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ON THE NOMOGRAPHIC CHART OF THREE COMPLEX VARIABLES IN THE LINE COORDINATES

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The methods of nomographing the functional relations among three complex variables which satisfy Massau's complex chart determinant: $\det(\mathbb{M}_3^c) = 0$, have been discussed [1], [2]. In this article, the author tries to investigate the methods of nomographing them in the line coordinates.

1. LINE COORDINATES

If we represent a point P by (x_1, x_2, x_3) in the homogeneous coordinates, the straight line through the point P is represented by

$$u_1x_1 + u_2x_2 + u_3x_3 = 0 \tag{1}$$

and $(u_1, u_2, u_3$ are called the homogeneous coordinates of the straight line or the line coordinates. If we put

$$x_1 : x_2 : x_3 = x : y : 1\,, \tag{2}$$

the homogeneous coordinates (x_1, x_2, x_3) of the point P are transformed into the Cartesian rectangular coordinates (x, y). Moreover, if we put

$$u_1 : u_2 : u_3 = \xi : \eta : -m\,, \tag{3}$$

the line coordinates (u_1, u_2, u_3) are transformed into the Cartesian rectangular coordinates (ξ, η). By these transformations the point P in the Cartesian rectangular coordinates (x, y) is represented by the straight line

$$x\xi + y\eta = m \tag{4}$$

in the Cartesian rectangular coordinates (ξ, η), where m is an arbitrary constant.

2. RELATIONS BETWEEN THE COORDINATES (x, y) AND (ξ, η)

If we represent a point $P_j(x_j, y_j)$ in the Cartesian rectangular coordinates (x, y) by the point $P_j(z_j)$ in the Gaussian complex plane, where $z_j = x_j + \mathrm{i}y_j$, $\mathrm{i} = \sqrt{-1}$, the point $P_j(z_j)$ is transformed into the straight line p_j in the Cartesian rectangular coordinates (ξ, η) where

$$(5) \qquad p_j : x_j\xi + y_j\eta = m .$$

We represent the intersecting point $P_{jk}(\xi_{jk}, \eta_{jk})$ of the straight lines p_j and p_k in the Cartesian rectangular coordinates (ξ, η) by the point $P_{jk}(z_{jk})$ in the Gaussian complex plane where

$$(6) \qquad P_{jk} = P_{kj}, \quad z_{jk} = z_{kj}, \quad z_{jk} = \xi_{jk} + \mathrm{i}\eta_{jk}, \quad j \neq k, \quad \mathrm{i} = \sqrt{-1} .$$

From the relations:

$$(7) \qquad z_{jk} = \xi_{jk} + \mathrm{i}\eta_{jk}, \quad x_j\xi_{jk} + y_j\eta_{jk} = m, \quad x_k\xi_{jk} + y_k\eta_{jk} = m$$

we have

$$\xi_{jk} = \frac{m(y_k - y_j)}{x_jy_k - x_ky_j}, \quad \eta_{jk} = \frac{m(x_j - x_k)}{x_jy_k - x_ky_j} .$$

Therefore,

$$(8) \qquad z_{jk} = \frac{m(y_k - y_j) + \mathrm{i}m(x_j - x_k)}{x_jy_k - x_ky_j} = \frac{\mathrm{i}m(x_j - x_k - \mathrm{i}y_k + \mathrm{i}y_j)}{x_jy_k - x_ky_j} = \frac{\mathrm{i}m(z_j - z_k)}{x_jy_k - x_ky_j},$$
$$j \neq k, \quad \mathrm{i} = \sqrt{-1} .$$

We have the following relation:

$$(9) \qquad \sphericalangle\, P_{jk}OP_{lj} = \arg\left(\frac{z_{lj}}{z_{jk}}\right) = \arg\left(\frac{\mathrm{i}m(z_l - z_j)}{x_ly_j - x_jy_l} \cdot \frac{x_jy_k - x_ky_j}{\mathrm{i}m(z_j - z_k)}\right) =$$
$$= \arg\left(\frac{z_l - z_j}{z_k - z_j} \cdot \frac{x_ky_j - x_jy_k}{x_ly_j - x_jy_l}\right) = \arg\left(\frac{z_l - z_j}{z_k - z_j}\right) + \arg\left(\frac{x_ky_j - x_jy_k}{x_ly_j - x_jy_l}\right),$$
$$j \neq k, \ k \neq l, \ l \neq j .$$

