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ON ASYMPTOTIC STABILITY OF PASSIVE LINEAR ELECTRICAL NETWORKS

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1. INTRODUCTION

Given an oriented graph G with branches (edges) $v_1, \ldots, v_r$ and nodes (vertices) $u_1, \ldots, u_s$, let us denote $\mathbf{v} = (v_1, \ldots, v_r)^\mathsf{T}$, $\mathbf{u} = (u_1, \ldots, u_s)^\mathsf{T}$. Let $\mathbf{c}$ be a real vector of type $(r, 1)$. Then the expression $K = \mathbf{c}^\mathsf{T}\mathbf{v}$ will be called a 1-*complex*. If $\tilde{K} = \tilde{\mathbf{c}}^\mathsf{T}\mathbf{v}$ si also a 1-complex, α, $\tilde{\alpha}$ real numbers, let us define $\alpha K + \tilde{\alpha}\tilde{K} = (\alpha\mathbf{c} + \tilde{\alpha}\tilde{\mathbf{c}})^\mathsf{T}\,\mathbf{v}$. We put $K = \mathbf{c}^\mathsf{T}\mathbf{v} = 0$ if and only if $\mathbf{c} = \mathbf{o}$. We call the complexes $K_1, \ldots, K_m$ *linearly independent*, if $\sum_{i=1}^{r} \delta_i K_i = 0$ implies that $\delta_i = 0$, $i = 1, \ldots, r$. Similarly, the expression $L = \mathbf{c}^\mathsf{T}\mathbf{u}$, where $\mathbf{c}$ is a real vector of type $(s, 1)$, will be called a 0-*complex*. The notions of $\alpha L + \tilde{\alpha}\tilde{L}$, $L = 0$ and linear independence are defined analogously.

For each branch v of G we define $\partial v = u_2 - u_1$, where $u_2(u_2)$ is the terminal (initial) node of the branch v. For an arbitrary 1-complex $K = \mathbf{c}^\mathsf{T}\mathbf{v}$ we define $\partial K = \sum_{i=1}^{r} c_i\, \partial v_i$. If $\partial K = 0$, then the 1-complex K will be called a *cycle*.

Lemma 1. *Let $K = \mathbf{c}^\mathsf{T}\mathbf{v}$ be a cycle. Then there exist loops $K_i = \mathbf{d}_i^\mathsf{T}\mathbf{v}$, $i = 1, \ldots, m$ such that*

1. $K = \sum_{i=1}^{m} \alpha_i \mathbf{d}_i^\mathsf{T}\mathbf{v}$,

2. *if we denote $\mathbf{d}_i = (d_{i1}, \ldots, d_{ir})^\mathsf{T}$, $\mathbf{c} = (c_1, \ldots, c_r)^\mathsf{T}$, then $d_{ij} \neq 0 \Rightarrow c_j \neq 0$ for $i = 1, \ldots, m$, $j = 1, \ldots, r$.*

Proof may be found in [2], Theorem 1.2. From Lemma 1 one obtains easily

Lemma 2. *Let $\mathbf{B}$ be a real diagonal positive semidefinite matrix of type (r, r). Then the condition $\mathbf{d}^\mathsf{T}\mathbf{B}\mathbf{d} > 0$ holds for every loop $\mathbf{d}^\mathsf{T}\mathbf{v}$ if and only if $\mathbf{c}^\mathsf{T}\mathbf{B}\mathbf{c} > 0$ holds for every nonzero cycle $\mathbf{c}^\mathsf{T}\mathbf{v}$.*

The *incidence matrix* $\mathbf{A} = (a_{ik})$ (of type (r, s)) of a graph G is defined by the conditions

$a_{ik} = 1$, if u_k is the terminal node of the branch v_i,
$a_{ik} = -1$, if u_k is the initial node of the branch v_i,
$a_{ik} = 0$, if u_k is not incident with the branch v_i.

Lemma 3. *Let* $K = \mathbf{c}^\mathsf{T}\mathbf{v}$; *then* K *is a cycle if and only if* $\mathbf{A}^\mathsf{T}\mathbf{c} = \mathbf{o}$.

Proof is evident.

