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ORTHOEXPONENTIAL POLYNOMIALS AND THE LEGENDRE POLYNOMIALS

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Orthogonal exponential polynomials or *orthoexponential polynomials* have interesting applications in Automatic Control – Laning-Battin [8], Electrical Circuits Theory – Čížek [2], Tuttle [10], Hydrometeorology – Dmitriyev [4], and as a means for the inversion of Laplace transforms – Jaroch [5]. A very simple connection will be shown to exist between the orthoexponential and the Legendre polynomials.

In this paper, the letters k, m, n denote nonnegative integers, t and x are real variables, δ_{mn} is the Kronecker delta function and, in Definition II only, $[n/2]$ denotes the integer part of $n/2$. As the notation and standardization of both the orthoexponential and the Legendre polynomials vary, definitions of the Legendre, Jacobi, and orthoexponential polynomials as used in this paper are re-stated here.

Definition I [6]. The functions $\varphi_n(t) = \sum_{k=1}^n b_{nk} e^{-kt}$, $n = 1, 2, 3, \dots$, are called *orthoexponential polynomials* if the coefficients b_{nk} are so chosen that for all m, n the following conditions are satisfied: $\varphi_n(0) = 1$ and $\int_0^\infty \varphi_m(t) \varphi_n(t) dt = \delta_{mn} \|\varphi_n\|^2$.

The coefficients of orthoexponential polynomials are $b_{nk} = (-1)^{n+k} \binom{n}{k} \binom{n+k-1}{k-1}$; $\varphi_1(t) = e^{-t}$. For all n we have $\varphi_n(+\infty) = 0$. Orthoexponential polynomials are bounded and $|\varphi_n(t)| \leq 1$ for all nonnegative t . Their norm is $\|\varphi_n\| = (2n)^{-1/2}$ and orthonormal exponential polynomials are therefore $t \mapsto (2n)^{1/2} \varphi_n(t)$.

Definition II [9]. The polynomials $P_n(x) = \sum_{k=0}^{[n/2]} a_{nk} x^{n-2k}$, $n = 0, 1, 2, \dots$, are called *Legendre polynomials* if the coefficients a_{nk} are so chosen that for all m, n the following conditions are satisfied: $P_n(1) = 1$ and $\int_{-1}^1 P_m(x) P_n(x) dx = \delta_{mn} \|P_n\|^2$.

The coefficients of the Legendre polynomials are $a_{nk} = (-1)^k 2^{-n} \binom{n}{k} \binom{2n-2k}{n}$; $P_0(x) \equiv 1$. Their norm is $\|P_n\| = (n + \frac{1}{2})^{-1/2}$, and $|P_n(x)| \leq 1$ if only $-1 \leq x \leq 1$.

The notation $P_n^{(\alpha,\beta)}(x)$ is used for the Jacobi polynomials, orthogonal over $(-1, 1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, $\alpha > -1$, $\beta > -1$, and standardized with $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. Legendre polynomials are a special case of the Jacobi polynomials for $\alpha = \beta = 0$. Many properties of the Jacobi polynomials originate from their close connection with the hypergeometric function. The following relation will be used in this paper:

$$(1) \quad (1+x)P_{n-1}^{(0,1)}(x) = P_n^{(0,0)}(x) + P_{n-1}^{(0,0)}(x), \quad n \geq 1,$$

where $P_n^{(0,0)}(x) \equiv P_n(x)$ are the Legendre polynomials [1], [9].

It is known [6] that orthoexponential polynomials can be expressed in terms of the Jacobi polynomials $P_n^{(0,1)}(x)$. However, no explicit mention has been found so far of a simple relation connecting the orthoexponential and the Legendre polynomials as expressed in the following two theorems.

