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# SOLUTION OF ELLIPTIC PROBLEM WITH NOT FULLY SPECIFIED DIRICHLET BOUNDARY VALUE CONDITIONS AND ITS APPLICATION IN HYDRODYNAMICS

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## 1. INTRODUCTION

In this paper we shall deal with the solution of a problem connected with the investigation of two-dimensional models describing stream fields of an ideal incompressible fluid in multiply connected domains. It concerns mainly the irrotational stationary flows round a group of profiles which are inserted into the fluid. We can meet this problem e.g. in the study of stream fields in diffuser channels of turbines with rings for the compensation of undesirable development of the boundary layer. See [3].

We consider not only plane stream fields as the authors who use the "singularity" or complex functions methods (see e.g. [4, 6, 8]), but we study the situation including plane and axially-symmetric stream fields and moreover, the flows in a layer of variable thickness, which represent the fundamental two-dimensional linear problems of inner hydrodynamics.

## 2. FORMULATION AND SOLUTION OF THE PROBLEM

First, let us introduce the notation. Let $\Omega \subset E_2$ be an $(r + 1)$-multiply connected bounded domain with a Lipschitz boundary $\partial\Omega$. $E_2(E_k)$ is the two-dimensional ($k$-dimensional) Euclidean space. The coordinates of points in $E_2$ will be denoted by $x$, $y$. Let the components $C_0, \ldots, C_r$ of $\partial\Omega$ be geometric images of Jordan curves and let $C_i \subset \operatorname{Int} C_0$ for $i = 1, \ldots, r$. Further, let $\partial\Omega$ be divided into three parts $(\partial\Omega)_n$, $(\partial\Omega)_t$ and $K$:

$$\partial\Omega = (\partial\Omega)_n \cup (\partial\Omega)_t \cup K ,$$

where the union is disjoint, $K$ is a finite set and the sets $C_i \cap (\partial\Omega)_n$ and $C_i \cap (\partial\Omega)_t$ have finite numbers of components, which are open arcs ($i = 0, \ldots, r$). We suppose that $C_i \cap (\partial\Omega)_n \neq \emptyset$ for $i = 0, \ldots, r$. The closure of the set $\Omega$ will be denoted by $\bar{\Omega}$. See Fig. 1, where $C_0 = L_1 \cup L_2 \cup \Gamma_1 \cup \Gamma_2$, $C_0 \cap (\partial\Omega)_n = L_1 \cup L_2$, $C_0 \cap (\partial\Omega)_t = \Gamma_1 \cup \Gamma_2$, $K = \{A, B, C, D\}$.

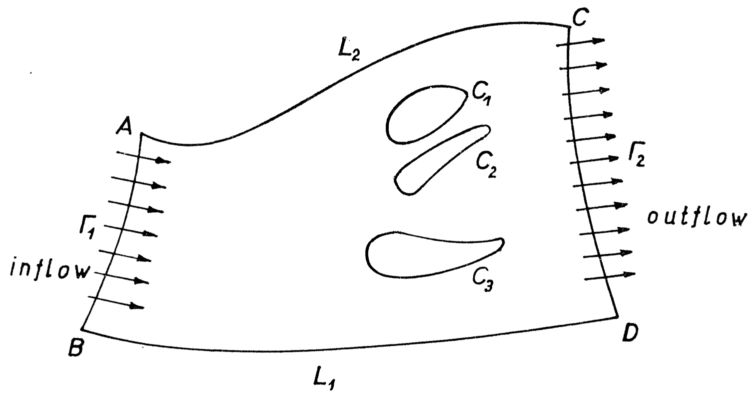


Fig. 1.