Similarly, a point $Q_j(w_j)$ in the Gaussian complex plane is transformed into the straight line q_j and the intersecting point Q_{jk} of the straight lines q_j and q_k is represented by the point $Q_{jk}(w_{jk})$ in the Gaussian complex plane, where

$$(10) \qquad w_j = u_j + \mathrm{i}v_j, \quad Q_{jk} = Q_{kj}, \quad w_{jk} = w_{kj}, \quad k \neq j .$$

Moreover, we have the relation

$$(11) \qquad \sphericalangle\, Q_{jk}OQ_{lj} = \arg\left(\frac{w_{lj}}{w_{jk}}\right) = \arg\left(\frac{w_l - w_j}{w_k - w_j}\right) + \arg\left(\frac{u_kv_j - u_jv_k}{u_lv_j - u_jv_l}\right),$$
$$j \neq k, \ k \neq l, \ l \neq j .$$

If we have the relation

$$\Delta P_1 P_2 P_3 \backsim \Delta Q_1 Q_2 Q_3 \,, \tag{12}$$

namely,

$$\begin{vmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{vmatrix} = 0 \,, \tag{13}$$

we have the relation:

$$\arg\left(\frac{z_l - z_j}{z_k - z_j}\right) = \arg\left(\frac{w_l - w_j}{w_k - w_j}\right), \tag{14}$$

$$j, k, l = 1, 2, 3,\ j \neq k,\ k \neq l,\ l \neq j \,.$$

Therefore,

$$\measuredangle\, P_{jk}OP_{lj} - \measuredangle\, Q_{jk}OQ_{lj} = \arg\left(\frac{x_k y_j - x_j y_k}{x_l y_j - x_j y_l}\right) - \arg\left(\frac{u_k v_j - u_j v_k}{u_l v_j - u_j v_l}\right). \tag{15}$$

As the values of

$$\left(\frac{x_k y_j - x_j y_k}{x_l y_j - x_j y_l}\right) \quad \text{and} \quad \left(\frac{u_k v_j - u_j v_k}{u_l v_j - u_j v_l}\right)$$

are real, their arguments are zero or π.

Therefore,

$$\measuredangle\, P_{jk}OP_{lj} = \measuredangle\, Q_{jk}OQ_{lj}\,, \quad \measuredangle\, P_{jk}OP_{lj} = \measuredangle\, Q_{jk}OQ_{lj} + \pi\,, \tag{16}$$

or

$$\measuredangle\, P_{jk}OP_{lj} = \measuredangle\, Q_{jk}OQ_{lj} - \pi\,.$$

If we superpose the vector OP_{jk} on the vector OQ_{jk}, the vector OP_{lj} and the vector OQ_{lj} are collinear and the point P_{lj} is the intersecting point of p_l and p_j while the point Q_{lj} is the intersecting point of q_l and q_j. If one of the values z_1, z_2 and z_3 is zero, for example, $z_1 = 0$, p_1 is the straight line through the point at infinity. The point $P_{12}(z_{12})$ is the point at infinity on p_2 and the point $P_{31}(z_{31})$ is the point at infinity on p_3. If we draw the straight lines p_2' and p_3' through the origin which are parallel to the straight lines p_2 and p_3, respectively, we have the following relations:

$$\measuredangle\, P_{12}OP_{23} = \text{the intersecting angle of } p_2' \text{ and } OP_{23}\,, \tag{17}$$

$$\measuredangle\, P_{31}OP_{23} = \text{the intersecting angle of } p_2' \text{ and } OP_{23}\,,$$

$$\measuredangle\, P_{12}OP_{31} = \text{the intersecting angle of } p_2' \text{ and } p_3'\,.$$

3. REPRESENTATION OF AN ANALYTIC FUNCTION IN THE LINE COORDINATES

If $w = f(z)$ is an analytic function of $z = x + iy$, we have the relation:

$$w = f(z) = u(x, y) + iv(x, y), \quad i = \sqrt{-1}\,. \tag{18}$$

The point $P(u, v)$ which is represented by $w = f(z)$ in the Gaussian complex plane, is shown by the intersection of the curvilinear nets

$$u = u(x, y) \quad \text{and} \quad v = v(x, y)\,.$$

The point $P(u, v)$ in the coordinates (u, v) is transformed into the straight line p in the coordinates (ξ, η),

$$p : u(x, y)\,\xi + v(x, y)\,\eta = m\,. \tag{19}$$

If y is constant, we have the following envelope of the straight lines p having the parameter x and index y:

$$F(x, \xi, \eta) = u(x, y)\,\xi + v(x, y)\,\eta - m = 0\,, \tag{20}$$

$$\frac{\partial F}{\partial x} = \frac{\partial u(x, y)}{\partial x}\,\xi + \frac{\partial v(x, y)}{\partial x}\,\eta = 0\,.$$

