Ľubor Malina

On generalized methods of the transfer of conditions

Aplikace matematiky, Vol. 24 (1979), No. 3, 184–198

Persistent URL: `http://dml.cz/dmlcz/103795`

ON GENERALIZED METHODS OF THE TRANSFER OF CONDITIONS

ĽUBOR MALINA

(Received May 5, 1977)

In [2] we have suggested a possible general approach to direct methods for solution of systems with band matrices. In the present paper we shall further develop *Algorithm* $\mathscr{T}1$ (in the notation of [2]). Namely, we define *Algorithm* $\mathscr{T}4$ which includes not only *Algorithm* $\mathscr{T}1$ but also methods leading to diagonalization of the matrix of the system, such as for example the process of Gauss-Jordan elimination. As we have promised in [2], we also slightly touch the question of numerical stability for the present algorithms. Eventually we shall briefly discuss the concept of "well conditioned" systems (cf. [1]) in Part 3. In the last part we show how one could obtain concrete methods from the general *Algorithm* $\mathscr{T}4$, namely the Gauss-Jordan elimination and the even-odd reduction closely connected with the fast Fourrier technique.

1. PRELIMINARIES

In this part we quote briefly some concepts and results from [2]. Consider the problem of solving a system

$$\mathbf{G}\mathbf{y} = \mathbf{b} \tag{1.1}$$

where $\mathbf{G} = (g_{ij})$ is a square band matrix of order N whose bandwidth is $2p + 1$, vectors $\mathbf{b} = [b_1, \ldots, b_N]^\mathrm{T}$ and $\mathbf{y} = [y_1, \ldots, y_N]^\mathrm{T}$ are N-dimensional. We choose a parameter j, $j \in \{0, 1, \ldots, 2p - 1\}$ supposing $J = (N - 2p)/(2p - j)$ to be an integer. At the end of the second part we show the changes which occur in the case that this hypothesis is not fulfilled. Let us denote

$$\mathbf{x}_i^{(j)} = [y_{I+1}, \ldots, y_{I+2p}]^\mathrm{T} \quad \text{for} \quad i = 1(1)\, J + 1\,, \tag{1.2}$$

where $I = (2p - j)(i - 1)$. If it is clear from the context which value of j has been chosen we shall simply write $\mathbf{x}_i$. From (1.2) it follows that we divided the vector $\mathbf{y}$

of unknowns into subvectors $\mathbf{x}_i$ in such a way that the last j elements of a vector $\mathbf{x}_i$ repeat as the first j elements of a vector $\mathbf{x}_{i+1}$. In an analogous way we divide the vector $\mathbf{b}$ into $2p$-dimensional vectors $\mathbf{f}_i$,

$$\mathbf{f}_i = [b_{I+p+1}, \ldots, b_{I+p+2p-j}, 0, \ldots, 0]^{\mathrm{T}}, \tag{1.3}$$

and the matrix $\mathbf{G}$ so that the $(I + p + 1)$-st equation up to the $(I + p + 2p - j)$-th equation of the system (1.1) can be written in a matrix form

$$\mathbf{A}_i \mathbf{x}_i + \mathbf{B}_i \mathbf{x}_{i+1} = \mathbf{f}_i$$

where both the matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ are square matrices of order $2p$. The first and the last p equations of the system (1.1) will be called the left and the right boundary condition, respectively. We write them in a matrix form

$$\mathbf{A}_0 \mathbf{x}_1 = \mathbf{f}_0 \quad \text{and} \quad \mathbf{A}_{J+1} \mathbf{x}_{J+1} = \mathbf{f}_{J+1},$$

where both the matrices $\mathbf{A}_0$ and $\mathbf{A}_{J+1}$ have p rows and $2p$ columns (they are $p \times 2p$ matrices). Thus the system (1.1) can be written in an equivalent form

$$\mathbf{A}_i \mathbf{x}_i + \mathbf{B}_i \mathbf{x}_{i+1} = \mathbf{f}_i, \tag{1.4}$$

$$\mathbf{A}_0 \mathbf{x}_1 = \mathbf{f}_0 \quad \text{and} \quad \mathbf{A}_{J+1} \mathbf{x}_{J+1} = \mathbf{f}_{J+1}. \tag{1.5}$$

Notation. A matrix $\mathbf{M}$ with i rows and j columns will be said to be an $i \times j$ matrix. The symbol $\mathbf{O}_{i,j}$ denotes the $i \times j$ null matrix while $\mathbf{I}_j$ denotes the identity matrix of order j. *Rank* $\mathbf{M}$ denotes the rank of the matrix $\mathbf{M}$.

The set $\mathfrak{M} = \{1, \ldots, J + 1\}$ is divided into four disjoint parts $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \mathfrak{M}_4$ where

$$\mathfrak{M}_1 = \{i \in \mathfrak{M} \mid \text{both } \mathbf{A}_i \text{ and } \mathbf{B}_i \text{ are regular, or } i = 1,\ i = J + 1\},$$

$$\mathfrak{M}_2 = \{i \in \mathfrak{M} \mid \mathbf{A}_i \text{ is singular, } \mathbf{B}_i \text{ is regular}\},$$

$$\mathfrak{M}_3 = \{i \in \mathfrak{M} \mid \mathbf{A}_i \text{ is regular, } \mathbf{B}_i \text{ is singular}\},$$

$$\mathfrak{M}_4 = \{i \in \mathfrak{M} \mid \text{both } \mathbf{A}_i \text{ and } \mathbf{B}_i \text{ are singular}\}.$$

The equation (1.4) can be rewritten into an equivalent form:

for $i \in \mathfrak{M}_2 \cup \mathfrak{M}_4$,

$$\begin{bmatrix} \mathbf{A}_{i,1} \\ \mathbf{O}_{n_i,2p} \end{bmatrix} \mathbf{x}_i + \begin{bmatrix} \mathbf{B}_{i,1} \\ \mathbf{B}_{i,2} \end{bmatrix} \mathbf{x}_{i+1} = \begin{bmatrix} \mathbf{f}_{i,1} \\ \mathbf{f}_{i,2} \end{bmatrix} \tag{1.6}$$

where *rank* $\mathbf{A}_i = 2p - n_i$ and *rank* $\mathbf{A}_{i,1}$ is equal to the number of its rows;

for $i \in \mathfrak{M}_1 \cup \mathfrak{M}_3$,

$$\begin{bmatrix} {}_1\mathbf{A}_i \\ {}_2\mathbf{A}_i \end{bmatrix} \mathbf{x}_i + \begin{bmatrix} \mathbf{O}_{m_i,2p} \\ {}_2\mathbf{B}_i \end{bmatrix} \mathbf{x}_{i+1} = \begin{bmatrix} {}_1\mathbf{f}_i \\ {}_2\mathbf{f}_i \end{bmatrix} \tag{1.7}$$

where *rank* $\mathbf{B}_i = 2p - m_i$ and *rank* ${}_2\mathbf{B}_i$ is equal to the number of its rows.