Let us denote by $\mathbf{X}$ the matrix of type (r, n) the columns of which form a complete system of linearly independent solutions of the equation $\mathbf{A}^\mathsf{T}\mathbf{x} = \mathbf{o}$. Then the following statement is true:

Lemma 4. *a) The elements of the vector* $\mathbf{X}^\mathsf{T}\mathbf{v}$ *form a complete set of linearly independent cycles of the graph* G.

b) If $\mathbf{c}^\mathsf{T}\mathbf{v}$ *is a cycle, then there exists a real vector* $\mathbf{w}$ *such that* $\mathbf{c} = \mathbf{X}\mathbf{w}$.

Proof see in [1], Theorem 1.1.

Let G be an oriented graph, $\mathbf{R}, \mathbf{L}, \mathbf{S}$ real matrices of type (r, r). Then the ordered tetrad $(G, \mathbf{R}, \mathbf{L}, \mathbf{S})$ will be called a *network*.

Let us denote by R the field of rational functions of complex variable p with real coefficients. If $\mathbf{M}$ is a matrix the elements of which belong to R, we call it a matrix *over* R.

Let a network $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$ be given, let $\mathbf{e}$ be a vector of type $(r, 1)$ over R, let $\mathbf{i}_0, \mathbf{q}_0$ be constant real vectors of type $(r, 1)$. Then a vector $\mathbf{i}$ of type $(r, 1)$ over R is said to be a *solution of the network* N *corresponding to the vector* $\mathbf{e}$ *and initial vectors* $\mathbf{i}_0, \mathbf{q}_0$, *if the following conditions are satisfied*:

$(K1)$ $\mathbf{A}^\mathsf{T}\mathbf{i} = \mathbf{o}$,

$(K2)$ $\mathbf{c}^\mathsf{T}(\mathbf{L}p + \mathbf{R} + \mathbf{S}p^{-1}) = \mathbf{c}^\mathsf{T}(\mathbf{e} + \mathbf{L}\mathbf{i}_0 - \mathbf{S}\mathbf{q}_0 p^{-1})$ for every cycle $\mathbf{c}^\mathsf{T}\mathbf{v}$ of the graph G.

Theorem 1. *Let a network* $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$ *be given, let* $\mathbf{X}$ *be a matrix the columns of which form a complete set of linearly independent solutions of the equation* $\mathbf{A}^\mathsf{T}\mathbf{x} = \mathbf{o}$. *Then the solution of the network* N *corresponding to the vector* $\mathbf{e}$ *and initial vectors* $\mathbf{i}_0, \mathbf{q}_0$ *(if it exists) is given by*

$$\mathbf{i} = \mathbf{X}[\mathbf{X}^\mathsf{T}(\mathbf{L}p + \mathbf{R} + \mathbf{S}p^{-1})\,\mathbf{X}]^{-1}\,\mathbf{X}^\mathsf{T}(\mathbf{e} + \mathbf{L}\mathbf{i}_0 - \mathbf{S}\mathbf{q}_0 p^{-1})\,.$$

Proof follows from [1], Theorem 1.3.

A network $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$ will be called *passive*, if the following conditions are fulfilled:

a) the matrices $\mathbf{R}, \mathbf{S}$ are diagonal,

b) the matrices $\mathbf{R}, \mathbf{L}, \mathbf{S}$ are positive semidefinite.

Theorem 2. *Let $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$ be a passive network. If*

$$\mathbf{c}^{\mathsf{T}}(\mathbf{R} + \mathbf{L} + \mathbf{S})\,\mathbf{c} > 0$$

for every nonzero cycle $\mathbf{c}^{\mathsf{T}}\mathbf{v}$ of the graph G, then for any vectors $\mathbf{e}, \mathbf{i}_0, \mathbf{q}_0$ there exists a unique solution $\mathbf{i}$ of N.

Proof follows from [1], Theorem 5.2.

Let $\mathbf{Z}$ be a matrix over R. A complex number α will be called a pole of m-th order of the matrix $\mathbf{Z}$, if α is a pole of m-th order of at least one element of $\mathbf{Z}$ and a pole of at most m-th order of each element of $\mathbf{Z}$.