Solving these expressions with respect to ξ and η, we have

$$\xi = \frac{\begin{vmatrix} m & v(x, y) \\ 0 & \dfrac{\partial v(x, y)}{\partial x} \end{vmatrix}}{\begin{vmatrix} u(x, y) & v(x, y) \\ \dfrac{\partial u(x, y)}{\partial x} & \dfrac{\partial v(x, y)}{\partial x} \end{vmatrix}} = \frac{m\,\dfrac{\partial v(x, y)}{\partial x}}{u(x, y)\,\dfrac{\partial v(x, y)}{\partial x} - v(x, y)\,\dfrac{\partial u(x, y)}{\partial x}}, \tag{21}$$

$$\eta = \frac{\begin{vmatrix} u(x, y) & m \\ \dfrac{\partial u(x, y)}{\partial x} & 0 \end{vmatrix}}{\begin{vmatrix} u(x, y) & v(x, y) \\ \dfrac{\partial u(x ,y)}{\partial x} & \dfrac{\partial v(x, y)}{\partial x} \end{vmatrix}} = \frac{-m\,\dfrac{\partial u(x, y)}{\partial x}}{u(x, y)\,\dfrac{\partial v(x, y)}{\partial x} - v(x, y)\,\dfrac{\partial u(x, y)}{\partial x}},$$

$$u(x, y)\,\frac{\partial v(x, y)}{\partial x} \mp v(x, y)\,\frac{\partial u(x, y)}{\partial x}\,.$$

If x is constant, we have the following envelope having the parameter y and index x:

(22)
$$\xi = \frac{m \dfrac{\partial v(x, y)}{\partial y}}{u(x, y) \dfrac{\partial v(x, y)}{\partial y} - v(x, y) \dfrac{\partial u(x, y)}{\partial y}},$$

$$\eta = \frac{-m \dfrac{\partial u(x, y)}{\partial y}}{u(x, y) \dfrac{\partial v(x, y)}{\partial y} - v(x, y) \dfrac{\partial u(x, y)}{\partial y}},$$

$$u(x, y) \frac{\partial v(x, y)}{\partial y} \neq v(x, y) \frac{\partial u(x, y)}{\partial y}.$$

Therefore, if the values of x and y are given, the straight line p is the common tangent of the envelopes (21) and (22) (See Fig. 1).

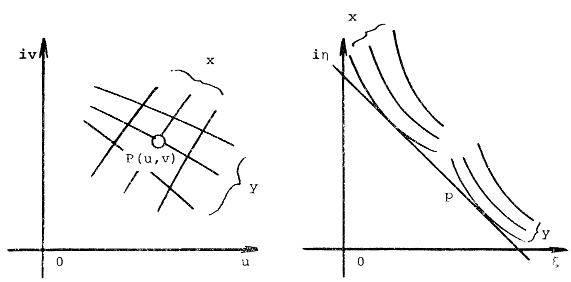

Fig. 1.

4. COMPLEX CHARTS OF THREE VARIABLES

If a given functional relation of three complex variables $F(z_1, z_2, z_3) = 0$ is represented by Massau's complex chart determinant of the third order or complex nomographic function:

(23)
$$\det(\mathcal{M}_3^c) = \begin{vmatrix} f_1(z_1) & f_2(z_2) & f_3(z_3) \\ g_1(z_1) & g_2(z_2) & g_3(z_3) \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

equation (23) is called a key equation or a type equation for the three complex variable charts. We put

(24)
$$w_j = f_j(z_j) = f_j(x_j + \mathrm{i}y_j) = u_j(x_j, y_j) + \mathrm{i}v_j(x_j, y_j),$$
$$w_j^* = g_j(z_j) = g_j(x_j + \mathrm{i}y_j) = u_j^*(x_j, y_j) + \mathrm{i}v_j^*(x_j, y_j),$$
$$j = 1, 2, 3, \mathrm{i} = \sqrt{-1}.$$

