Aplikace matematiky

Manfred Müller; Joachim Naumann

On evolution inequalities of a modified Navier-Stokes type. II

Aplikace matematiky, Vol. 23 (1978), No. 6, 397–407

Persistent URL: http://dml.cz/dmlcz/103768

ON EVOLUTION INEQUALITIES OF A MODIFIED NAVIER-STOKES TYPE, II

MANFRED MÜLLER and JOACHIM NAUMANN

(Received February 21, 1977)

The aim of the present part of our paper is to continue the study of the abstract evolution inequality that we have begun in [4].

For that purpose we maintain the assumptions upon the spaces W, V and H (where $W \subset V \subset H$) which are formulated in [4]. The functional $\varphi : V \to (-\infty, +\infty]$ is assumed to be proper, convex and lower semi-continuous. Since our subsequent discussion is essentially based on [4] we also keep the assumption that $\varphi(x) \geqq \varphi(0)$ for all $x \in V$. Let $D(\varphi)$ denote the effective domain of φ, i.e.

$$D(\varphi) = \{x \in V : \varphi(x) < +\infty\} .$$

Finally, given $u \in L^2(0, T; V)$ we set

$$\Phi(u) = \int_0^T \varphi(u) \,\mathrm{d}t \quad \textit{if} \quad \varphi(u(\cdot)) \in L^1(0, T) , \qquad +\infty \quad \textit{otherwise} .$$

We then consider the evolution problem

$$\int_0^T (u' + Au + B(u, u) , v - u) \,\mathrm{d}t + \Phi(v) - \Phi(u) \geqq$$

$$\geqq \int_0^T (f, v - u) \,\mathrm{d}t \quad \forall v \in L^p(0, T; W) ,$$

$$u(0) = u_0$$

where $A : W \to W^*$ is a (nonlinear) mapping, $B : W \times W \to W^*$ a bilinear mapping, f and u_0 are given data. As in [4], the conditions we are going to impose upon A and B are motivatied by a type of modified Navier-Stokes equations under certain unilateral boundary conditions.

In the first section of the present part of our paper we state the results. Theorem 1 yields the existence and uniqueness of a weak solution to the above inequality (i.e. u' is replaced by the derivative of the test function v). This concept of solution permits to weaken the assumptions upon the data. Theorem 2 presents two regularity results (with respect to t) for the solution u to the above inequality. In particular, we obtain a regularity property of the type $t^\alpha u' \in L^\infty(0, T; H) \cap L^2(0, T; V)$ $(\alpha \geqq \frac{1}{2})$ which represents a smoothing effect upon the initial value u_0 [1]). The proofs of Theorem 1 and Theorem 2 are then given in Section 2 and Section 3, respectively.

1. STATEMENT OF RESULTS

In the present part of our paper we subject the mapping $A : W \to W^*$ to the following conditions:

(1.1) $$A \text{ is hemi-continuous};$$

(1.2) $$(Ax - Ay, x - y) \geqq c_1 \|x - y\|^2 \quad \forall x, y \in W, \quad c_1 = \text{const} > 0;$$

(1.3) $$(Ax, x) \geqq c_2 |||x|||^p \quad \forall x \in W, \quad c_2 = \text{const} > 0, \quad p > 3;$$

(1.4) $$\begin{cases} \textit{there exists a functional } F : W \to \mathbb{R} \textit{ such that} \\ A = \operatorname{grad} F; \end{cases}$$

(1.5) $$|||Ax|||_* \leqq c_3(|||x|||^{p-1} + 1) \; \forall x \in W, \quad c_3 = \text{const} > 0.$$

One easily deduces from (1.1) and (1.4) that the functional F admits the representation

$$F(x) = F(0) + \int_0^1 (A(sx), x) \, \mathrm{d}s, \quad x \in W.$$

Thus, our hypotheses (1.1)–(1.5) imply those imposed upon the mapping A in [4].

The conditions upon the bilinear mapping $B : W \times W \to W^*$ will be strengthened as follows:

(1.6) $$\begin{aligned} &|(B(x, y), z)| \leqq c_4 |x| \, \|y\| \, |||z||| , \\ &|(B(x, y), z)| \leqq c_4 \|x\| \, |||y||| \, |z| , \\ &|(B(x, y), z)| \leqq c_4 |||x||| \, \|y\| \, |z| \\ &\forall x, y, z \in W, \quad c_4 = \text{const}. \end{aligned}$$

The main results of the present part of our paper are the following two theorems.

