sufficiently smooth is possible.

6.9. The boundary value problems for second order ordinary differential equations with jumping nonlinearities were investigated in [5]. A generalization and an almost complete description of various parameters μ, ν is given in [2]. For further generalization see [6], [3].

6.10. It seems that to give such a complete result for the problem (37) as in Theorem 5.1 is impossible since it is very difficult to express the analogue of the set $\mathfrak{M}$. We can describe the situation concerning the solvability of the problem (37) with g given by (28) for such parameters μ, ν which are near to the diagonal $\{(\lambda, \lambda) \in \mathbf{R}^2;\ \lambda \in \mathbf{R}^1\}$.

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Souhrn

SLABÁ PERIODICKÁ ŘEŠENÍ OKRAJOVÉ ÚLOHY PRO NELINEÁRNÍ ROVNICI VEDENÍ TEPLA

VĚNCESLAVA ŠŤASTNOVÁ, SVATOPLUK FUČÍK

V práci jsou dokázány věty o existenci periodických řešení u okrajové úlohy pro rovnici vedení tepla s nelineárním členem $g(u)$, kde g je spojitá reálná funkce a za předpokladu, že existují konečné limity $\lim_{\xi\to\infty} g(\xi)/\xi$, $\lim_{\xi\to-\infty} g(\xi)/\xi$. Důkazy těchto vět jsou založeny na použití Lerayovy-Schauderovy teorie stupně zobrazení.

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