We need also the following lemma.

Lemma. *Let* $\mathbf{C}_1$ *be an* $a_1 \times n$ *matrix*, rank $\mathbf{C}_1 = h_1$, *let* $\mathbf{C}_2$ *be an* $a_2 \times n$ *matrix*, rank $\mathbf{C}_2 = h_2$ *and let* rank $\mathbf{C}_3 = h_3$ *where*

$$\mathbf{C}_3 = [\mathbf{C}_1^{\mathsf{T}}, \mathbf{C}_2^{\mathsf{T}}]^{\mathsf{T}} .$$

Then there are matrices $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *such that*

(1) $\mathbf{S}_1\mathbf{C}_1 = \mathbf{S}_2\mathbf{C}_2$,

(2) rank $\mathbf{S}_1\mathbf{C}_1 \leqq h_1 + h_2 - h_3$ *and there is a pair of* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *such that rank* $\mathbf{S}_1\mathbf{C}_1 = h_1 + h_2 - h_3$.

Algorithm $\mathcal{T}1$.

(a) ***Transfer from the left to the right:***
for $i \in \mathfrak{M}_1 \cup \mathfrak{M}_3$,

$$\text{(1.8)} \qquad \begin{aligned} \mathbf{D}_1 &= \mathbf{A}_0 , \quad \mathbf{d}_1 = \mathbf{f}_0 , \\ \mathbf{D}_{i+1} &= \mathbf{Z}_i\mathbf{D}_i\mathbf{H}_i , \quad \mathbf{d}_{i+1} = \mathbf{Z}_i(-\mathbf{d}_i + \mathbf{D}_i\mathbf{h}_i) \end{aligned}$$

where $\mathbf{H}_i = \mathbf{A}_i^{-1}\mathbf{B}_i$ and $\mathbf{h}_i = \mathbf{A}_i^{-1}\mathbf{f}_i$ and $\mathbf{Z}_i$ always stands for a regular matrix of such order that the multiplication is well defined.

For $i \in \mathfrak{M}_2 \cup \mathfrak{M}_4$,

$$\text{(1.9)} \qquad \mathbf{D}_{i+1} = \mathbf{Z}_i \begin{bmatrix} \mathbf{S}_2\mathbf{B}_{i,1} \\ \mathbf{B}_{i,2} \end{bmatrix} \quad \text{and} \quad \mathbf{d}_{i+1} = \mathbf{Z}_i \begin{bmatrix} \mathbf{S}_2\mathbf{f}_{i,1} - \mathbf{S}_1\mathbf{d}_i \\ \mathbf{f}_{i,2} \end{bmatrix}$$

where the matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ are such that $\mathbf{S}_2\mathbf{A}_{i,1} = \mathbf{S}_1\mathbf{D}_i$.

(b) ***Tranfer from the right to the left:***
For $i \in \mathfrak{M}_1 \cup \mathfrak{M}_3$, the matrix $\mathbf{R}_i$ is an arbitrary matrix such that $[\mathbf{D}_i^{\mathsf{T}}, \mathbf{R}_i^{\mathsf{T}}]^{\mathsf{T}}$ is a regular, square matrix of order $2p$,

$$\text{(1.10)} \qquad \mathbf{r}_i = \mathbf{R}_i \left(\mathbf{h}_i - \mathbf{H}_i \begin{bmatrix} \mathbf{D}_{i+1} \\ \mathbf{R}_{i+1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_{i+1} \\ \mathbf{r}_{i+1} \end{bmatrix} \right) .$$

For $i = J + 1$,

$$\text{(1.11)} \qquad \mathbf{r}_{J+1} = \mathbf{R}_{J+1} \begin{bmatrix} \mathbf{D}_{J+1} \\ \mathbf{A}_{J+1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_{J+1} \\ \mathbf{f}_{J+1} \end{bmatrix} .$$

For $i \in \mathfrak{M}_2$,

$$\text{(1.12)} \qquad \mathbf{R}_i = \mathbf{W}_i\mathbf{R}_{i+1}\mathbf{B}_i^{-1}\mathbf{A}_i \quad \text{and} \quad \mathbf{r}_i = \mathbf{W}_i(-\mathbf{r}_{i+1} + \mathbf{R}_{i+i}\mathbf{B}_i^{-1}\mathbf{f}_i) .$$

For $i \in \mathfrak{M}_4$,

$$\mathbf{R}_i \begin{bmatrix} \mathbf{S}^{(1)} {}_2\mathbf{A}_i \\ {}_1\mathbf{A}_i \end{bmatrix} \quad \text{and} \quad \mathbf{r}_i = \begin{bmatrix} \mathbf{S}^{(1)} {}_2\mathbf{f}_i - \mathbf{S}^{(2)} \mathbf{r}_{i+1} \\ {}_1\mathbf{f}_i \end{bmatrix} \tag{1.13}$$

where

$$\mathbf{S}^{(1)} {}_2\mathbf{B}_i = \mathbf{S}^{(2)} \mathbf{R}_{i+1} \,.$$

Vectors $\mathbf{x}_i$ for $i = 1(1)\, J + 1$ are defined as solutions of systems

$$\mathbf{Q}_i \mathbf{x} = \mathbf{q}_i \quad \text{for} \quad i = 1(1)\, J + 1 \tag{1.14}$$

where

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{D}_i \\ \mathbf{R}_i \end{bmatrix} \quad \text{and} \quad \mathbf{q}_i = \begin{bmatrix} \mathbf{d}_i \\ \mathbf{r}_i \end{bmatrix}.$$

It is proved (cf. Theorem 2.1 from [2]) that every solution $\{\mathbf{x}_i\}_{i=1}^{J+1}$ of the system (1.4)–(1.5), i.e., of the system (1.1) consists of solutions of the systems (1.14) and vice versa.