---

[1]) For a detailed discussion of this problem within the framework of monotone operators we refer to the book: *Barbu, V.:* Nonlinear semigroups and differential equations in Banach spaces. Bucharest, Leyden 1976. — However, note that the theory developed in this book does not cover our above evolution problem.

Theorem 1. *Let the mapping A satisfy* (1.1)–(1.5), *while the bilinear mapping B is assumed to fulfil* (1.6).

Let the data satisfy the following conditions:

$$f = f_1 + f_2 : f_1 \in L^2(0, T; V^*)\,, \quad f_2, f_2' \in L^{p'}(0, T; W^*)\,;$$

$$u_0 \in \overline{W \cap D(\varphi)}^{H}\ ^{2)}\,.$$

Then there exists exactly one function $u \in L^p(0, T; W) \cap C([0, T]; H)$ such that

$$(1.7) \qquad \Phi(u) < +\infty\,;$$

$$(1.8) \qquad \int_0^T (v' + Au + B(u, u),\ v - u)\,\mathrm{d}t + \Phi(v) - \Phi(u) \geqq$$

$$\geqq \int_0^T (f, v - u)\,\mathrm{d}t - \tfrac{1}{2}|v(0) - u_0|^2$$

$$\forall v \in L^p(0, T; W) \quad \text{with} \quad v' \in L^{p'}(0, T; W^*)\,;$$

$$(1.9) \qquad u(0) = u_0\,.$$

Theorem 2. *Suppose that the mapping A satisfies the conditions* (1.1)–(1.5), *while the bilinear mapping B fulfils the condition* (1.6).

(i) *Let*

$$f \in L^2(0, T; V^*)\,, \quad f' \in L^{p'}(0, T; W^*)\,, \quad t^\alpha f' \in L^2(0, T; V^*)\,;$$

$$u_0 \in W \cap D(\varphi)$$

where $\alpha \geqq \frac{1}{2}$. Then there exists exactly one function $u \in L^\infty(0, T; W) \cap C([0, T]; H)$ such that

$$(1.10) \qquad \Phi(u) < +\infty\,, \quad u' \in L^2(0, T; H)\,;$$

$$(1.11) \qquad u \in C((0, T]; V)\,, \quad t^\alpha u' \in L^\infty(0, T; H) \cap L^2(0, T; V)\,;$$

$$(1.12) \qquad \int_0^T (u' + Au + B(u, u),\ v - u)\,\mathrm{d}t + \Phi(v) - \Phi(u) \geqq$$

$$\geqq \int_0^T (f, v - u)\,\mathrm{d}t \quad \forall v \in L^p(0, T; W)\,;$$

$$(1.13) \qquad u(0) = u_0\,.$$

(ii) *If the data satisfy the conditions*

$$f, f' \in L^2(0, T; V^*)\,, \quad u_0 \in W \cap (\partial\varphi)^{-1}(0)\,,$$

$$(f(0) - Au_0 - B(u_0, u_0)) \in H$$

---

[2]) $\overline{W \cap D(\varphi)}^H$ = closure of $W \cap D(\varphi)$ in H.

then the function u from (i) satisfies in addition

$$u \in C([0, T]; V), \quad u' \in L^\infty(0, T; H) \cap L^2(0, T; V). \tag{1.14}$$

2. PROOF OF THEOREM 1

Uniqueness. Let $u_i \in L^p(0, T; W) \cap C([0, T]; H)$ $(i = 1, 2)$ be two functions satisfying (1.7)–(1.9). We then conclude from [1; Théorème II.3] that

$$\tfrac{1}{2}|u_1(t) - u_2(t)|^2 + c_1 \int_0^t \|u_1 - u_2\|^2 \, ds \leqq$$

$$\leqq \int_0^t (-B(u_1, u_1) + B(u_2, u_2), \; u_1 - u_2) \, ds \leqq$$

$$\leqq \text{const} \int_0^t (|||u_1||| + |||u_2|||) \, \|u_1 - u_2\| \, |u_1 - u_2| \, ds$$

for all $t \in [0, T]$. Therefore

$$|u_1(t) - u_2(t)|^2 \leqq \text{const} \int_0^t (|||u_1|||^2 + |||u_2|||^2) \, |u_1 - u_2|^2 \, ds$$

for all $t \in [0, T]$; hence, by Gronwall's lemma, $u_1 \equiv u_2$.