Let us denote by $\mathfrak{G}$ the set of all complex numbers with positive real part and by $\overline{\mathfrak{G}}$ its closure (∞ belongs to $\overline{\mathfrak{G}}$). Let $\mathfrak{S}_n$ be the set of all symmetrical matrices $\mathbf{Z}$ over R of type (n, n) which fulfil the condition

$$\operatorname{Re} \mathbf{x}^{\mathsf{T}}\mathbf{Z}\mathbf{x} \geqq 0$$

for every real vector $\mathbf{x}$ of type $(n, 1)$ and for any $p \in \mathfrak{G}$ which is not a pole of $\mathbf{Z}$. Let $\mathfrak{P}_n$ be the set of all matrices belonging to $\mathfrak{S}_n$ which fulfil the condition

$$\operatorname{Re} \mathbf{x}^{\mathsf{T}}\mathbf{Z}\mathbf{x} > 0$$

for every real nonzero vector $\mathbf{x}$ of type $(n, 1)$ and for every $p \in \mathfrak{G}$ which is not a pole of $\mathbf{Z}$.

Obviously: a) $\mathbf{Z}_1, \mathbf{Z}_2 \in \mathfrak{S}_n \Rightarrow \alpha_1\mathbf{Z}_1 + \alpha_2\mathbf{Z}_2 \in \mathfrak{S}_n$ provided $\alpha_1, \alpha_2 \geqq 0$,

b) $\mathbf{Z}_1 \in \mathfrak{S}_n,\ \mathbf{Z}_2 \in \mathfrak{P}_n \Rightarrow \mathbf{Z}_1 + \mathbf{Z}_2 \in \mathfrak{P}_n$,

c) in particular, every positive (semi-)definite matrix belongs to $(\mathfrak{S}_n)\ \mathfrak{P}_n$.

Theorem 3. *If $\mathbf{Z} \in \mathfrak{S}_n$, then there exist real numbers $\omega_1, \ldots, \omega_m$ and constant matrices $\mathbf{H}_k \in \mathfrak{S}_n$, $k = 0, 1, \ldots, m$, such that*

$$\mathbf{Z}(p) = \tilde{\mathbf{Z}}(p) + \mathbf{H}_0 p + \sum_{k=1}^{m} \mathbf{H}_k \frac{p}{p^2 + \omega_k^2},$$

where $\tilde{\mathbf{Z}} \in \mathfrak{S}_n$ has no poles in $\overline{\mathfrak{G}}$.

Theorem 4. *Let $\mathbf{Z} \in \mathfrak{S}_n$. Then $\mathbf{Z} \in \mathfrak{P}_n$ if and only if* $\det \mathbf{Z} \neq 0$ *for every $p \in \mathfrak{G}$.*

Theorem 5. *If $\mathbf{Z} \in \mathfrak{P}_n$ then $\mathbf{Z}^{-1}$ exists and $\mathbf{Z}^{-1} \in \mathfrak{P}_n$.*

Theorem 6. *If $\mathbf{Z} \in \mathfrak{S}_n$ and $\mathbf{C}$ is any real constant matrix of type (n, k), then $\mathbf{C}^{\mathsf{T}}\mathbf{Z}\mathbf{C} \in \mathfrak{S}_k$.*

Proofs of Theorems 3–6 can be found in [1], Chap. 4.

2. A CRITERION OF ASYMPTOTIC STABILITY OF PASSIVE NETWORK

Let $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$ be a passive network. The network N will be called *asymptotically stable* if for any real vectors $\mathbf{i}_0, \mathbf{q}_0$ the solution $\mathbf{i}$ of the network N corresponding to the vector $\mathbf{e} = \mathbf{o}$ and initial conditions $\mathbf{i}_0, \mathbf{q}_0$ exists and has no poles in $\overline{\mathfrak{G}}$.