Let us consider the following boundary value problem:

$$\text{(1)} \qquad \frac{\partial}{\partial x}\left(h(x, y)\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(h(x, y)\frac{\partial\psi}{\partial y}\right) = -\tilde{\omega}(x, y) \text{ in } \Omega\,,$$

$$\text{(2)} \qquad \frac{\partial\psi}{\partial n}\bigg| (\partial\Omega)_t = \varphi_t\,,$$

$$\text{(3)} \qquad \psi \mid (\partial\Omega)_n \cup K = \tilde{\psi}\,.$$

Here $h$, $\tilde{\omega}$ are functions defined in the domain $\Omega$, $h$ is continuous and bounded and satisfies the inequalities

$$\text{(4)} \qquad 0 < h_0 \leqq h(x, y) \leqq h_1 < +\infty \text{ in } \Omega\,,$$

where $h_0$, $h_1$ are constants. $\partial/\partial n$ denotes the derivative in the direction of the outer normal to $\partial\Omega$. $\varphi_t$ and $\tilde{\psi}$ are given functions defined in the set $(\partial\Omega)_t$ and $(\partial\Omega)_n$, respectively. Let $W_2^1(\Omega)$ be the well-known Sobolev space. Then, provided there exists a function $\psi_0 \in W_2^1(\Omega)$ such that $\psi_0 \mid (\partial\Omega)_n = \tilde{\psi}$ and $\varphi_t \in L_2((\partial\Omega)_t)$, $\tilde{\omega} \in L_2(\Omega)$, the problem (1), (2), (3) has a unique weak solution $\psi \in W_2^1(\Omega)$. See [5, 7].

If the function $h$ has continuous and bounded first order partial derivatives in $\Omega$, then we can write the equation (1) in the form

$$\Delta\psi + \frac{1}{h}\nabla h\,.\,\nabla\psi = -\omega\,,$$

which is a special case of the uniformly elliptic equation

(5) $$a(x, y)\frac{\partial^2\psi}{\partial x^2} + 2b(x, y)\frac{\partial^2\psi}{\partial x\,\partial y} + c(x, y)\frac{\partial^2\psi}{\partial y^2} + + d(x, y)\frac{\partial\psi}{\partial x} + e(x, y)\frac{\partial\psi}{\partial y} = -\omega(x, y)\,.$$

Functions $a, b, \ldots, e$ are continuous and bounded in $\Omega$ and

$$a(x, y)\,\xi^2 + 2b(x, y)\,\xi\eta + c(x, y)\,\eta^2 \geqq C(\xi^2 + \eta^2)$$

for all $(x, y) \in \Omega$ and all $\xi, \eta \in E_1$. $C$ is a constant independent of $x, y, \xi, \eta$. Let us now consider the problem (5), (2), (3).

In hydrodynamics we can meet the problem of the determination of the Dirichlet condition (3) ([2, 3]). The function $\tilde{\psi}$ is in every set $C_i \cap (\partial\Omega)_n$ given by the relation

(6) $$\tilde{\psi} \mid C_i \cap (\partial\Omega)_n = \tilde{\psi}_i + q_i\,, \quad i = 0, \ldots, r\,,$$

where the functions $\tilde{\psi}_0, \ldots, \tilde{\psi}_r$ and the constant $q_0$ are given, but $q_1, \ldots, q_r$ are unknown real constants, which must be determined so that a solution $\psi$ of the problem (5), (2), (3) satisfies some other demands. There exist various criteria for the determination of the constants $q_i$. One of the most common will be used here.

It is necessary to impose some additional smoothness conditions on a solution of our problem. Therefore we shall introduce the following assumptions.

Assumption (A1). Let $a_i \in C_i \cap (\partial\Omega)_n$, $i = 1, \ldots, r$, be given points. Let the outer normal to $\partial\Omega$ exist at every point $u \in (\partial\Omega)_t \cup \{a_1, \ldots, a_r\}$. For every $u \in (\partial\Omega)_t \cup \cup \{a_1, \ldots, a_r\}$ let there exist a closed circle $K_u \subset \bar{\Omega}$ such that $u \in \partial K_u$.

For given $\Omega, (\partial\Omega)_n, (\partial\Omega)_t, a, \ldots, e, \omega, \varphi_t$ let $\mathscr{P}(\Omega, (\partial\Omega)_n, (\partial\Omega)_t, a, \ldots, e, \omega, \varphi_t)$ denote the class of functions $\tilde{\psi}$ defined in $(\partial\Omega)_n$, for which the solution $\psi \in C(\bar{\Omega}) \cap \cap C^2(\Omega)$ of the problem (5), (2), (3) exists and has a finite derivative $\partial\psi/\partial n$ at every point $a_i$, $i = 1, \ldots, r$.