Existence. Let $\{f_{1m}\} \subset L^2(0, T; V^*)$ and $\{u_{0m}\} \subset W \cap D(\varphi)$ $(m = 1, 2, \ldots)$ be sequences such that

$$f'_{1m} \in L^2(0, T; V^*) \quad (m = 1, 2, \ldots),$$

$$f_{1m} \to f_1 \quad \textit{strongly in} \quad L^2(0, T; V^*),$$

$$u_{0m} \to u_0 \quad \textit{strongly in} \quad H$$

as $m \to \infty$. Applying the basic existence result in [4] one obtains for each m a function $u_m \in L^\infty(0, T; W) \cap C([0, T]; H)$ such that

$$\Phi(u_m) < +\infty, \quad u'_m \in L^2(0, T; H);$$

$$\int_0^T (u'_m + Au_m + B(u_m, u_m), \; v - u_m) \, dt + \tag{2.1}$$

$$+ \Phi(v) - \Phi(u_m) \geqq \int_0^T (f_{1m} + f_2, \; v - u_m) \, dt$$

$$\forall v \in L^p(0, T; W);$$

$$u_m(0) = u_{0m}. \tag{2.2}$$

From [1; App. I, Prop. 3] we conclude that (2.1) is equivalent to

$$(2.1_1) \qquad (u_m'(t) + A\,u_m(t) + B(u_m(t), u_m(t)),\ x - u_m(t)) + \\ + \varphi(x) - \varphi(u_m(t)) \geqq (f_{1m}(t) + f_2(t),\ x - u_m(t))$$

for all $x \in W$ and a.a. $t \in [0, T]$ $(m = 1, 2, \ldots)$. Setting $x = 0$ in (2.1_1) one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_m(t)|^2 + c_2|||u_m(t)|||^p \leqq$$

$$\leqq -(B(u_m(t), u_m(t)), u_m(t)) + (f_{1m}(t) + f_2(t), u_m(t)) \leqq$$

$$\leqq \text{const}\,[|||u_m(t)|||^3 + (\|f_{1m}(t)\|_* + |||f_2(t)|||_*)\,|||u_m(t)|||]$$

for a.a. $t \in [0, T]$. Thus

$$(2.3) \qquad |u_m(t)| \leqq k_1 \quad \forall t \in [0, T], \quad m = 1, 2, \ldots,$$
$$\|u_m\|_{L^p(0,T;W)} \leqq k_1 \quad \textit{for} \quad m = 1, 2, \ldots .$$

Next, observing our hypotheses (1.2) and (1.6) we easily derive from (2.1_1) that

$$\tfrac{1}{2}|u_m(t) - u_n(t)|^2 + c_1\int_0^t \|u_m - u_n\|^2\,\mathrm{d}s \leqq$$

$$\leqq \tfrac{1}{2}|u_{0m} - u_{0n}|^2 - \int_0^t (B(u_m, u_m) - B(u_n, u_n),\ u_m - u_n)\,\mathrm{d}s +$$

$$+ \int_0^t (f_{1m} - f_{1n},\ u_m - u_n)\,\mathrm{d}s \leqq \tfrac{1}{2}|u_{0m} - u_{0n}|^2 + \int_0^t (f_{1m} - f_{1n},\ u_m - u_n)\,\mathrm{d}s +$$

$$+ \text{const}\int_0^t (|||u_m||| + |||u_n|||)\,\|u_m - u_n\|\,|u_m - u_n|\,\mathrm{d}s$$

for all $t \in [0, T]$ and $m, n = 1, 2, \ldots$. Thus, by (2.3) and Gronwall's lemma,

$$|u_m(t) - u_n(t)|^2 + \int_0^t \|u_m - u_n\|^2\,\mathrm{d}s \leqq \text{const}\left(|u_{0m} - u_{0n}|^2 + \int_0^T \|f_{1m} - f_{1n}\|_*^2\,\mathrm{d}s\right)$$

for all $t \in [0, T]$ and $m, n = 1, 2, \ldots$, i.e. $\{u_m\}$ is a Cauchy sequence both in $C([0, T]; H)$ and $L^2(0, T; V)$.