2. ALGORITHM $\mathcal{T}4$

The idea leading to the *Algorithm* $\mathcal{T}4$ is the following one. As soon as we have computed the matrix $\mathbf{D}_i$ and the vector $\mathbf{d}_i$ such that

$$\mathbf{D}_i \mathbf{x}_i = \mathbf{d}_i \,,$$

we consider the vector $\mathbf{x}_{i+1}$ to be known and perform a transfer to the left. Thus the vectors $\mathbf{r}_n$ for $n < i$ depend on $\mathbf{x}_{i+1}$ and they are written in the form of a sum of a vector and a matrix multiplied by the vector $\mathbf{x}_{i+1}$.

Let integers $\{i_k\}_{k=1}^{K}$ be given,

$$0 = i_0 < i_1 \ldots < i_{K-1} < i_K = J + 1 \,. \tag{2.1}$$

Denote

$$I_k = \{i_{k-1} + 1, \ldots, i_k\} \quad \text{for} \quad k = 1(1)\, K \,.$$

Algorithm $\mathcal{T}4$.

For every $k \in \{1, \ldots, K\}$ define

(a_k) ***the k-th transfer to the right:***

The matrix $\mathbf{D}_i$ and the vector $\mathbf{d}_i$ for $i \in \mathbf{I}_k \cap (\mathfrak{M}_1 \cup \mathfrak{M}_3)$ and for $i \in I_k \cap (\mathfrak{M}_2 \cup \mathfrak{M}_4)$ are given by (1.8) and (1.9), respectively.

(b_k) ***the k-th transfer to the left:***

The vector $\mathbf{r}_i$ is of the form

$$\mathbf{r}_i = \mathbf{U}_i \mathbf{x}_{i_k+1} + \mathbf{u}_i \tag{2.2}$$

where for $i \in I_k \cap (\mathfrak{M}_1 \cup \mathfrak{M}_3)$ the matrix $\boldsymbol{R}_i$ is an arbitrary matrix such that the matrix $\boldsymbol{Q}_i = [\boldsymbol{D}_i^\mathsf{T}, \boldsymbol{R}_i^\mathsf{T}]^\mathsf{T}$ is regular,

$$\boldsymbol{u}_i = \boldsymbol{R}_i(\boldsymbol{h}_i - \boldsymbol{H}_i\boldsymbol{Q}_{i+1}^{-1}\boldsymbol{q}_{i+1,k}) \tag{2.3}$$

and

$$\boldsymbol{U}_i = -\boldsymbol{R}_i\boldsymbol{H}_i\boldsymbol{Q}_{i+1}^{-1}\begin{bmatrix}\boldsymbol{O}_{n,2p}\\ \boldsymbol{U}_{i+1}\end{bmatrix} \quad \text{for} \quad i < i_k\,, \tag{2.4}$$

where n is equal to the number of rows of the matrix $\boldsymbol{D}_{i+1}$ and

$$\boldsymbol{Q}_{i+1} = \begin{bmatrix}\boldsymbol{D}_{i+1}\\ \boldsymbol{R}_{i+1}\end{bmatrix}, \quad \boldsymbol{q}_{i+1,k} = \begin{bmatrix}\boldsymbol{d}_{i+1}\\ \boldsymbol{u}_{i+1}\end{bmatrix}.$$

The "initial condition" $\boldsymbol{u}_i$ and $\boldsymbol{U}_i$ for $i = i_k$ is

$$\begin{aligned}\boldsymbol{u}_i &= \boldsymbol{R}_i\boldsymbol{h}_i\,,\\ \boldsymbol{U}_i &= -\boldsymbol{R}_i\boldsymbol{H}_i\,.\end{aligned} \tag{2.5}$$

For $i \in I_k \cap \mathfrak{M}_2$ and $i < i_k$ the matrix $\boldsymbol{R}_i$ is given by (1.12) and

$$\boldsymbol{u}_i = \boldsymbol{W}_i(-\boldsymbol{u}_{i+1} + \boldsymbol{R}_{i+1}\boldsymbol{B}_i^{-1}\boldsymbol{f}_i)\,, \quad \boldsymbol{U}_i = -\boldsymbol{W}_i\boldsymbol{U}_{i+1} \tag{2.6}$$

under the notation from Part 1.

For $i \in I_k \cap \mathfrak{M}_4$ and $i < i_k$ the matrix $\boldsymbol{R}_i$ is given by (1.13) and

$$\boldsymbol{u}_i = \begin{bmatrix}\boldsymbol{S}^{(1)}\,{}_2\boldsymbol{f}_i - \boldsymbol{S}^{(2)}\boldsymbol{u}_{i+1}\\ {}_1\boldsymbol{f}_i\end{bmatrix}, \quad \boldsymbol{U}_i = \begin{bmatrix}\boldsymbol{S}^{(2)}\boldsymbol{U}_{i+1}\\ \boldsymbol{O}_{m,2p}\end{bmatrix} \tag{2.7}$$

under the notation from Part 1; m is equal to the dimension of the vector ${}_1\boldsymbol{f}_i$.

As for the "initial conditions", if $i_k \in \mathfrak{M}_2 \cup \mathfrak{M}_4$ and *rank* $[\boldsymbol{D}_i^\mathsf{T}, \boldsymbol{A}_i^\mathsf{T}]^\mathsf{T} = 2p$ for $i = i_k$, then there is a matrix $\boldsymbol{A}_{i,3}$ (submatrix of $\boldsymbol{A}_i$) such that the matrix

$$\begin{bmatrix}\boldsymbol{D}_i\\ \boldsymbol{A}_{i,3}\end{bmatrix} \quad \text{for} \quad i = i_k$$

is regular. Denoting by $\boldsymbol{B}_{i,3}$ and $\boldsymbol{f}_{i,3}$ the matrix and the vector which result from $\boldsymbol{B}_i$ and $\boldsymbol{f}_i$, respectively, in the same way as the matrix $\boldsymbol{A}_{i,3}$ from $\boldsymbol{A}_i$, i.e., by crossing out some rows of the matrix $\boldsymbol{A}_i$ to obtain the regular matrix $[\boldsymbol{D}_i^\mathsf{T}, \boldsymbol{A}_{i,3}^\mathsf{T}]^\mathsf{T}$, we define

$$\boldsymbol{u}_i = \begin{bmatrix}\boldsymbol{D}_i\\ \boldsymbol{A}_{i,3}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{d}_i\\ \boldsymbol{f}_{i,3}\end{bmatrix} \quad \text{and} \quad \boldsymbol{U}_i = -\begin{bmatrix}\boldsymbol{D}_i\\ \boldsymbol{A}_{i,3}\end{bmatrix}^{-1}\boldsymbol{B}_{i,3} \tag{2.8}$$

for $i = i_k$.