Assumption (A2). Let $\mathscr{P}(\Omega, (\partial\Omega)_n, (\partial\Omega)_t, a, \ldots, e, \omega, \varphi_t) \neq \emptyset$. If $\tilde{\psi} : (\partial\Omega)_n \to E_1$ and $\tilde{\psi} \mid C_i \cap (\partial\Omega)_n$ is constant ($i = 0, \ldots, r$), then let

$$\tilde{\psi} \in \mathscr{P}(\Omega, (\partial\Omega)_n, (\partial\Omega)_t, a, \ldots, e, 0, 0)\,.$$

In the case $(\partial\Omega)_t = \emptyset$ (Dirichlet problem), we can find in [5] sufficient conditions imposed on the domain $\Omega$ and on the functions $a, \ldots, e$ and $\omega$, under which the assumption (A2) is satisfied.

Our problem (we shall denote it by (P)) consists in the following: Let $v_1, \ldots, v_r \in E_1$ and $\tilde{\psi}_i : C_i \cap (\partial\Omega)_n \to E_1$ ($i = 0, \ldots, r$) be given, $q_0 = 0$. We want to find a function

$\psi : \bar{\Omega} \to E_1$ and constants $q_1, \ldots, q_r \in E_1$ satisfying (2), (3), (5), (6) and

$$\frac{\partial \psi}{\partial n}(a_i) = v_i\,, \quad i = 1, \ldots, r\,. \tag{7}$$

In the investigation of this problem we shall use the well-known maximum principle for solutions of elliptic equations.

**Theorem 1.** *Let $(x_0, y_0) \in \bar{\Omega}$ be a maximum point of a solution $\varphi \in C(\bar{\Omega}) \cap C^2(\Omega)$ of the equation (5), where $\omega = 0$. 1) If $(x_0, y_0) \in \Omega$, then $\varphi$ is constant in $\bar{\Omega}$. 2) If $(x_0, y_0) \in \partial\Omega$ is such that the derivative $(\partial\varphi/\partial n)(x_0, y_0)$ and a closed circle $K \subset \bar{\Omega}$ with $(x_0, y_0) \in \partial K$ exist, then $\varphi$ is either constant in $\bar{\Omega}$ or $(\partial\varphi/\partial n)(x_0, y_0) > 0$. (See [1].)*

Remark 1. As a simple consequence of Theorem 1 we get the uniqueness of the solution of the problem (5), (2), (3) in the space $C(\bar{\Omega}) \cap C^2(\Omega)$.

Let (A1) and (A2) be valid. Given functions $\tilde{\psi}_i : C_i \cap (\partial\Omega)_n \to E_1$, $i = 0, \ldots, r$, denote by $\vartheta : (\partial\Omega)_n \to E_1$ the function defined by the relations $\vartheta \mid C_i \cap (\partial\Omega)_n = \tilde{\psi}_i$, $i = 0, \ldots, r$. We assume that

$$\vartheta \in \mathscr{P}(\Omega, (\partial\Omega)_n, (\partial\Omega)_t, a, \ldots, e, \omega, \varphi_t)\,. \tag{8}$$

Let $\psi_0$ be the solution of the equation (5) with the boundary value conditions

$$\psi_0 \mid (\partial\Omega)_n = \vartheta\,, \qquad \frac{\partial \psi_0}{\partial n}\Big| (\partial\Omega)_t = \varphi_t\,, \tag{9a}$$

let $\psi_i$ $(i = 1, \ldots, r)$ be the solution of the equation (5) with $\omega = 0$, satisfying the boundary value conditions of the form

$$\psi_i \mid C_j \cap (\partial\Omega)_n = \delta_{ij}\,, \quad j = 0, .., r\,, \qquad \frac{\partial \psi_i}{\partial n}\Big| (\partial\Omega)_t = 0\,. \tag{9b}$$