From (2.3) and the latter estimate we may conclude (by passing to a subsequence if necessary) that

$$(2.4) \qquad u_m \to u \quad \textit{strongly in} \quad C([0, T]; H),$$

$$(2.5) \qquad u_m \to u \quad \textit{strongly in} \quad L^2(0, T; V),$$

$$(2.6) \qquad u_m \to u \quad \textit{weakly in} \quad L^p(0, T; W),$$

$$(2.7) \qquad \mathscr{A}u_m \to \chi \quad \textit{weakly in} \quad L^{p'}(0, T; W^*)$$ [3])

[3]) Recall that $(\mathscr{A}v)(t) = A\,v(t)$ for a.a. $t \in [0, T]$ and any $v \in L_p(0, T; W)$ (cf. [4]).

as $m \to \infty$. Taking into account the estimates

$$\left|\int_0^T (B(u_m, u_m) - B(u, u), v)\,\mathrm{d}t\right| \leqq$$

$$\leqq c_4(\|u - u_m\|_{C([0,T];H)} \|u_m\|_{L^2(0,T;V)} + \|u\|_{C([0,T];H)} \|u - u_m\|_{L^2(0,T;V)}) \|v\|_{L^2(0,T;W)}$$

and

$$\left|\int_0^T (B(u_m, u_m),\ u - u_m)\,\mathrm{d}t\right| \leqq \text{const}\ \|u - u_m\|_{C([0,T];H)} \|u_m\|^2_{L^2(0,T;W)}$$

where $v \in L^p(0, T; W)$, one obtains by virtue of (2.3) and (2.4), (2.5) that

$$\int_0^T (B(u_m, u_m),\ v - u_m)\,\mathrm{d}t \to \int_0^T (B(u, u),\ v - u)\,\mathrm{d}t \tag{2.8}$$

as $m \to \infty$.

Combining (2.2) and (2.4) we find $u(0) = u_0$. Further, (2.1_1) implies

$$\int_0^T (v' + Au_m + B(u_m, u_m),\ v - u_m)\,\mathrm{d}t + \Phi(v) - \Phi(u_m) \geqq$$

$$\geqq \int_0^T (f_{1m} + f_2,\ v - u_m)\,\mathrm{d}t - \tfrac{1}{2}|v(0) - u_{0m}|^2$$

or equivalently,

$$\int_0^T (Au_m, u_m)\,\mathrm{d}t \leqq \int_0^T (v', v - u_m)\,\mathrm{d}t + \int_0^T (Au_m, v)\,\mathrm{d}t +$$

$$+ \int_0^T (B(u_m, u_m),\ v - u_m)\,\mathrm{d}t + \Phi(v) - \Phi(u_m) +$$

$$+ \int_0^T (f_{1m} + f_2,\ u_m - v)\,\mathrm{d}t + \tfrac{1}{2}|v(0) - u_{0m}|^2$$

for all $v \in L^p(0, T; W)$ with $v' \in L^{p'}(0, T; W^*)$ $(m = 1, 2, \ldots)$. By (2.6)–(2.8),

$$\limsup \int_0^T (Au_m, u_m)\,\mathrm{d}t \leqq \int_0^T (v', v - u)\,\mathrm{d}t + \int_0^T (\chi, v)\,\mathrm{d}t + \tag{2.9}$$

$$+ \int_0^T (B(u, u),\ v - u)\,\mathrm{d}t + \Phi(v) - \Phi(u) +$$

$$+ \int_0^T (f, u - v)\,\mathrm{d}t + \tfrac{1}{2}|v(0) - u_0|^2$$

for the v's having the above properties (obviously $\Phi(u) < +\infty$).

