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TABLES FOR THE TWO-SAMPLE HAGA TEST OF LOCATION

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Description of the test. Let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be two random samples with densities f_1, f_2, respectively. Suppose that the samples are denoted so that we have $m \leqq n$. We want to test the hypothesis H_0 that f_1 and f_2 are identical (but otherwise arbitrary) against the alternatives of shift in location which may be expressed by $f_1(x) = f(x - \Delta)$, $f_2(x) = f(x)$, where $\Delta > 0$, or $\Delta < 0$ (one-sided alternatives), or $\Delta \neq 0$ (two-sided alternative).

When testing this hypothesis by means of the Haga test we proceed as follows: First, we find the quantities A, and B', being equal to the number of observations among $X_1, \ldots, X_m$ larger than $\max\limits_{1 \leqq j \leqq n} Y_j$, or smaller than $\min\limits_{1 \leqq j \leqq n} Y_j$, respectively, and the quantities A', and B, being equal to the number of observations among $Y_1, \ldots, Y_n$ larger than $\max\limits_{1 \leqq i \leqq m} X_i$, or smaller than $\min\limits_{1 \leqq i \leqq m} X_i$, respectively. (Of course, only one of the numbers A, A' (or B, B') is positive, while the other must be equal to zero.) The statistic for the Haga test is then equal to

$$H = A + B - A' - B' .$$

This test based on H can be applied both against two-sided alternatives and against one-sided alternatives. However, against one-sided alternatives $\Delta > 0$ we could apply a simpler statistic $A + B$, which generates the locally most powerful rank test of the hypothesis H_0 against a shift Δ of a uniform distribution on the interval (α, β) for Δ close to $\beta - \alpha$, i.e. in some region $\beta - \alpha - \varepsilon < \Delta < \beta - \alpha$. (Cf. Hájek - Šidák [2], Section III. 1.2 and Problem II.13.)

Description of the table. In Table 1 we tabulate (in per cents) the upper tails $100\, P\{H \geqq k\}$ of the distribution of H under the hypothesis H_0, i.e. the one-sided significance levels for the Haga test in percents for

$$k = 7, 8, \ldots, 11; \quad \max(2, n - 10) < m \leqq n \leqq 25 ,$$

$$k = 9, 10, \ldots, 13; \quad \max(2, n - 15) < m \leqq n - 10; \quad 13 \leqq n \leqq 25 ,$$

$$k = 11, 12, \ldots, 15; \quad 2 < m \leqq n - 15 ; \quad 18 \leqq n \leqq 25 .$$

This range of levels includes almost all practically used significance levels for $3 \leqq m \leqq n \leqq 25$.

For the computation of Table 1 we have used the formulas

$$P\{H = m + n\} = P\{H = -m - n\} = \binom{m+n}{m}^{-1},$$

$$P\{H = m + n - 1\} = P\{H = -m - n + 1\} = 0\,,$$

$$P\{H = t\} = \left\{\sum_{i=1}^{|t|-1} \binom{m+n-|t|-2}{m-i-1} + \sum_{i=1}^{[(m-|t|)/2]} \binom{m+n-|t|-2i-2}{n-2} + \right.$$

$$\left. + \sum_{i=1}^{[(n-|t|)/2]} \binom{m+n-|t|-2i-2}{m-2}\right\} \binom{m+n}{n}^{-1}$$

$$\text{for} \quad t = -m - n + 2, \ldots, m + n - 2\,,$$

valid for $m \geqq 2$, $n \geqq 2$. (Cf. Hájek - Šidák [2], Theorem IV. 2.2.a, where however the restriction of the validity to $m \geqq 2$, $n \geqq 2$ is not mentioned (for $m = 1$ or $n = 1$ the formula is not true); similarly, this formula with a detailed derivation may be found in Haga [1], where the mentioned restriction is also missing.)

The distribution of the statistic H under the hypothesis H_0 is symmetric about 0, so that $P\{H \leqq -k\} = P\{H \geqq k\}$.

If we are testing H_0 against the one-sided alternative $\Delta > 0$ (i.e. f_1 is shifted to the right with respect to f_2), we use the critical region $\{H \geqq k\}$ and its significance level may be found in Table 1. If we are testing against the alternative $\Delta < 0$ (i.e. f_1 is shifted to the left with respect to f_2), we use the critical region $\{H \leqq -k\}$, but its significance level is equal to $P\{H \geqq k\}$ so that it may be found again directly in Table 1. If we are testing against the two-sided alternative $\Delta \neq 0$, we use the critical region $\{|H| \geqq k\}$ whose significance level is equal to $2P\{H \geqq k\}$.