If $q_1, \ldots, q_r \in E_1$ are arbitrary numbers, then

$$\psi = \psi_0 + \sum_{i=1}^{r} q_i \psi_i \tag{10}$$

is a unique solution of the problem (2), (3), (5), $\psi \in C(\bar{\Omega}) \cap C^2(\Omega)$ and a finite derivative $(\partial\psi/\partial n)(a_i)$ exists for $i = 1, \ldots, r$, as follows from (A2) and (8). Now, it is evident that the problem (P) is equivalent to the determination of constants $q_1, \ldots, q_r$

so that the function (10) fulfils the conditions (7). If we put (10) into (7), we get a system of linear algebraic equations

(11) $$\sum_{j=1}^{r} \alpha_{ij} q_j = \beta_i \,, \quad i = 1, \ldots, r \,,$$

where

(12) $$\alpha_{ij} = \frac{\partial \psi_j}{\partial n} (a_i) \,, \quad \beta_i = v_i - \frac{\partial \psi_0}{\partial n} (a_i) \,.$$

Let us prove the assertion on the solvability of the problem (P).

**Theorem 2.** *If we suppose that the assumptions* (A1), (A2) *and* (8) *are fulfilled, then the matrix of the system* (11) *is regular and the problem* (P) *has a unique solution. This solution is given by the formula* (10) *with constants* $q_1, \ldots, q_r$ *which are the solution of the system* (11).

Proof. Relations (7) and (10) define a mapping $f : E_r \to E_r$, $f(q_1, \ldots, q_r) = (v_1, \ldots, v_r)$. It is evident that the problem (P) has a unique solution for every $(v_1, \ldots, v_r) \in E_r$ if and only if $f$ is a $1 - 1$ mapping and $f(E_r) = E_r$. This property is obviously equivalent to the regularity of the matrix $(\alpha_{ij})_{i,j=1}^{r}$ of the system (11).

Let the matrix $(\alpha_{ij})$ be singular. Then we can suppose that there exist numbers $\lambda_1, \ldots, \lambda_r \in E_1$ such that e.g. $\lambda_1 = 1$ and

$$\sum_{j=1}^{r} \alpha_{ij} \lambda_j = 0 \quad \text{for} \quad i = 1, \ldots, r \,,$$

or, in view of (12)

$$\sum_{j=1}^{r} \lambda_j \frac{\partial \psi_j}{\partial n} (a_i) = 0 \quad \text{for} \quad i = 1, \ldots, r \,.$$

Let us put

$$\varphi = \sum_{j=1}^{r} \lambda_j \psi_j \,.$$

It is easy to see that $\varphi$ is a solution of the equation (5) with $\omega = 0$, $\varphi \in C(\bar{\Omega}) \cap C^2(\Omega)$ and

(13) $$\varphi \mid C_0 \cap (\partial \Omega)_n = 0 \,, \quad \varphi \mid C_1 \cap (\partial \Omega)_n = 1 \,,$$

(14) $$\frac{\partial \varphi}{\partial n} (\partial \Omega)_t = 0 \,, \quad \frac{\partial \varphi}{\partial n} (a_i) = 0 \quad \text{for} \quad i = 1, \ldots, r \,.$$

$\varphi$ is not constant and achieves its maximum in $\bar{\Omega}$. Theorem 1, the assumption (A1) and the first equality in (14) imply that the maximum point of $\varphi$ lies in the set $(\partial \Omega)_n \cup K$. Since $\varphi$ is constant in every set $C_i \cap (\partial \Omega)_n$, $i = 0, \ldots, r$ (see (9b)), one of $a_i$

($i = 1, \ldots, r$) is necessarily a maximum point of $\varphi$ in view of (13). From Theorem 1 it follows that the inequality

$$\frac{\partial \varphi}{\partial n}(a_i) > 0$$

holds for this point $a_i$, which is a contradiction to (14).

Remark 2. If $C_{i_0} \cap (\partial\Omega)_n = \emptyset$ for some $i_0 \in \{1, \ldots, r\}$, then we do not consider the condition (7) for this $i_0$. The function $\psi$ satisfies the Neumann condition in the whole set $C_{i_0}$ and (11) reduces to a system of $s$ equations for $s$ unknowns, where $s < r$ is the number of all curves $C_i$ such that $C_i \cap (\partial\Omega)_n \neq \emptyset$. Theorem 2 remains valid.