Let $x \in D(\varphi)$ be arbitrary. Set

$$v_k(t) = e^{-kt}x + k\int_0^t e^{k(s-t)}\,u(s)\,\mathrm{d}s\,, \quad t \in [0, T]$$

$(k = 1, 2, \ldots)$. In other words, v_k is the solution to the initial value problem

$$v_k(t) + \frac{1}{k}\,v_k'(t) = u(t) \quad \textit{for a.a.} \quad t \in [0, T]$$

$$v_k(0) = x\,.$$

Then it holds

$$v_k' \in L^p(0, T; W) \quad (k = 1, 2, \ldots)\,,$$

$$v_k \to u \quad \textit{strongly in} \quad L^p(0, T; W) \quad \textit{as} \quad k \to \infty\,,$$

$$\Phi(v_k) \leqq \Phi(u) + \frac{1}{k}\left[\varphi(x) + (z^*, x) + c_0\right] + \int_0^T (z^*, u - v_k)\,\mathrm{d}t$$

for $k = 1, 2, \ldots$ where $z^* \in W^*$ and $c_0 \in \mathbb{R}$ depend only on φ itself but neither on u, x nor on k (cf. [1; Lemme II.2] and its proof).

Inserting $v = v_k$ in (2.9) and letting $k \to \infty$ we get

$$\limsup \int_0^T (Au_m, u_m)\,\mathrm{d}t \leqq \int_0^T (\chi, u)\,\mathrm{d}t + \tfrac{1}{2}|x - u_0|^2\,.$$

Hence

$$\limsup \int_0^T (Au_m, u_m - u)\,\mathrm{d}t \leqq 0\,.$$

The operator $\mathscr{A}$ being monotone and hemi-continuous, we finally obtain

$$\int_0^T (Au, u - v)\,\mathrm{d}t \leqq \liminf \int_0^T (Au_m, u_m - v)\,\mathrm{d}t \leqq$$

$$\leqq \int_0^T (v' + B(u, u),\ v - u)\,\mathrm{d}t + \Phi(v) - \Phi(u) +$$

$$+ \int_0^T (f, u - v)\,\mathrm{d}t + \tfrac{1}{2}|v(0) - u_0|^2$$

for all $v \in L^p(0, T; W)$ with $v' \in L^{p'}(0, T; W^*)$.

3. PROOF OF THEOREM 2

The existence of a function $u \in L^\infty(0, T; W) \cap C([0, T]; H)$ which satisfies (1.10), (1.12), (1.13) follows from our basic existence result in [4]. If $u_1, u_2 \in L^\infty(0, T; W) \cap$

$\cap\, C([0, T]; H)$ are two functions satisfying (1.10), (1.12), (1.13) we have

$$(u_i'(t) + A\,u_i(t) + B(u_i(t), u_i(t)),\ x - u_i(t)) +$$
$$+\ \varphi(x) - \varphi(u_i(t)) \geqq (f(t),\ x - u_i(t))$$

for all $x \in W$ and a.a. $t \in [0, T]$ $(i = 1, 2)$ (cf. [1; App. I]). Observing that $|||u_i(t)||| \leqq$ $\leqq$ const for a.a. $t \in [0, T]$ $(i = 1, 2)$ one readily obtains

$$|u_1(t) - u_2(t)|^2 \leqq \text{const} \int_0^t |u_1 - u_2|^2\, \mathrm{d}s$$

for all $t \in [0, T]$, hence $u_1 \equiv u_2$.

Thus, it remains to prove the second regularity property in (1.11) and (1.14), respectively.

Part (i). Let $u \in L^\infty(0, T; W) \cap C([0, T]; H)$ be the function which satisfies (1.10), (1.12), (1.13).

Let $h \in (0, T)$. From the inequality

$$(u'(t) + A\,u(t) + B(u(t), u(t)),\ x - u(t)) +$$
$$+\ \varphi(x) - \varphi(u(t)) \geqq (f(t),\ x - u(t))$$

which holds for all $x \in W$ and for a.a. $t \in [0, T]$ we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t + h) - u(t)|^2 + c_1\|u(t + h) - u(t)\|^2 \leqq$$
$$\leqq \text{const}\, \|u(t + h) - u(t)\|\, |u(t + h) - u(t)| +$$
$$+\ \|f(t + h) - f(t)\|_*\, \|u(t + h) - u(t)\|$$