Remark. If the conditions $(K1), (K2)$ are interpreted as Laplace transforms of Kirchhoff's laws, then one can easily prove that for any solutions $\mathbf{i}_1, \mathbf{i}_2$ of N corresponding to the same vector $\mathbf{e}_0$ the difference $\mathbf{i}_1 - \mathbf{i}_2$ (which is a solution of N corresponding to $\mathbf{e} = \mathbf{o}$) has no poles in $\overline{\mathfrak{G}}$ if and only if

$$\lim_{t \to \infty} \|\mathscr{L}^{-1}(\mathbf{i}_1)(t) - \mathscr{L}^{-1}(\mathbf{i}_2)(t)\| = 0 .$$

Theorem 7. *Let $N = (G, \mathbf{R}, \mathbf{L}, \mathbf{S})$ be a passive network. Suppose that the following conditions are fulfilled for every nonzero cycle $\mathbf{c}^{\mathsf{T}}\mathbf{v}$ of the graph G:*

1. $\mathbf{c}^{\mathsf{T}}(\mathbf{R} + \mathbf{S})\, \mathbf{c} > 0$,

2. $\mathbf{c}^{\mathsf{T}}(\mathbf{R} + \mathbf{L})\, \mathbf{c} > 0$,

3. *if $\mathbf{c}^{\mathsf{T}}\mathbf{R}\mathbf{c} = 0$ then there exists a nonzero cycle $\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{v}$ of G such that the conditions $\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{S}\mathbf{c} \neq 0$ and $\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{L}\mathbf{c} = 0$ are simultaneously fulfilled.*

Then the network N is asymptotically stable.

Proof.

Lemma 5. *Under the same assumptions as in Theorem 7,*

$$\mathbf{W}(p) = \mathbf{X}^{\mathsf{T}}\mathbf{Z}(p)\, \mathbf{X} \in \mathfrak{P}_n .$$

Proof. The network N is passive and hence by Theorem 6 $\mathbf{W} \in \mathfrak{S}_n$. It follows from Theorem 4 that $\mathbf{W} \in \mathfrak{P}_n$ if and only if $\det \mathbf{W} \neq 0$ in $\mathfrak{G}$. Suppose that there exists $p_0 \in \mathfrak{G}$ such that $\det \mathbf{W}(p_0) = 0$. Then there exists a nonzero vector $\mathbf{w}$ such that $\mathbf{W}(p_0)\, \mathbf{w} = \mathbf{o}$, hence $\operatorname{Re}(\mathbf{w}^{\mathsf{T}}\mathbf{X}^{\mathsf{T}}\mathbf{Z}(p_0)\, \mathbf{X}\mathbf{w}) = 0$, which for $p_0 = p_0' + ip_0''$, $\mathbf{c} = \mathbf{X}\mathbf{w}$ and nonzero cycle $\mathbf{c}^{\mathsf{T}}\mathbf{v}$ yields

$$\mathbf{c}^{\mathsf{T}}\mathbf{R}\mathbf{c} + p_0'\mathbf{c}^{\mathsf{T}}\mathbf{L}\mathbf{c} + \frac{p_0'}{|p_0|^2}\, \mathbf{c}^{\mathsf{T}}\mathbf{S}\mathbf{c} = 0 . \tag{1}$$

By hypothesis, all terms on the left-hand side of (1) are non-negative and cannot be simultaneously zero, which is a contradiction.

Lemma 6. *Under the same assumptions as in Theorem 7,*

$$\det \mathbf{W}(i\omega_0) \neq 0$$

for every real $\omega_0 \neq 0$.

Proof. Suppose $\det \mathbf{W}(i\omega_0) = 0$, ω_0 being a real nonzero number. Then there exists a real nonzero vector $\mathbf{w}$ such that

(2) $$\mathbf{W}(i\omega_0)\,\mathbf{w} = \mathbf{o}$$

and therefore for a nonzero cycle $\mathbf{c}^{\mathsf{T}}\mathbf{v}$, where $\mathbf{c} = \mathbf{X}\mathbf{w}$, it holds

$$\mathbf{c}^{\mathsf{T}}\mathbf{R}\mathbf{c} + i\left(\omega_0\mathbf{c}^{\mathsf{T}}\mathbf{L}\mathbf{c} - \frac{1}{\omega_0}\,\mathbf{c}^{\mathsf{T}}\mathbf{S}\mathbf{c}\right) = 0$$

and hence $\mathbf{c}^{\mathsf{T}}\mathbf{R}\mathbf{c} = 0$.