## 3. APPLICATION IN HYDRODYNAMICS

Let us consider again the boundary value problem (1), (2), (3). We shall easily find out that for every solution $\psi$ of the equation (1) with $\tilde{\omega} = 0$ in $\Omega$ the functions

$$v_x = h\frac{\partial \psi}{\partial y}, \quad v_y = -h\frac{\partial \psi}{\partial x} \tag{15}$$

are the solution of the system of equations

$$\frac{\partial\left(\frac{1}{h}v_x\right)}{\partial x} + \frac{\partial\left(\frac{1}{h}v_y\right)}{\partial y} = 0\,, \tag{16}$$

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0 \tag{17}$$

for the irrotational flow of an ideal incompressible fluid. (16) is the continuity equation, (17) is the condition of the irrotational flow. $h = 1$ or $h = 1/y$ in $\Omega$ for plane or axially-symmetric flows respectively. In the latter case the assumption (4) implies that the axis of symmetry $x$ does not intersect the stream field. If $h = h(x, y)$ is a continuous function satisfying (4), then the equation (1) describes two-dimensional or axially-symmetric flows in a layer of a variable thickness $1/h$.

If the function $\tilde{\omega}$ is not identically equal to zero, then the functions defined by the relations (15) fulfil the equations (16) and

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \tilde{\omega}\,. \tag{18}$$

$\tilde{\omega}$ can be characterized as the vorticity of a stream field. It is necessary to emphasize that the system (16), (18) does not describe rotational stream fields in general, since it is necessary to satisfy the Euler equations of motion (see [2]). $\psi$ is the so called stream function.

The condition (2) corresponds to the tangential component $V_t$ of velocity at the points of the set $(\partial\Omega)_t$. The function $\tilde{\psi}$ in (3) can be determined from the given normal component $V_n$ of velocity in $(\partial\Omega)_n$ uniquely in every set $C_i \cap (\partial\Omega)_n$ $(i = 1, \ldots, r)$ up to an additive constant $q_i$, which is not known in advance. As a simple example, the plane flow in a bounded domain round $r$ fixed and impermeable profiles can be used. The stream function $\psi$ is on the boundary of each profile equal to a constant, which is, however, unknown. One of the possible criteria for the determination of these constants is the knowledge of the so called trailing points $a_i$, where the velocity is supposed to be equal to zero, from which we get the set of conditions

$$\frac{\partial\psi}{\partial n}(a_i) = 0\,, \quad i = 1, \ldots, r\,.$$

In more general situations, e.g. for moving profiles, we get the conditions (7).

Remark 3. In paragraph 2, the method of solution of the problem (P) is included among other. It is possible to find the functions $\psi_0, \ldots, \psi_r$ and then, in view of Theorem 2, to get the solution sought in the form (10). We can add that in the report [3] a direct numerical method for the approximate solution of the problem (P) was suggested, which was also applied to the solution of an analogous nonlinear problem of subsonic compressible flow.

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## Souhrn

# ŘEŠENÍ ELIPTICKÉHO PROBLÉMU S NESPECIFIKOVANÝMI DIRICHLETOVSKÝMI OKRAJOVÝMI PODMÍNKAMI A JEHO APLIKACE V HYDRODYNAMICE

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V článku je řešen smíšený okrajový problém pro lineární parciální diferenciální rovnici eliptického typu ve vícenásobně souvislé oblasti. Dirichletov ké podmínky jsou na komponentách hranice oblasti dané až na aditivní konstanty, předem neznámé. Tyto konstanty je třeba určit spolu s řešením okrajové úlohy tak, aby byly splněny jisté doplňující podmínky. Byla dokázána existence a jednoznačnost řešení této úlohy za předpokladu splnění jisté hladkosti řešení okrajové úlohy se známými okrajovými podmínkami. Výsledky mají bezprostřední aplikace v hydrodynamice při řešení obtékání soustavy profilů.



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