for a.a. $t \in [0, T - h]$. We now multiply each term of the inequality by $t^{2\alpha}$, add the term $\alpha t^{2\alpha - 1}|u(t + h) - u(t)|^2$ on both sides and integrate over the interval $[0, t]$ (where $t \in [0, T - h]$). Then

$$\tfrac{1}{2}(t^\alpha|u(t + h) - u(t)|)^2 + c_1 \int_0^t (s^\alpha\|u(s + h) - u(s)\|)^2\, \mathrm{d}s \leqq$$
$$\leqq \text{const} \int_0^t s^{2\alpha}\|u(s + h) - u(s)\|\, |u(s + h) - u(s)|\, \mathrm{d}s +$$
$$+ \int_0^t s^{2\alpha}\|f(s + h) - f(s)\|_*\, \|u(s + h) - u(s)\|\, \mathrm{d}s +$$
$$+\ \alpha T^{2\alpha - 1} \int_0^{T - h} |u(s + h) - u(s)|^2\, \mathrm{d}s\,,$$

and therefore

$$(3.1)\qquad (t^\alpha |u(t+h)-u(t)|)^2 + \int_0^t (s^\alpha \|u(s+h)-u(s)\|)^2 \, ds \leqq$$

$$\leqq \text{const} \left[\int_0^{T-h} |u(s+h)-u(s)|^2 \, ds + \int_0^{T-h} (s^\alpha \|f(s+h)-f(s)\|_*)^2 \, ds \right]$$

for all $t \in [0, T-h]$.

Let us now note the following criterion:

Let X be a real reflexive Banach space with norm $\| \ \|$, α *a real number* $\geqq 0$. *Then the following conditions are equivalent: if* $1 < r < +\infty$:

a) $u \in L^r(0, T; X)$, $t^\alpha u' \in L^r(0, T; X)$;

b) *there exists a positive constant C such that*

$$\int_0^{T-h} (t^\alpha \|u(t(t+h) - u(t)\|)^r \, dt \leqq Ch^r \quad \forall h \in (0, T),$$

and if $r = +\infty$:

a) $u \in L^\infty(0, T; X)$, $t^\alpha u' \in L^\infty(0, T; X)$;

b) *there exists a positive constant C such that*

$$\sup_{t \in [0, T-h]} \text{ess}\, t^\alpha \|u(t+h) - u(t)\| \leqq Ch \quad \forall h \in (0, T)$$

(*cf.* [2; App.] *for* $\alpha = 0$, [3] *for* $\alpha > 0$).

Since $f, t^\alpha f' \in L^2(0, T; V^*)$ and $u, u' \in L^2(0, T; H)$, the criterion just presented implies the existence of positive constants C_i $(i = 1, 2)$ such that

$$\int_0^{T-h} |u(t+h) - u(t)|^2 \, dt \leqq C_1 h^2 \quad \forall h \in (0, T),$$

$$\int_0^{T-h} (t^\alpha \|f(t+h) - f(t)\|_*)^2 \, dt \leqq C_2 h^2 \quad \forall h \in (0, T).$$

Inserting these estimates into (3.1) yields

$$(t^\alpha |u(t+h) - u(t)|)^2 + \int_0^t (s^\alpha \|u(s+h) - u(s)\|)^2 \, ds \leqq C_3 h^2$$

for all $t \in [0, T-h]$ and any $h \in (0, T)$ $(C_3 = \text{const} > 0)$.

The second regularity property in (1.11) is now easily seen when combining the above criterion with the latter estimate.

Part (*ii*). Let us consider the initial value problem

$$(3.2)\qquad (u_n'(t), w_i) + (A\, u_n(t), w_i) + (B(u_n(t), u_n(t)), w_i) + \\ + (C_\varepsilon(u_n(t)), w_i) = (f(t), w_i) \quad (i = 1, \ldots, n),$$

$$(3.3)\qquad u_n(0) = u_0$$

where $\{w_1, w_2, \ldots\}$ denotes the system of elements introduced in [4], where $u_n(t) = \sum_{i=1}^{n} g_{ni}(t)\, w_i$ and $u_0 \in \operatorname{span}\{w_1, \ldots, w_n\}$ for a certain natural number n_0 $(n \geqq n_0)$; the mapping C_ε $(\varepsilon > 0)$ has the same meaning as in [4].