By assumption 3) of Theorem 7 there exists a cycle $\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{v}$ of G such that $\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{S}\mathbf{c} \neq 0$ and $\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{L}\mathbf{c} = 0$. By Lemma 4 there exists a nonzero vector $\tilde{\mathbf{w}}$ such that $\tilde{\mathbf{c}} = \mathbf{X}\tilde{\mathbf{w}}$. Then (2) implies $\tilde{\mathbf{w}}^{\mathsf{T}}\mathbf{W}(i\omega_0)\,\mathbf{w} = 0$, consequently

$$\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{R}\mathbf{c} + i\left(\omega_0\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{L}\mathbf{c} - \frac{1}{\omega_0}\,\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{S}\mathbf{c}\right) = 0$$

and hence

$$\omega_0^2\tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{L}\mathbf{c} = \tilde{\mathbf{c}}^{\mathsf{T}}\mathbf{S}\mathbf{c}\,.$$

This contradiction proves our lemma.

Lemma 7. *Under the same assumptions as in Theorem 7 the matrix $\mathbf{W}^{-1}$ has no poles in $\overline{\mathfrak{G}}$.*

Proof. Lemma 5 and Theorem 5 guarantee the existence of the matrix $\mathbf{W}^{-1} \in \mathfrak{P}_n$; by Theorem 3, $\mathbf{W}^{-1}$ has no poles in $\mathfrak{G}$ and the poles on the imaginary axis and at infinity are simple. Lemma 6 then implies that the only poles of $\mathbf{W}^{-1}$ in $\overline{\mathfrak{G}}$ can be 0 and ∞.

a) Suppose 0 is a pole of $\mathbf{W}^{-1}$. By Theorem 3 there exist matrices $\mathbf{H}, \mathbf{K} \in \mathfrak{S}_n$ such that $\mathbf{W}^{-1} = \mathbf{H}p^{-1} + \mathbf{K}(p)$, where $\mathbf{H}$ is a constant nonzero matrix and $\mathbf{K}(p)$ has no pole in 0. Simultaneously

$$\mathbf{W}(p) = \mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X}\frac{1}{p} + \mathbf{X}^{\mathsf{T}}(\mathbf{R} + \mathbf{L}p)\,\mathbf{X}\,.$$

The obvious identity $\mathbf{W}\mathbf{W}^{-1} = \mathbf{I}$ ($\mathbf{I}$ is the unit matrix) then yields

$$\mathbf{I} = \mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X}\mathbf{H}\,\frac{1}{p^2} + \mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X}\mathbf{K}(p)\,\frac{1}{p} + \mathbf{X}^{\mathsf{T}}(\mathbf{R} + \mathbf{L}p)\,\mathbf{X}\mathbf{H}\,\frac{1}{p} + \mathbf{X}^{\mathsf{T}}(\mathbf{R} + \mathbf{L}p)\,\mathbf{X}\mathbf{K}(p)\,.$$

This implies that $\mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X}\mathbf{H} = \mathbf{H}\mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X} = \mathbf{0}$. Multiplying by p and letting $p \to 0$ one obtains

$$\mathbf{X}^{\mathsf{T}}\mathbf{S}\mathbf{X}\mathbf{K}_0 + \mathbf{X}^{\mathsf{T}}\mathbf{R}\mathbf{X}\mathbf{H} = \mathbf{0}$$

(where $\boldsymbol{K}_0 = \lim_{p \to 0} \boldsymbol{K}(p)$). Consequently, $\boldsymbol{H}\boldsymbol{X}^{\mathsf{T}}(\boldsymbol{R} + \boldsymbol{S})\,\boldsymbol{X}\boldsymbol{H} = \boldsymbol{0}$. Suppose $\boldsymbol{H}$ has a nonzero column $\boldsymbol{h}$. Then for a nonzero cycle $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{v} = (\boldsymbol{X}\boldsymbol{h})^{\mathsf{T}}\,\boldsymbol{v}$ of G one obtains

$$\boldsymbol{c}^{\mathsf{T}}(\boldsymbol{R} + \boldsymbol{S})\,\boldsymbol{c} = 0\,,$$

which contradicts assumption 1 of Theorem 7.

b) Suppose ∞ is a pole of $\boldsymbol{W}^{-1}$. Similarly, from $\boldsymbol{W}^{-1} = \boldsymbol{H}p + \boldsymbol{K}(p)$ and $\boldsymbol{W} = \boldsymbol{X}^{\mathsf{T}}\boldsymbol{L}\boldsymbol{X}p + \boldsymbol{X}^{\mathsf{T}}(\boldsymbol{R} + \boldsymbol{S}p^{-1})\,\boldsymbol{X}$ one obtains $\boldsymbol{H}\boldsymbol{X}^{\mathsf{T}}(\boldsymbol{R} + \boldsymbol{L})\,\boldsymbol{X}\boldsymbol{H} = \boldsymbol{0}$, which contradicts assumption 2 of Theorem 7.