Example. Let us have two samples, and let one of the samples consists of values 27, 36, 35, 42, 37, 32, while the other sample consists of values 18, 31, 20, 34. For the use of Table 1 we must have $m \leqq n$, and therefore the latter sample with 4 values will be denoted as $X_1, \ldots, X_4$, while the former sample with 6 values as $Y_1, \ldots, Y_6$. We can easily find $A = 0$, $B' = 2$, $A' = 4$, $B = 0$, so that $H = -6$. If we want to test against the alternative $\Delta < 0$, we may use e.g. the critical region $\{H \leqq -6\}$, and in Table 1 for $n = 6$, $m = 4$, $k = 6$ we find its significance level $P\{H \geqq 6\} = 3.810\%$. If we want to test against the alternative $\Delta \neq 0$, we may use e.g. the critical region $\{|H| \geqq 6\}$ whose significance level is twice the tabulated level, that is 7.620%.

Remark on the asymptotic distribution. Let $m, n \to \infty$, and $m/(n + m) \to p$, $q = 1 - p$. Then it is not difficult to prove (cf. Haga [1]) that

$$\lim_{m,n\to\infty} P\{H \geqq k\} = \begin{cases} \dfrac{(p^2+q)\,p}{(p-q)(1+p)}\,p^k - \dfrac{(p^2+q)\,q}{(p-q)(1+q)}\,q^k & \text{for} \quad p \neq \frac{1}{2} \\ \left(\dfrac{k}{2}+\dfrac{1}{3}\right)2^{-k} & \text{for} \quad p = \frac{1}{2}, \end{cases}$$

for $k \geqq 1$. These values are given at the end of Table 1 for $k = 5, 6, \ldots, 13$ and for $p = 0.25; 0.30; \ldots; 0.50$; the values of these limits are symmetric with respect to p about the point $p = 0.5$.

For the tied observations an analogous remark may be applied as for E-test (cf. Šidák 3). Moreover, in Šidák [3] one can find also a bibliographical remark concerning the test based on the number of exceeding observations.

References

[1] *T. Haga:* A two-sample rank test on location. Ann. Inst. Statist. Math. 11 (1959/60), 211—219.

[2] *J. Hájek, Z. Šidák:* Theory of rank tests. Academia, Prague & Academic Press, New York—London, 1967.

[3] *Z. Šidák:* Tables for the two-sample location E-test based on exceeding observations. Apl. mat. 22 (1977), 166—175.

Souhrn

TABULKY PRO DVOUVÝBĚROVÝ HAGŮV TEST POLOHY

STANISLAV HOJEK

Pořadová statistika H založená na počtu přesahujících pozorování ve dvou výběrech je vhodná pro testování polohy dvou výběrů. Tyto tabulky obsahují jednostranné hladiny významnosti
$P\{H \geqq k\}$ pro

$$k = 7, 8, \ldots, 11; \ \max(2, n-10) < m \leqq n \leqq 25$$
$$k = 9, 10, \ldots, 13; \ \max(2, n-15) < m \leqq n-10; \ 13 \leqq n \leqq 25$$
$$k = 11, 12, \ldots, 15; \ 2 < m \leqq n-15; \ 18 \leqq n \leqq 25$$

což zahrnuje téměř všechny obvykle užívané hladiny významnosti pro $3 \leqq m \leqq n \leqq 25$, kde m, n jsou rozsahy výběrů.

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Table 1. One-sided significance levels 100 $P\{H \geqq k\}$ (i.e. in percents)