From (23), we have the relation

$$(25)\qquad \begin{vmatrix} w_1 & w_2 & w_3 \\ w_1^* & w_2^* & w_3^* \\ 1 & 1 & 1 \end{vmatrix} = 0\,,$$

and from (25), we have the relation

$$(26)\qquad \Delta P_1 P_2 P_3 \infty \Delta Q_1 Q_2 Q_3\,,$$

where vertices P_j and Q_j are represented by w_j and w_j^* in the Gaussian complex plane and they are shown by the intersections of curves of the curvilinear nets $u_j = u_j(x_j, y_j)$, $v_j = v_j(x_j, y_j)$ and $u_j^* = u_j^*(x_j, y_j)$, $v_j^* = v_j^*(x_j, y_j)$, respectively. By (21) and (22), the point P_j $(j = 1, 2, 3)$ is transformed into the straight line p_j in the line coordinates (ξ, η), and p_j is the common tangent of the respective curve in a family of curves which have index y_j and parameter x_j and the respective curve in a family of curves which have index x_j and parameter y_j. Similarly, the point Q_j $(j = 1, 2, 3)$ is transformed into the straight line q_j, where q_j is the common tangent of the respective curve in a family of curves which have index y_j and parameter x_j and the respective curve in a family of curves which have index x_j and parameter y_j. The points P_{jk} and Q_{jk} are the intersecting points of p_j, p_k and q_j, q_k, respectively, where $P_{jk} = P_{kj}$, $Q_{jk} = Q_{kj}$, $j, k = 1, 2, 3$, $j \neq k$.

5. METHOD OF SOLUTION

If a given functional relation $F(z_1, z_2, z_3) = 0$ is represented by the expression (23), we have a pair of figures, namely, the first partial chart where the family of curves has a common tangent p_j and the second partial chart where the family of curves has a common tangent q_j $(j = 1, 2, 3)$. If the values z_1 and z_2 are known, we

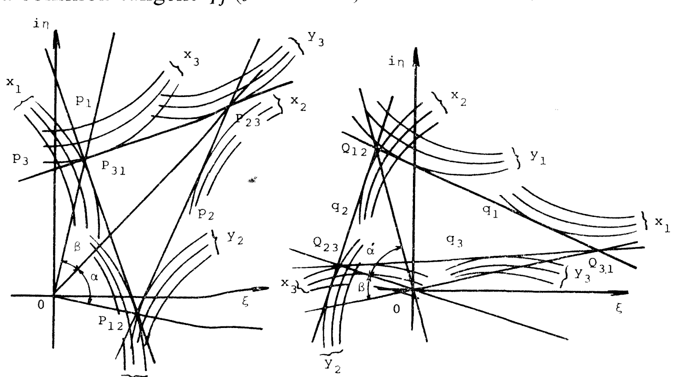

Fig. 2.

The first partial chart.

The second partial chart.

superpose the vector OQ_{12} on the vector OP_{12}, cf. Section 2, the vectors OQ_{23}, OP_{23} and the vectors OQ_{31}, OP_{31} are collinear, respectively, and the points P_{23}, P_{31} and Q_{23}, Q_{31} lie on the straight lines p_3 and q_3, respectively.

Therefore, if we seek for the straight lines p_3 and q_3 which satisfy the above conditions and are the common tangents of curves having the same indices x_3 and y_3, the value $z_3 = x_3 + iy_3$ is the required third quantity (See Fig. 2).

6. AFFINE TRANSFORMATION OF THE COMPLEX CHART

We multiply the given complex chart matrix $\mathbb{M}_3^c$ from the left by a matrix $\mathbb{A}$, where

(27)
$$\mathbb{A} = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{Vmatrix}, \quad \det(\mathbb{A}) \neq 0,$$

and every element a_{ij} is a complex number.

Then

(28)
$$\mathbb{A}\mathbb{M}_3^c = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{Vmatrix} \begin{Vmatrix} f_1(z_1) & f_2(z_2) & f_3(z_3) \\ g_1(z_1) & g_2(z_2) & g_3(z_3) \\ 1 & 1 & 1 \end{Vmatrix}$$

$$= \begin{Vmatrix} a_{11} f_1(z_1) + a_{12} g_1(z_1) + a_{13} & a_{11} f_2(z_2) + a_{12} g_2(z_2) + a_{13} & a_{11} f_3(z_3) + a_{12} g_3(z_3) + a_{13} \\ a_{21} f_1(z_1) + a_{22} g_1(z_1) + a_{23} & a_{21} f_2(z_2) + a_{22} g_2(z_2) + a_{23} & a_{21} f_3(z_3) + a_{22} g_3(z_3) + a_{23} \\ 1 & 1 & 1 \end{Vmatrix} = \overline{\mathbb{M}}_3^c .$$

The matrices $\mathbb{A}$ and $\overline{\mathbb{M}}_3^c$ are called the complex affine transformation matrix and the transformed complex chart matrix, respectively.

When $\det(\mathbb{M}_3^c) = 0$, we have $\det(\overline{\mathbb{M}}_3^c) = 0$ and vice versa. By an adequate affine transformation, we have other new charts which are convenient to use.