Proof of Theorem 7.

Let $\boldsymbol{i}(p)$ be a solution of N corresponding to the vector $\boldsymbol{e} = \boldsymbol{o}$ and initial conditions $\boldsymbol{i}_0, \boldsymbol{q}_0$ (its existence follows from Theorem 2). By Theorem 1,

$$\boldsymbol{i}(p) = \boldsymbol{A}(p)\left(\boldsymbol{L}\boldsymbol{i}_0 - \boldsymbol{S}\boldsymbol{q}_0\,\frac{1}{p}\right) \tag{3}$$

where

$$\boldsymbol{A}(p) = \boldsymbol{X}[\boldsymbol{X}^{\mathsf{T}}\boldsymbol{Z}(p)\,\boldsymbol{X}]^{-1}\,\boldsymbol{X}^{\mathsf{T}} = \boldsymbol{X}\boldsymbol{W}^{-1}\boldsymbol{X}^{\mathsf{T}}.$$

From Lemma 7 it follows that $\boldsymbol{A}$ has no poles in $\overline{\mathfrak{G}}$ and hence the only pole of $\boldsymbol{i}$ in $\overline{\mathfrak{G}}$ can be 0.

Suppose 0 is a pole of $\boldsymbol{W}^{-1}\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}p^{-1}$. Then there exist matrices $\boldsymbol{H}, \boldsymbol{K}$ of type (n, r) such that

$$\boldsymbol{W}^{-1}\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}p^{-1} = \boldsymbol{H}p^{-1} + \boldsymbol{K}(p)\,,$$

where $\boldsymbol{H}$ is a constant matrix and $\boldsymbol{K}(p)$ is regular at 0 (and hence $\boldsymbol{K}_0 = \lim_{p \to 0} \boldsymbol{K}(p)$ exists). This implies further that

$$\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}p^{-1} = \boldsymbol{W}(\boldsymbol{H}p^{-1} + \boldsymbol{K}(p))\,,$$

which yields

$$\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}p^{-1} = \boldsymbol{X}^{\mathsf{T}}\boldsymbol{L}\boldsymbol{X}\boldsymbol{K}(p)\,p + \boldsymbol{X}^{\mathsf{T}}\boldsymbol{L}\boldsymbol{X}\boldsymbol{H} + \boldsymbol{X}^{\mathsf{T}}\boldsymbol{R}\boldsymbol{X}\boldsymbol{K} + (\boldsymbol{X}^{\mathsf{T}}\boldsymbol{R}\boldsymbol{X}\boldsymbol{H} + \boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{X}\boldsymbol{K})\,p^{-1} + \\ + \boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{X}\boldsymbol{H}p^{-2}\,.$$

This implies that $\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{X}\boldsymbol{H} = \boldsymbol{0}$ and therefore

$$\boldsymbol{H}^{\mathsf{T}}\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{X}\boldsymbol{H} = \boldsymbol{0}\,. \tag{4}$$

Multiplying by p and letting $p \to 0$ one obtains $\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S} = \boldsymbol{X}^{\mathsf{T}}\boldsymbol{R}\boldsymbol{X}\boldsymbol{H} + \boldsymbol{X}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{X}\boldsymbol{K}_0$ and hence

$$\boldsymbol{H}^{\mathsf{T}}\boldsymbol{X}^{\mathsf{T}}\boldsymbol{S} = \boldsymbol{H}^{\mathsf{T}}\boldsymbol{X}^{\mathsf{T}}\boldsymbol{R}\boldsymbol{X}\boldsymbol{H}\,. \tag{5}$$