When

$$\operatorname{rank}[\boldsymbol{D}_i^\mathsf{T}, \boldsymbol{A}_i^\mathsf{T}]^\mathsf{T} < 2p \quad \text{for} \quad i = i_k$$

we choose an arbitrary matrix $\mathbf{C}$ and a vector $\mathbf{c}$ such that

$$\mathbf{x}_i = \mathbf{C}\mathbf{x}_{i+1} + \mathbf{c}$$

for arbitrary $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ for which the following identity holds:

$$\mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{x}_{i+1} = \mathbf{f}_i \quad \text{for} \quad i = i_k \,.$$

We set in this case

$$\mathbf{u}_i = \mathbf{R}_i\mathbf{c} \quad \text{and} \quad \mathbf{U}_i = \mathbf{R}_i\mathbf{C} \quad \text{for} \quad i = i_k \,, \tag{2.9}$$

where $\mathbf{R}_i$ in both cases is a matrix such that the matrix $[\mathbf{D}_i^{\mathrm{T}}, \mathbf{R}_i^{\mathrm{T}}]^{\mathrm{T}}$ is regular.

For $i_k = J + 1$ we set

$$\mathbf{u}_{J+1} = \mathbf{r}_{J+1} \quad \text{and} \quad \mathbf{U}_{J+1} = \mathbf{O} \,.$$

Denoting

$$\mathbf{q}_{i,k} = \begin{bmatrix} \mathbf{d}_i \\ \mathbf{u}_i \end{bmatrix}, \quad \mathbf{Q}_{i,k} = \begin{bmatrix} \mathbf{O} \\ \mathbf{U}_i \end{bmatrix}, \quad \mathbf{Q}_i = \begin{bmatrix} \mathbf{D}_i \\ \mathbf{R}_i \end{bmatrix} \quad \text{for} \quad i \in I_k \,, \tag{2.10}$$

we define vectors $\mathbf{x}_i$ for $i \in I_k$ to be solutions of the systems

$$\mathbf{Q}_i\mathbf{x}_i = \mathbf{q}_{i,k} + \mathbf{Q}_{i,k}\mathbf{x}_{i_k+1} \,. \tag{2.11}$$

We compute $\mathbf{x}_{i_{k-1}+1}$ from the system (2.11) for $i = i_{k-1} + 1$ and put it into the systems (2.11) for all $i \leqq i_{k-1}$.

Hence for $i = 1(1)\, i_{k-1}$ we have the systems (from the $(k - 1)$-st transfer)

$$\mathbf{Q}_i\mathbf{x}_i = \mathbf{q}_{i,k-1} + \mathbf{Q}_{i,k-1}\mathbf{x}_{i_{k-1}+1} \tag{2.12}$$

and by (2.11)

$$\mathbf{Q}_i\mathbf{x} = \mathbf{q}_{i,k-1} + \mathbf{Q}_{i,k-1}\mathbf{Q}_j^{-1}(\mathbf{q}_{j,k} + \mathbf{Q}_{j,k}\mathbf{x}_{i_k+1})$$

where j stands for $i_{k-1} + 1$. Denoting

$$\mathbf{q}_{i,k} = \mathbf{q}_{i,k-1} + \mathbf{Q}_{i,k-1}\mathbf{Q}_j^{-1}\mathbf{q}_{j,k} \tag{2.13}$$

and

$$\mathbf{Q}_{i,k} = \mathbf{Q}_{i,k-1}\mathbf{Q}_j^{-1}\mathbf{Q}_{j,k} \quad \text{for} \quad i \leqq i_{k-1} \,, \tag{2.14}$$

the system can be written also in the form of the systems (2.11). Thus after the k-th transfer from the left to the right and vice versa we have for the vectors $\mathbf{x}_i$ and $i \leqq i_k$ the systems

$$\mathbf{Q}_i\mathbf{x}_i = \mathbf{q}_{i,k} + \mathbf{Q}_{i,k}\mathbf{x}_{i_k+1} \,. \tag{2.15}$$

Finally, we come to the systems

$$\mathbf{Q}_i\mathbf{x}_i = \mathbf{q}_{i,K} \,. \tag{2.16}$$

Theorem 2.1. *Every solution* $\{\mathbf{x}_i\}_{i=1}^{J+1}$ *of the system* (1.4)–(1.5) *is composed of solutions of the systems* (2.16) *and vice versa. The system* (1.4)–(1.5) *has a unique solution iff the systems* (2.16) *have unique solutions.*

The proof of the theorem repeats almost literally the proof of the analogous theorem from [2] (Theorem 2.1); therefore we omit it.

Remark 1. *Algorithm* $\mathscr{T}4$ is of the Gauss-Jordan elimination form. However, there is an another possibility. After the "k-th transfer" we need not compute the vector $\mathbf{x}_{i_{k-1}+1}$ and put it into all systems (2.12) for $i \leqq i_{k-1}$ but can go through to the "$(k+1)$-st transfer" immediately. Thus we can obtain eventually a system equivalent to the system (1.1) but possibly with a smaller number of unknowns (such are the even-odd reduction like methods). This algorithm will be referred to as *Algorithm* $\mathscr{T}5$.

Up to now, both in the present paper and in [2] we have supposed $(N-2p)/(2p-j)$ to be an integer. Let the parameter j be chosen such that

$$\frac{N-2p}{2p-j} = J + \frac{n}{2p-j}$$

where J is an integer and $n < 2p - j$ is an integer as well. Definition (1.2) of vectors $\mathbf{x}_i$ is not changed for $i = 1(1)\,J$. However, the vector $\mathbf{x}_{J+1}$ is defined as the last $2p$ elements of the vector $\mathbf{y}$. Thus the vector $\mathbf{x}_{J+1}$ repeats the last $2p - n$ elements of the vector $\mathbf{x}_J$ as its first $2p - n$ elements. Hence we must change the definition of the matrices $\mathbf{A}_J$ and $\mathbf{B}_J$ and of the vector $\mathbf{f}_J$ in the following way:

$$\mathbf{A}_J = \left[\begin{array}{c|c} \begin{matrix} g_{N-p-n,N-2p-2}, & \ldots, & g_{N-p-n,N-2p-1} \\ & & \\ & & g_{N-p-1,N-2p-1} \end{matrix} & \mathbf{O}_{n,2p-n} \\ \hline \mathbf{O}_{2p-n,n} & \mathbf{I}_{2p-n} \end{array}\right],$$

$$\mathbf{B}_J = \left[\begin{array}{c|c} \multicolumn{2}{c}{\begin{matrix} g_{N-p-n,N-2p}, & \ldots, & g_{N-p-n,N-n-1}, & 0, & \ldots, & 0 \\ g_{N-p-1,N-2p}, & \ldots, & & & & g_{N-p-1,N-1} \end{matrix}} \\ \hline -\mathbf{I}_{2p-n} & \mathbf{O}_{2p-n,n} \end{array}\right]$$

and

$$\mathbf{f}_J = [y_{N-p-n}, \ldots, y_{N-p-1}, 0, \ldots, 0]^{\mathrm{T}}$$

is a $2p$-dimensional vector. Nothing else is changed, not even the definition of the *Algorithms* $\mathscr{T}1$, $\mathscr{T}3$, $\mathscr{T}4$ and $\mathscr{T}5$. All theorems proved till now hold also in this case.