7. SOME TYPE EQUATIONS

1. Type equation

(29)
$$\frac{f_1(z_1) + f_2(z_2)}{g_1(z_1) + g_2(z_2)} = \frac{f_1(z_1) + f_3(z_3)}{g_1(z_1) + g_3(z_3)} .$$

The corresponding chart matrix is

$$(30) \qquad \begin{Vmatrix} -a\,f_1(z_1) & a\,f_2(z_2) & a\,f_3(z_3) \\ -b\,g_1(z_1) & b\,g_2(z_2) & b\,g_3(z_3) \\ 1 & 1 & 1 \end{Vmatrix},$$

where a and b are the chart factors, and the skeleton of the corresponding complex chart is similar as in Fig. 2. If we put $f_j(z_j) = z_j$, $g_j(z_j) = z_j^2$ $(j = 1, 2, 3)$ in the expression (29), we have the relation:

$$(31) \qquad \frac{z_1 + z_2}{z_1^2 + z_2^2} = \frac{z_1 + z_3}{z_1^2 + z_3^2}.$$

As a practical example, we put $a = 4$, $b = 1$ and $m = 240$ in the rectangular section paper of 1000×700 mm and obtained nomographically $z_1 = 3.70 + 2{\cdot}86i$ or $z_1 = -2{\cdot}10 - 4{\cdot}86i$ for the exact solution $z_1 = 3{\cdot}7016 + 2{\cdot}8599i$ or $z_1 = -2{\cdot}1016 - 4{\cdot}8599i$, respectively, when the given values are $z_2 = -3{\cdot}2 + 2{\cdot}2i$ and $z_3 = 4{\cdot}8 - 4{\cdot}2i$, $i = \sqrt{-1}$.

2. Type equation

$$(32) \qquad \frac{1}{f_1(z_1)} + \frac{1}{f_2(z_2)} = \frac{1}{f_3(z_3)}.$$

The corresponding chart matrix is

$$(33) \qquad \begin{Vmatrix} a\,f_1(z_1) & a\,f_2(z_2) & a\,f_3(z_3) \\ b\,f_1^2(z_1) & b\,f_2^2(z_2) & 0 \\ 1 & 1 & 1 \end{Vmatrix},$$

and the skeleton of the chart is shown in Fig. 3.

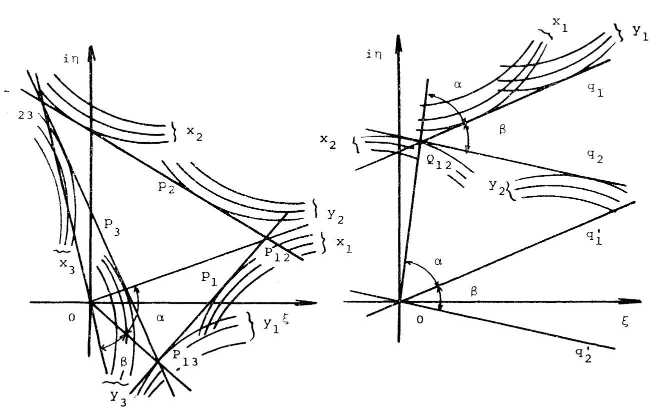

Fig. 3.

The first partial chart. The second partial chart.

If we put $f_j(z_j) = z_j$ $(j = 1, 2, 3)$ in the expression (32), we have the relation:

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{z_3}. \tag{34}$$

As a practical example, we put $a = 4$, $b = 1$ and $m = 60$ in the rectangular section paper of 1000 × 700 mm and obtained nomographically $z_3 = 0{\cdot}83 + 0{\cdot}91i$ for the exact solution $z_3 = 0{\cdot}8255 + 0{\cdot}9088i$ when the given values are $z_1 = 1{\cdot}4 + 2{\cdot}1i$ and $z_2 = 1{\cdot}8 + 1{\cdot}5i$, $i = \sqrt{-1}$.

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Souhrn

O KONSTRUKCI NOMOGRAMŮ S TŘEMI KOMPLEXNÍMI PROMĚNNÝMI POMOCÍ PŘÍMKOVÝCH SOUŘADNIC

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V článku se pojednává o nomografickém zobrazení vztahu mezi třemi komplexními proměnnými, jestliže tento vztah lze zapsat ve tvaru determinantu

$$\begin{vmatrix} f_1(z_1), & f_2(z_2), & f_3(z_3) \\ g_1(z_1), & g_2(z_2), & g_3(z_3) \\ 1 & 1 & 1 \end{vmatrix} = 0 .$$

Soustavy křivek, tvořících nomogram, jsou obálky soustav přímek, a proto se v článku s výhodou používá aparátu přímkových souřadnic. Nomogram má charakter dotykového nomogramu.

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