Suppose that the j-th column $\boldsymbol{h}$ of $\boldsymbol{H}$ is nonzero. Then $\boldsymbol{d}^{\mathsf{T}}\boldsymbol{v} = (\boldsymbol{X}\boldsymbol{h})^{\mathsf{T}}\,\boldsymbol{v}$ is a nonzero cycle of G and it follows from (4) that $\boldsymbol{d}^{\mathsf{T}}\boldsymbol{S}\boldsymbol{d} = 0$, therefore by assumption 1 $\boldsymbol{d}^{\mathsf{T}}\boldsymbol{R}\boldsymbol{d} > 0$

and hence the element (j, j) of the matrix $\boldsymbol{H}^T\boldsymbol{X}^T\boldsymbol{R}\boldsymbol{X}\boldsymbol{H}$ is nonzero. However, from $\boldsymbol{d}^T\boldsymbol{S}\boldsymbol{d} = 0$ and from the fact that $\boldsymbol{S}$ is a diagonal positive semidefinite matrix it follows that $\boldsymbol{d}^T\boldsymbol{S} = \boldsymbol{o}$, and hence the j-th row of the matrix $\boldsymbol{H}^T\boldsymbol{X}^T\boldsymbol{S}$ is zero, which contradicts (5). This contradiction proves that $\boldsymbol{W}^{-1}\boldsymbol{X}^T\boldsymbol{S}p^{-1}$ has no poles in $\overline{\mathfrak{G}}$ and it follows from (3) that $\boldsymbol{i}$ has the same property.

From Theorem 7 one can immediately obtain the following well-known theorem:

Theorem 8. *Let $N = (G, \boldsymbol{R}, \boldsymbol{L}, \boldsymbol{S})$ be a passive network. Suppose $\boldsymbol{d}^T\boldsymbol{R}\boldsymbol{d} > 0$ for each loop $\boldsymbol{d}^T\boldsymbol{v}$ of G. Then N is asymptotically stable.*

Proof follows from Theorem 7, Lemma 2 and from the diagonality of $\boldsymbol{R}$.

For networks with a diagonal matrix $\boldsymbol{L}$ one can obtain the following

Theorem 9. *Let $N = (G, \boldsymbol{R}, \boldsymbol{L}, \boldsymbol{S})$ be a passive network with a diagonal matrix $\boldsymbol{L}$. Suppose the following conditions are fulfilled for every nonzero loop $\boldsymbol{d}^T\boldsymbol{v}$ of G:*

1. $\boldsymbol{d}^T(\boldsymbol{R} + \boldsymbol{S})\,\boldsymbol{d} > 0$,
2. $\boldsymbol{d}^T(\boldsymbol{R} + \boldsymbol{L})\,\boldsymbol{d} > 0$,
3. *if $\boldsymbol{d}^T\boldsymbol{R}\boldsymbol{d} = 0$, then there exists a loop $\tilde{\boldsymbol{d}}^T\boldsymbol{v}$ of G such that simultaneously $\tilde{\boldsymbol{d}}^T\boldsymbol{S}\boldsymbol{d} \neq 0$ and $\tilde{\boldsymbol{d}}^T\boldsymbol{L}\boldsymbol{d} = 0$.*

Then the network N is asymptotically stable.

Proof. Theorem 9 can be proved in a similar manner as Theorem 7. By Lemma 2, assumptions 1 and 2 of Theorem 9 are equivalent with those of Theorem 7. Assumption 3 is used only in the proof of Lemma 6, which can be proved analogously using assumption 3 of Theorem 9, Lemma 1 and the diagonality of the matrices $\boldsymbol{R}$, $\boldsymbol{L}$, $\boldsymbol{S}$.

Remark. From the physical view-point, Theorem 9 gives sufficient conditions of asymptotic stability which can be used for networks with loops without nonzero resistors. Such a loop without nonzero resistors must contain a nonzero capacitor and an inductor (assumptions 1 and 2) and the capacitor must be contained in another loop (assumption 3). Theorem 7 is a generalization of this condition to networks with inductive couplings.

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Souhrn

O ASYMPTOTICKÉ STABILITĚ PASIVNÍCH LINEÁRNÍCH ELEKTRICKÝCH OBVODŮ

ZDENĚK RYJÁČEK

V práci je uvedeno kriterium asymptotické stability řešení lineárního elektrického obvodu se soustředěnými parametry, jež je oslabením podmínek dosud známých — kriterium lze použít i na obvody, jejichž některé smyčky neobsahují nenulový ohmický odpor.

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