Theorem 1. *Let $\varphi_n(t)$ be the orthoexponential polynomials and $P_n(x)$ the Legendre polynomials in accordance with Definitions I and II; $n = 1, 2, 3, \dots$. Then, for arbitrary t ,*

$$(2) \quad \varphi_n(t) = \frac{1}{2}[P_n(2e^{-t} - 1) + P_{n-1}(2e^{-t} - 1)].$$

Proof. In Eq. (2), orthoexponential polynomials are expressed, in accordance with Definitions I and II, as linear combinations of exponential functions e^{-t} , e^{-2t} , ..., e^{-nt} . The standardization $\varphi_n(0) = 1$, $\varphi_n(+\infty) = 0$, is satisfied at the same time. It remains to prove that $\int_0^\infty \varphi_m(t) \varphi_n(t) dt = \delta_{mn}(2n)^{-1}$. Substituting $x = 2e^{-t} - 1$ in Eq. (1) we have

$$2e^{-t}P_{n-1}^{(0,1)}(2e^{-t} - 1) = P_n(2e^{-t} - 1) + P_{n-1}(2e^{-t} - 1)$$

and Eq. (2) is therefore equivalent to

$$(3) \quad \varphi_n(t) = e^{-t}P_{n-1}^{(0,1)}(2e^{-t} - 1).$$

The norm of the Jacobi polynomials is known [1] [9] and for the case under consideration we have $\|P_n^{(0,1)}\| = 2^{1/2}(n+1)^{-1/2}$. As a consequence,

$$\int_0^\infty \varphi_m(t) \varphi_n(t) dt = \int_0^\infty e^{-2t}P_{m-1}^{(0,1)}(2e^{-t} - 1)P_{n-1}^{(0,1)}(2e^{-t} - 1) dt$$

and, if the substitution $x = 2e^{-t} - 1$ is used,

$$\int_0^\infty \varphi_m(t) \varphi_n(t) dt = \frac{1}{4} \int_{-1}^{+1} (1+x)P_{m-1}^{(0,1)}(x)P_{n-1}^{(0,1)}(x) dx = \delta_{mn}(2n)^{-1}.$$

Q.e.d.

Remark. Jacobi polynomials are defined by Courant and Hilbert [3] [6] as polynomials orthogonal over $(0, 1)$ with respect to the weight function $x^{q-1}(1-x)^{p-q}$, $q > 0$, $p - q > -1$, and denoted by $G_n(p, q, x)$. The standardization is $G_n(p, q, 0) = 1$ for all n . In this case Eq. (3) becomes

$$(4) \quad \varphi_n(t) = (-1)^{n-1} n e^{-t} G_{n-1}(2, 2, e^{-t})$$

and Eq. (2) assumes the form

$$(5) \quad \varphi_n(t) = (-1)^n 2^{-1} [G_n(1, 1, e^{-t}) - G_{n-1}(1, 1, e^{-t})].$$

Example 1. The value of $\varphi_{10}(1)$ is to be computed from Eq. (2). For $t = 1$ we have $2e^{-t} - 1 = -0.264241$ and thus $\varphi_{10}(1) = \frac{1}{2}[P_{10}(-0.264241) + P_9(-0.264241)] = \frac{1}{2}(0.237018 - 0.148194)$. The result is $\varphi_{10}(1) = 0.044412$. A calculation of values of the orthoexponential polynomials based on Definition I involves, for large n , computations with prohibitively large numbers. In this example, if Definition I is used for the computation of $\varphi_{10}(1)$ then the largest coefficient involved is $b_{10,7} = -960960$.

Theorem 2. Let $\varphi_n(t)$ be the orthoexponential polynomials and $P_n(x)$ the Legendre polynomials in accordance with Definitions I and II; $n = 1, 2, 3, \dots$. Then, for arbitrary t ,

$$(6) \quad P_n(2e^{-t} - 1) = 2 \varphi_n(t) - 2 \varphi_{n-1}(t) + \dots + (-1)^{n-1} 2 \varphi_1(t) + (-1)^n.$$

Proof. For $n = 1$ we have $\varphi_1(t) = \frac{1}{2}[P_1(2e^{-t} - 1) + 1]$ as a consequence of Theorem 1 and thus $P_1(2e^{-t} - 1) = 2 \varphi_1(t) - 1$. For $n = 2$ we find $\varphi_2(t) = \frac{1}{2}[P_2(2e^{-t} - 1) + P_1(2e^{-t} - 1)]$ or $\varphi_2(t) = \frac{1}{2}[P_2(2e^{-t} - 1) + 2 \varphi_1(t) - 1]$ and that too agrees with Eq. (6). The proof proceeds by induction. Let us assume that Eq. (6) is valid for a certain integer $n > 1$. It follows from Theorem 1 that $\varphi_{n+1}(t) = \frac{1}{2}[P_{n+1}(2e^{-t} - 1) + P_n(2e^{-t} - 1)]$ or $P_{n+1}(2e^{-t} - 1) = 2 \varphi_{n+1}(t) - P_n(2e^{-t} - 1)$. Substituting for $P_n(2e^{-t} - 1)$ from Eq. (6) we have $P_{n+1}(2e^{-t} - 1) = 2 \varphi_{n+1}(t) - 2 \varphi_n(t) + \dots + (-1)^n 2 \varphi_1(t) + (-1)^{n+1}$. Q.e.d.