Observing that f is continuous (from $[0, T]$ into V^*) we obtain by an analogous argument as in [4] the existence of functions $g_{ni} \in C^1([0, T])$ $(i = 1, \ldots, n)$ which satisfy (3.2) for all $t \in [0, T]$, and the initial condition (3.3). Further, as in [4] we get the estimates

$$|||u_n(t)||| \leqq \text{const} \quad \forall t \in [0, T]\,, \quad \forall n \geqq n_0\,, \quad \forall \varepsilon > 0\,;$$

$$\|u_n'\|_{L^2(0,T;H)} \leqq \text{const} \quad \forall n \geqq n_0\,, \quad \forall \varepsilon > 0\,.$$

Thus, in order to complete the proof of part (ii) it suffices to establish appropriate additional a-priori-estimates for $\{u_n'\}$ that yield the second regularity property in (1.14) when $n \to \infty$ $(\varepsilon > 0$ fixed) and then $\varepsilon \to 0$.

To this end, let $t \in [0, T)$ be arbitrary, and let $h > 0$ such that $t + h \in [0, T]$. From (3.2) we conclude that

$$|u_n(t + h) - u_n(t)|^2 + \int_0^t \|u_n(s + h) - u_n(s)\|^2 \,\mathrm{d}s \leqq$$

$$\leqq \text{const} \left(|u_n(h) - u_n(0)|^2 + \int_0^{T-h} \|f(s + h) - f(s)\|_*^2 \,\mathrm{d}s \right) +$$

$$+ \text{const} \int_0^t |u_n(s + h) - u_n(s)|^2 \,\mathrm{d}s$$

where the constants depend neither on t nor on n and ε. Dividing each term of the inequality by h^2 and applying the above criterion, one gets after letting $h \to 0$

$$(3.4) \qquad |u_n'(t)|^2 + \int_0^t \|u_n'(s)\|^2 \,\mathrm{d}s \leqq \text{const} \left(1 + |u_n'(0)|^2 + \int_0^t |u_n'(s)|^2 \,\mathrm{d}s \right)$$

for all $t \in [0, T]$.

Next, it is readily verified that our hypothesis $0 \in \partial\varphi(u_0)$ implies $C_\varepsilon(u_0) = 0$ for all $\varepsilon > 0$. We then infer from (3.2) that

$$|u_n'(0)| \leqq |f(0) - Au_0 - B(u_0, u_0)|$$

for all $n \geqq n_0$ and all $\varepsilon > 0$. Inserting the estimate into (3.4) we finally obtain

$$|u_n'(t)| \leqq \text{const} \quad \forall t \in [0, T]\,, \quad \forall n \geqq n_0\,, \quad \forall \varepsilon > 0\,;$$

$$\|u_n'\|_{L^2(0,T;V)} \leqq \text{const} \quad \forall n \geqq n_0\,, \quad \forall \varepsilon > 0\,.$$

This completes the proof of Theorem 2.

References

[1] *Brézis, H.:* Problémes unilatéraux. J. Math. Pures Appl., 51 (1972), 1—168.

[2] *Brézis, H.:* Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert. Math. Studies 5, North Holland, 1973.

[3] *Gröger, K.:* Regularitätsaussagen für Evolutionsgleichungen mit stark monotonen Operatoren. Math. Nachr. 67 (1975), 21—34.

[4] *Müller, M.* and *Naumann, J.:* On evolution inequalities of a modified Navier-Stokes type, I. Apl. Mat. 23 (1978), 174—184.

Souhrn

O EVOLUČNÍCH NEROVNOSTECH MODIFIKOVANÉHO NAVIEROVA-STOKESOVA TYPU, II

MANFRED MÜLLER, JOACHIM NAUMANN

V této části práce pokračují autoři ve studiu abstraktní evoluční nerovnosti, započatém v části I. Věta 1 se týká existence o jednoznačnosti slabého řešení vyšetřované evoluční nerovnosti. Důkaz je založen na metodě aproximace slabého řešení posloupnosti silných řešení. Věta 2 podává dva výsledky o regularitě silného řešení.

Authors' addresses: Dr. *Manfred Müller,* Dr. *Joachim Naumann,* Sektion Mathematik der Humboldt-Universität zu Berlin, 1086 Berlin, PF 1297.