3. NUMERICAL STABILITY

For the sake of brevity we suppose both the matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ to be regular for all $i \in \mathfrak{M}$. It is only a technical matter to extend the results also to the case when this is not fulfilled using ideas analogous to that of [4]. Only the transfer from the left to the right will be discussed because the backward transfer is analogous.

Let the matrix $\mathbf{D}_{i+1}$ be computed by (1.8). Due to the round-off and other errors the equation

$$\mathbf{D}_{i+1} - \mathbf{Z}_i\mathbf{D}_i\mathbf{H}_i = \mathbf{O},$$

where $\mathbf{D}_i$, $\mathbf{Z}_i$, $\mathbf{H}_i$ are computed numerically, is not exactly fulfilled.

Definition 3.1. *The difference*

$$\Delta_i = \mathbf{D}_{i+1} - \mathbf{Z}_i\mathbf{D}_i\mathbf{H}_i,$$

where $\mathbf{D}_1$ is given exactly, is called the error of the numerical realization (of the method of *Algorithm* $\mathscr{T}1$, given by a choice of the matrices $\mathbf{Z}_i$).

One of the methods how to assess quantitatively the influence of such an error on exact solution of the system (1.1) is to replace both the matrix $\mathbf{G}$ and the vector $\mathbf{b}$ by a "perturbed" matrix $\mathbf{G} + \delta\mathbf{G}$ and a "perturbed" vector $\mathbf{b} + \delta\mathbf{b}$, respectively and to choose these perturbation so that the "numerical", inaccurate solution of the system (1.1) is the exact solution of the system with these perturbed data obtained by an exact method of the transfer of conditions. Thus we can assume

$$\mathbf{D}_{i+1} = \mathbf{D}_i\mathbf{H}_i + \mathbf{V}_i\mathbf{D}_{i+1} + \delta_i^{(1)} \tag{3.1}$$

$$\mathbf{d}_{i+1} = -\mathbf{d}_i + \mathbf{D}_i\mathbf{h}_i + \mathbf{V}_i\mathbf{d}_{i+1} + \delta_{i,1}\mathbf{d}_{i+1} + \delta_i^{(2)} \tag{3.2}$$

where $\mathbf{V}_i = \mathbf{I} - \mathbf{Z}_i^{-1}$. This equation can be rewritten as

$$\mathbf{D}_{i+1} = \mathbf{D}_i\mathbf{H}_i + (\mathbf{V}_i + \delta_{i,1})\,\mathbf{D}_{i+1} + \delta_i^{(1)} - \delta_{i,1}\,\mathbf{D}_{i+1}, \tag{3.3}$$

$$\mathbf{d}_{i+1} = -\mathbf{d}_i + \mathbf{D}_i\mathbf{h}_i + (\mathbf{V}_i + \delta_{i,1})\,\mathbf{d}_{i+1} + \delta_i^{(2)}. \tag{3.4}$$

Without loss of generality we can suppose the *rank* $\mathbf{D}_i$ to be equal to the number of its rows. Thus the matrix $\mathbf{D}_i\mathbf{D}_i^{\mathrm{T}}$ is regular. Let us denote

$$\delta_{i,2} = \mathbf{D}_i^{\mathrm{T}}(\mathbf{D}_i\mathbf{D}_i^{\mathrm{T}})^{-1}\,(\delta_i^{(1)} - \delta_{i,1}\mathbf{D}_{i+1}), \tag{3.5}$$

$$\delta_{i,3} = \mathbf{D}_i^{\mathrm{T}}(\mathbf{D}_i\mathbf{D}_i^{\mathrm{T}})^{-1}\,\delta_i^{(2)}. \tag{3.6}$$

Then

$$\mathbf{D}_{i+1} = \mathbf{D}_i(\mathbf{H}_i + \delta_{i,2}) + (\mathbf{V}_i + \delta_{i,1})\,\mathbf{D}_{i+1},$$

$$\mathbf{d}_{i+1} = -\mathbf{d}_i + \mathbf{D}_i(\mathbf{h}_i + \delta_{i,3}) + (\mathbf{V}_i + \delta_{i,1})\,\mathbf{d}_{i+1}.$$

Theorem 3.1. *Let the system* (1.1) *have a solution, let both the matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ be regular for $i = 1(1)\,J$, and let the errors of numerical realization be of the form* (3.1)–(3.2). *Then these errors can be regarded as perturbations of the initial data, i.e., of the matrix $\mathbf{G}$ and the vector $\mathbf{b}$, and of the method of the transfer of* *conditions, i.e., of the matrices $\mathbf{Z}_i$. Moreover, denoting by $\|\cdot\|$ the usual Euclidean norm of a matrix and by $\|\cdot\|_0$ the spectral norm, we have the following estimates*:

$$\|\delta_{i,2}\|_0 \leqq \sqrt{\left|\frac{\lambda_{\max}}{\lambda_{\min}}\right|} \left(\frac{\|\delta_i^{(1)}\|_0}{\|\mathbf{D}_i\|_0} + \|\delta_{i,1}\|_0\right) \leqq$$

$$\leqq \sqrt{\frac{n|\lambda_{\max}|}{|\lambda_{\min}|}} \left(\frac{\|\delta_i^{(1)}\|}{\|\mathbf{D}_i\|} + \frac{\|\delta_{i,1}\|}{\sqrt{n}}\right),$$

$$\|\delta_{i,3}\|_0 \leqq \sqrt{\left|\frac{\lambda_{\max}}{\lambda_{\min}}\right|} \frac{\|\delta_i^{(2)}\|_0}{\|\mathbf{D}_i\|_0} \leqq \sqrt{(n)} \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right| \frac{\|\delta_i^{(2)}\|}{\|\mathbf{D}_i\|},$$

where $n = \operatorname{rank} \mathbf{D}_i\mathbf{D}_i^{\mathsf{T}}$, $\lambda_{\max}$ *and* $\lambda_{\min}$ *stand for the maximal and minimal eigenvalues of the matrix* $\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}}$, *respectively.*

Proof. It is sufficient to prove only the estimates.