Remark. For the Jacobi polynomials in the Courant and Hilbert notation [3] an analogous relation follows from Eq. (5):

$$(7) \quad G_n(1, 1, e^{-t}) = 1 - 2 \varphi_1(t) + 2 \varphi_2(t) - 2 \varphi_3(t) + \dots + (-1)^n 2 \varphi_n(t).$$

Example 2. The series $1 - 2 \varphi_1(t) + 2 \varphi_2(t) - 2 \varphi_3(t) + \dots + (-1)^n 2 \varphi_n(t) + \dots = 0$ is convergent for any positive t . The n -th partial sum of this series is $(-1)^n \cdot P_n(2e^{-t} - 1)$ as a consequence of Theorem 2. Laplace's theorem on the asymptotics of the Legendre polynomials [9] states that $P_n(x) = O(n^{-1/2})$ for $n \rightarrow \infty$ if only $-1 < x < 1$. Thus, the n -th partial sum of the series under consideration is $(-1)^n \cdot$

$P_n(2e^{-t} - 1) = o(1)$ and the series therefore converges. Accordingly, the series $2 \sum_{n=1}^{\infty} (-1)^{n-1} \varphi_n(t) = 1$ converges for any positive t ; for $t = 0$ the series is divergent.

CONCLUSION

The theorems proved in this paper are useful if a formulation is sought, for orthoexponential polynomials, of some of the many known results in the theory of classical orthogonal polynomials. Furthermore, Theorem 1 may be convenient for the computation of values of orthoexponential polynomials if the values of the Legendre polynomials are available. In addition to the recurrence formula

$$(8) \quad \begin{aligned} (n+1)(2n-1)\varphi_{n+1} &= \\ &= [(4n^2-1)e^{-t} - 2n^2]2\varphi_n - (n-1)(2n+1)\varphi_{n-1}, \end{aligned}$$

$n = 1, 2, 3, \dots$; $\varphi_1 = e^{-t}$, $\varphi_0 \equiv 0$ (see Jaroch-Novotný [7]) we have here another relation, namely $\varphi_n(t) = \frac{1}{2}[P_n(2e^{-t} - 1) + P_{n-1}(2e^{-t} - 1)]$, which is useful when an effective algorithm is required for the computation of values of orthoexponential polynomials.

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Souhrn

O SOUVISLOSTI ORTOEXPONENCIÁLNÍCH MNOHOČLENŮ S LEGENDREOVÝMI MNOHOČLENY

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Vyjádření ortoexponenciálních mnohočlenů $\varphi_n(t) = \sum_{k=1}^n b_{nk} e^{-kt}$ podle definice I je pro velká n nevhodné pro numerický výpočet funkčních hodnot, protože k výsledku přicházíme odčítáním velkých čísel. V tomto případě je užitečný rekurentní vzorec (8) nebo v tomto článku dokázaná věta 1, totiž $\varphi_n(t) = \frac{1}{2}[P_n(2e^{-t} - 1) + P_{n-1}(2e^{-t} - 1)]$, kde $P_n(x)$ jsou Legendreovy mnohočleny. Věty, ukazující na souvislost ortoexponenciálních mnohočlenů s Legendreovými mnohočleny jsou užitečné jednak pro numerické výpočty jednak proto, že umožňují jednoduchým způsobem přenést na ortoexponenciální mnohočleny některé výsledky z teorie klasických ortogonálních mnohočlenů. Ortoexponenciální mnohočleny jsme definovali se standardizací $\varphi_n(0) = 1$ pro všechna n , protože při této volbě zvláště vyniknou souvislosti s klasickou teorií ortogonálních mnohočlenů.

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