The equation (3.5)–(3.6) imply

$$\|\delta_{i,2}\| \leqq \|\mathbf{D}_i^{\mathsf{T}}(\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}})^{-1}\| \left(\|\delta_i^{(1)}\| + \|\delta_{i,1}\mathbf{D}_{i+1}\|\right),$$

$$\|\delta_{i,3}\| \leqq \|\mathbf{D}_i^{\mathsf{T}}(\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}})^{-1}\| \, \|\delta_i^{(2)}\|.$$

But

$$\|(\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}})^{-1} \mathbf{D}_i\|_0 \leqq \sqrt{\frac{1}{|\lambda_{\min}|}}$$

and

$$\|\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}}\|_0 \leqq \sqrt{|\lambda_{\max}|}.$$

This two inequalities together with

$$\|\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}}\|_0 \leqq \sqrt{(n)} \, \|\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}}\|$$

complete the proof.

Theorem 3.1 displays the important role of matrices $\mathbf{D}_i$ and $\mathbf{D}_i^{\mathsf{T}}(\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}})^{-1}$ for the numerical stability of the method of the transfer of conditions. We can expect that for "large" systems (1.1) such methods of the transfer of conditions will be "good" for which both the norms $\|\mathbf{D}_i\|$ and $\|\mathbf{D}_i^{\mathsf{T}}(\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}})^{-1}\|$ less than one at least for "sufficiently great" values of i.

In the book [1], the following concept of well conditioned systems is discussed.

Definition 3.2. *The system* (1.1) *with* $g_{12} = \ldots = g_{1p} = g_{N,N-p} = \ldots = g_{N,N-1} = 0$ *is said to be well conditioned if the norm of the solution is bounded independently of* N.

Although this type of numerical stability is appropriate for invariant imbedding like methods, for great values of N and methods of the transfer of conditions we have to discuss also this stability. Consider a tridiagonal matrix $\mathbf{G}$ and denote

$$g_{i,i-1} = c_i, \quad g_{i,i} = a_i \quad \text{and} \quad g_{i,i+1} = w_i.$$

We choose $j = 1$, hence $J = N - 2$ and

$$\mathbf{x}_i = [y_i, y_{i+1}]^{\mathrm{T}} \quad \text{for} \quad i = 1(1)\, J + 1\,,$$

$$\mathbf{A}_i = \begin{bmatrix} c_i & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}_i = \begin{bmatrix} a_1 & w_i \\ -1 & 0 \end{bmatrix}, \quad \mathbf{f}_i = \begin{bmatrix} b_i \\ 0 \end{bmatrix},$$

$$\mathbf{H}_i = \mathbf{A}_i^{-1}\mathbf{B}_i = \begin{bmatrix} \dfrac{a_i}{c_i} & \dfrac{w_i}{c_i} \\ -1 & 0 \end{bmatrix}, \quad \mathbf{h}_i = \mathbf{A}_i^{-1}\mathbf{f}_i = \begin{bmatrix} \dfrac{b_i}{c_i} \\ 0 \end{bmatrix}.$$

Thus the equations (1.8) imply

$$D_{i+1,1} = Z_i\left(D_{i,1}\frac{a_i}{c_i} - D_{i,2}\right) \quad \text{and} \quad D_{i+1,2} = Z_i D_{i,1}\frac{w_i}{c_i}, \tag{3.7}$$

$$d_{i+1} = Z_i\left(-d_i + D_{i,1}\frac{b_i}{c_i}\right) \tag{3.8}$$

and $D_{1,1} = 1$, $D_{1,2} = 0$, $d_1 = b_1$, $\mathbf{D}_i = [D_{i,1}, D_{i,2}]$.

It is shown (cf. [2]) that choosing

$$Z_i = \frac{c_i}{a_i - c_i D_{i,2}}$$

for every i such that the denominator is nonzero, the equations (3.7)–(3.8) are the equations of one step of Gaussian elimination. In particular,

$$D_{i,1} = 1\,.$$

Thus

$$\|\mathbf{D}_i^{\mathrm{T}}(\mathbf{D}_i\mathbf{D}_i^{\mathrm{T}})^{-1}\| = \frac{|1 + D_{i,2}|}{1 + D_{i,2}^2}$$

and this is less than one for $|D_{i,2}| > 1$ and less than or equal to two for $|D_{i,2}| < 1$. In each case this norm is bounded by a constant independently of i. We show that a necessary condition for the system (1.1) to be well conditioned is

$$|D_{i,2}| < 1\,. \tag{3.9}$$

Consider special vectors $\boldsymbol{b}$, namely $\boldsymbol{b} = \mathbf{e}_i$ where all elements of the vector $\mathbf{e}_i$ equal zero except for the i-th element which is equal to one. Denote by $\mathbf{y}^{(i)}$ the solution of the system (1.1) for this right hand side. Then any solution of the system (1.1) can be written in the form

$$\mathbf{y} = \sum_{i=1}^{N} b_i \mathbf{y}^{(i)}\,.$$

Equations (3.7)–(3.8) for $\boldsymbol{b} = \mathbf{e}_{N-1}$ imply

$$y_i^{(N-1)} = -y_{i+1}^{(N-1)} \frac{D_{i,2}}{D_{i,1}} \tag{3.10}$$

for $i < N - 1$ and

$$d_i = \frac{1}{a_i - c_i m_i}$$

for $i = N$, where m_i stands for $D_{i,2}/D_{i,1}$.

The first thing we should like to mention is that the necessary condition for the system to be well conditioned does not depend on the matrices $\boldsymbol{Z}_i$, i.e., on the method as the equation (3.10) implies. (The quantities m_i do not depend on $\boldsymbol{Z}_i$). Thus we can restrict ourselves to the Gaussian elimination. Also the equation (3.10) implies that the condition (3.9) is really necessary because if it is not fulfilled then $y_i^{(N-1)}$ is not bounded independently of $i < N - 1$.

It is not difficult to show that a sufficient condition to guarantee validity of (3.9) is

$$|c_i| + |w_i| \leqq |a_i|$$

and

$$|c_i| < |w_i|$$

for sufficiently great i. In this case also $|Z_i| < 1$.

If

$$|a_i| + |D_{i,2}|\,|c_i| < |w_i|$$

where $|D_{i,2}| > 1$ then also $|D_{i+1,2}| > 1$, and the system (1.1) is not well conditioned. Such systems exist as is shown in [3]. Hence we can conjecture that for some types of the matrix $\boldsymbol{G}$ there is hardly any possibility to construct direct methods which can compensate "ill conditioned" features of the system.

4. SOME EXAMPLES

In this last we show that the process of Gauss-Jordan elimination is a method of *Algorithm* $\mathcal{T}4$. For the sake of brevity we consider a tridiagonal matrix $\boldsymbol{G}$ which is symmetric and positive definite. Let us denote

$$g_{i,i} = a_i \quad \text{for} \quad i = 1(1)\,N$$

and

$$g_{i,i+1} = g_{i+1,i} = c_i \quad \text{for} \quad i = 1(1)\,N - 1\,.$$

We choose the parameter j to be $j = 1$ and $i_k = k$ for $k = 1(1)\,N - 1$, i.e., $K =$ $= N - 1$ and

$$\boldsymbol{x}_i = [y_i, y_{i+1}]^{\mathrm{T}} \quad \text{for} \quad i = 1(1)\,N - 1\,.$$

Then

$$\mathbf{H}_i = \begin{bmatrix} \dfrac{a_{i+1}}{c_i} & \dfrac{c_{i+1}}{c_i} \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{h}_i = \begin{bmatrix} \dfrac{b_{i+1}}{c_i} \\ 0 \end{bmatrix},$$

the matrices $\mathbf{Z}_i$ are chosen such that

$$D_{i,2} = c_i$$

where $\mathbf{D}_i = [D_{i,1}, D_{i,2}]$, i.e.,

$$Z_i = \frac{c_i}{D_{i,1}}, \quad \mathbf{Z}_i = [Z_i],$$

and

$$D_{i+1,1} = a_{i+1} - \frac{c_i^2}{D_{i,1}}, \quad d_{i+1} = b_{i+1} - c_i \frac{d_i}{D_{i,1}}$$

and under our assumptions it is easy to prove that $D_{i,1} \neq 0$. Then the matrices $\mathbf{R}_i$ are

$$\mathbf{R}_i = [0, 1].$$

Because $I_k = \{k\}$, (2.5) implies

$$u_k = 0 \quad \text{and} \quad \mathbf{U}_k = [1, 0].$$

Hence according to (2.10)

$$\mathbf{q}_{k,k} = \begin{bmatrix} d_k \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Q}_{k,k} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

while

$$\mathbf{Q}_k = \begin{bmatrix} D_{k,1} & c_k \\ 0 & 1 \end{bmatrix}.$$

Thus

(4.1)
$$\mathbf{Q}_k^{-1}\mathbf{Q}_{k,k} = \begin{bmatrix} -\dfrac{c_k}{D_{k,1}} & 0 \\ 1 & 0 \end{bmatrix}$$

and

(4.2)
$$\mathbf{Q}_k^{-1}\mathbf{q}_{k,k} = \begin{bmatrix} \dfrac{d_k}{D_{k,1}} \\ 0 \end{bmatrix}.$$

From (4.1) and (2.14) it follows that the matrix $\mathbf{Q}_{i,k}$ is of the form

(4.3)
$$\mathbf{Q}_{i,k} = \begin{bmatrix} 0 & 0 \\ Q_{i,k} & 0 \end{bmatrix} \quad \text{for} \quad i < k$$

and

$$Q_{i,k} = -c_k Q_{i,k-1}/D_{k,1}$$

because

$$\mathbf{Q}_{i,i} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and

$$Q_{i,k} = (-1)^{k-i} \frac{c_i}{D_{i,1}} \cdots \frac{c_k}{D_{k,i}} . \tag{4.4}$$

From (2.13) and (4.2) it follows that

$$\mathbf{q}_{i,k} = \begin{bmatrix} d_i \\ d_i^{(k-1)} + Q_{i,k-1} \dfrac{d_k}{D_{k,1}} \end{bmatrix}$$

where we have denoted

$$\mathbf{q}_{i,k-1} = \begin{bmatrix} d_i \\ d_i^{(k-1)} \end{bmatrix} .$$

The first element of the vector $\mathbf{q}_{i,k}$ does not change because (4.3) and (4.2) imply

$$\mathbf{Q}_{i,k-1} \mathbf{Q}_k^{-1} \mathbf{q}_{k,k} = \begin{bmatrix} 0 \\ \dfrac{Q_{i,k} d_k}{D_{k,1}} \end{bmatrix} .$$

Thus we can define

$$d_i^{(k)} = d_i^{(k-1)} + Q_{i,k-1} \frac{d_k}{D_{k,1}} \quad \text{for} \quad i < k , \tag{4.5}$$

$$d_i^{(i)} = d_i .$$

Hence the systems (2.15) are of the form

$$\begin{bmatrix} D_{i,1} & c_i \\ 0 & 1 \end{bmatrix} \mathbf{x}_i = \begin{bmatrix} d_i \\ d_i^{(k)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ Q_{i,k} & 0 \end{bmatrix} \mathbf{x}_{k+1} \quad \text{for} \quad i < k$$

and

$$\mathbf{x}_i = \begin{bmatrix} \dfrac{d_i}{D_{i,1}} - d_i^{(k)} \dfrac{c_i}{D_{i,1}} \\ d_i^{(k)} \end{bmatrix} + \begin{bmatrix} -c_i \dfrac{Q_{i,k}}{D_{i,1}} & 0 \\ Q_{i,k} & 0 \end{bmatrix} \mathbf{x}_{k+1}$$

For $k = K$ this can be written in the form

$$D_{i,1} y_i = d_i - d_i^{(K)} c_i \tag{4.6}$$

and (4.6) together with (4.5) and (4.4) is the process of Gauss-Jordan elimination.

The last method we should like to mention is the process of the so called even-odd reduction (cf. [5]). We show that this process is a method of *Algorithm* $\mathcal{T}5$.

Let the matrix $\mathbf{G}$ be block tridiagonal

$$\mathbf{G} = \begin{bmatrix} \mathbf{A} & \mathbf{I}_n & & & \\ \mathbf{I}_n & \mathbf{A} & \mathbf{I}_n & & \\ & \cdots & & & \\ & & \mathbf{I}_n & \mathbf{A} & \mathbf{I}_n \\ & & & \mathbf{I}_n & \mathbf{A} \end{bmatrix},$$

$$\mathbf{y} = [\mathbf{y}_0^{\mathrm{T}}, \ldots, \mathbf{y}_{N-1}^{\mathrm{T}}]^{\mathrm{T}} \quad \text{and} \quad \mathbf{b} = [\mathbf{b}_0^{\mathrm{T}}, \ldots, \mathbf{b}_{N-1}^{\mathrm{T}}]^{\mathrm{T}},$$

where $\mathbf{A}$ is a square matrix of order n and both $\mathbf{y}_i$ and $\mathbf{b}_i$ are n-dimensional vectors, $N = 2^t$, t is an integer. We choose $j = 1$ and $i_k = k$ for $k = 1(1)\,N - 1$, i.e.,

$$\mathbf{x}_i = [\mathbf{y}_{i-1}^{\mathrm{T}}, \mathbf{y}_i^{\mathrm{T}}]^{\mathrm{T}},$$

$$\mathbf{H}_i = \begin{bmatrix} \mathbf{A} & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{O}_{n,n} \end{bmatrix} \quad \text{and} \quad \mathbf{h}_i = \begin{bmatrix} \mathbf{b}_i \\ \mathbf{O} \end{bmatrix}.$$

The matrices $\mathbf{Z}_i$ are

$$\mathbf{Z}_i = -\mathbf{I}_n .$$

For every k we choose a special "boundary condition" for the k-th transfer to the left, namely

$$\mathbf{D}_k = [\mathbf{A}, \mathbf{I}_n] \quad \text{and} \quad \mathbf{d}_k = \mathbf{b}_{k-1} - \mathbf{y}_{k-2} .$$

Thus

$$\mathbf{D}_{k+1} = -\mathbf{D}_k \mathbf{H}_k = [-\mathbf{A}^2 + \mathbf{I}_n, -\mathbf{A}],$$

$$\mathbf{d}_{k+1} = \mathbf{b}_{k-1} - \mathbf{y}_{k-2} + \mathbf{A}\mathbf{b}_k$$

and we choose

$$\mathbf{R}_{k+1} = [\mathbf{I}_n, \mathbf{A}] .$$

Hence

$$\mathbf{u}_{k+1} = \mathbf{b}_{k+1} \quad \text{and} \quad \mathbf{U}_{k+1} = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_n \end{bmatrix}.$$

Then for $\mathbf{x}_{k+1}$ we have the equation

$$\begin{bmatrix} -\mathbf{A}^2 + \mathbf{I}_n, & -\mathbf{A} \\ \mathbf{I}_n & \mathbf{A} \end{bmatrix} \mathbf{x}_{k+1} = \begin{bmatrix} \mathbf{b}_{k-1} - \mathbf{y}_{k-2} + \mathbf{A}\mathbf{b}_k \\ \mathbf{b}_{k+1} \end{bmatrix} + \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I}_n \end{bmatrix} \mathbf{x}_{k+2} . \tag{4.7}$$

Choosing

$$\mathbf{R}_k = [\mathbf{I}_n, \mathbf{A}] \quad \text{and} \quad \mathbf{r}_k = \mathbf{b}_k - \mathbf{y}_{k+1}$$

we have for $\mathbf{x}_k$

$$\mathbf{Q}_k \mathbf{x}_k = \begin{bmatrix} \mathbf{A} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{A} \end{bmatrix} \mathbf{x}_k = \begin{bmatrix} \mathbf{b}_{k-1} - \mathbf{y}_{k-2} \\ \mathbf{b}_k \;\; - \mathbf{y}_{k+1} \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} \mathbf{O} & \mathbf{I}_n - \mathbf{A}^2 \\ \mathbf{I}_n & \mathbf{A} \end{bmatrix} \mathbf{x}_k = \begin{bmatrix} \mathbf{b}_{k-1} - \mathbf{y}_{k-2} - \mathbf{A}\mathbf{b}_k + \mathbf{A}\mathbf{y}_{k+1} \\ \mathbf{b}_k - \mathbf{y}_{k+1} \end{bmatrix}. \tag{4.8}$$

Thus (4.7) and (4.8) together yield

$$\mathbf{y}_{k-2} + (2\mathbf{I}_n - \mathbf{A}^2)\,\mathbf{y}_k + \mathbf{y}_{k+2} = \mathbf{b}_{k-1} - \mathbf{A}\mathbf{b}_k + \mathbf{b}_{k+1}\,, \tag{4.9}$$

$$\mathbf{y}_k + \mathbf{A}\mathbf{y}_{k+1} + \mathbf{y}_{k+2} = \mathbf{b}_{k+1}\,. \tag{4.10}$$

Thus the equations (4.9) form a system of $N/2$ equations for $N/2$ unknowns and continuing in this way we obtain just the so called even-odd reduction process.

References

[1] *Godunov S. K.* and *Rjabenkij V. S.*: Difference schemes. (Russian) Nauka, Moscow (1973).

[2] *Malina Ľ.*: General theory of direct methods for solving systems of equations with band matrices. (To appear in Aplikace matematiky).

[3] *Malina Ľ.*: Methods of the transfer of conditions and conditions of "well conditionedness". (Russian) In Numerical methods of linear algebra, ed. G. I. Marčuk, Novosibirsk (1977), 87—96.

[4] *Taufer J.*: Lösung der Randwertprobleme für Systeme von Linearen Differentialgleichungen. Rozpravy ČSAV, 83 (1973).

[5] *Hockney R. W.*: A fast direct solution of Poisson's equation using Fourrier analysis. J. of Association for Computing Machinery, 12 (1965), 95—113.

Souhrn

ZOBECNĚNÉ METODY PŘESUNU OKRAJOVÝCH PODMÍNEK

ĽUBOR MALINA

Tato práce je zobecněním autorovy práce [2]. Zobecňují se tzv. metody přesunu okrajových podmínek tak, aby algoritmus zahrnoval i přímé metody řešení soustav lineárních rovnic s pásovou maticí soustavy, které vedou na diagonalizaci původní matice soustavy. Je zde také zkoumána otázka numerické stability metod popsaných v [2]. Na závěr je ukázáno, jak lze volbou parametru zobecněného algoritmu dostat některé známé přímé metody (Gauss-Jordanova redukce).

Authors' address: Dr. *Ľubor Malina*, CSc., Ústav aplikovanej matematiky a výpočtovej techniky PF UK, Matematický pavilón, Mlynská dolina, 816 31 Bratislava.