n	m	k 5	6	7	8	9	10	11	12	13
3	3	5·000	5·000	—	—	—	—	—	—	—
4	3	5·714	2·857	2·857	—	—	—	—	—	—
	4	5·714	2·857	1·429	1·429	—	—	—	—	—
5	3	7·143	3·571	1·786	1·786	—	—	—	—	—
	4	6·349	3·175	1·587	0·794	0·794	—	—	—	—
	5	6·349	3·175	1·587	0·794	0·397	0·397	—	—	—
6	3	8·333	4·762	2·381	1·190	1·190	—	—	—	—
	4	7·143	3·810	1·905	0·952	0·476	0·476	—	—	—
	5	6·710	3·463	1·732	0·866	0·433	0·216	0·216	—	—
	6	6·710	3·463	1·732	0·866	0·433	0·216	0·108	0·108	—
7	3	10·000	5·833	3·333	1·667	0·833	0·833	—	—	—
	4	8·182	4·545	2·424	1·212	0·606	0·303	0·303	—	—
	5	7·323	3·914	2·020	1·010	0·505	0·253	0·126	0·126	—
	6	6·993	3·671	1·865	0·932	0·466	0·233	0·117	0·058	0·058
	7	6·993	3·671	1·865	0·932	0·466	0·233	0·117	0·058	0·029
8	3	11·515	7·273	4·242	2·424	1·212	0·606	0·606	—	—
	4	9·293	5·455	3·030	1·616	0·808	0·404	0·202	0·202	—
	5	8·081	4·507	2·409	1·243	0·622	0·311	0·155	0·078	0·078
	6	7·459	4·029	2·098	1·066	0·533	0·266	0·133	0·067	0·033
	7	7·211	3·838	1·974	0·995	0·497	0·249	0·124	0·062	0·031
	8	7·211	3·838	1·974	0·995	0·497	0·249	0·124	0·062	0·031
9	3	13·182	8·636	5·455	3·182	1·818	0·909	0·455	0·455	—
	4	10·490	6·434	3·776	2·098	1·119	0·559	0·280	0·140	0·140
	5	8·941	5·195	2·897	1·548	0·799	0·400	0·200	0·100	0·050
	6	8·052	4·496	2·418	1·259	0·639	0·320	0·160	0·080	0·040
	7	7·579	4·126	2·168	1·110	0·559	0·280	0·140	0·070	0·035
	8	7·384	3·974	2·065	1·049	0·527	0·263	0·132	0·066	0·033
	9	7·384	3·974	2·065	1·049	0·527	0·263	0·132	0·066	0·033
10	3	14·685	10·140	6·643	4·196	2·448	1·399	0·699	0·350	0·350
	4	11·688	7·493	4·595	2·697	1·499	0·799	0·400	0·200	0·100
	5	9·857	5·961	3·463	1·931	1·032	0·533	0·266	0·133	0·067
	6	8·729	5·045	2·810	1·511	0·787	0·400	0·200	0·100	0·050
	7	8·052	4·504	2·432	1·275	0·653	0·329	0·165	0·082	0·041
	8	7·681	4·210	2·228	1·150	0·583	0·293	0·146	0·073	0·037

Table 1. (*Continued*)

n	m	k								
		5	6	7	8	9	10	11	12	13
	9	7·525	4·085	2·142	1·097	0·553	0·277	0·139	0·069	0·035
	10	7·525	4·085	2·142	1·097	0·553	0·277	0·139	0·069	0·035
11	3	16·209	11·538	7·967	5·220	3·297	1·923	1·099	0·549	0·275
	4	12·894	8·571	5·495	3·370	1·978	1·099	0·586	0·293	0·147
	5	10·806	6·777	4·098	2·381	1·328	0·710	0·366	0·183	0·092
	6	9·462	5·656	3·264	1·818	0·978	0·509	0·259	0·129	0·065
	7	8·600	4·952	2·756	1·486	0·779	0·399	0·210	0·101	0·050
	8	8·068	4·522	2·450	1·291	0·666	0·337	0·169	0·085	0·042
	9	7·769	4·282	2·281	1·184	0·604	0·304	0·152	0·076	0·038
	10	7·641	4·179	2·208	1·138	0·577	0·290	0·145	0·073	0·036
	11	7·641	4·179	2·208	1·138	0·577	0·290	0·145	0·073	0·036
12	3	17·582	12·967	9·231	6·374	4·176	2·637	1·538	0·879	0·440
	4	14·066	9·670	6·429	4·121	2·527	1·484	0·824	0·440	0·220
	5	11·765	7·628	4·783	2·893	1·681	0·937	0·501	0·259	0·129
	6	10·229	6·313	3·771	2·176	1·212	0·652	0·339	0·172	0·086
	7	9·201	5·454	3·130	1·740	0·939	0·492	0·252	0·127	0·064
	8	8·521	4·895	2·722	1·471	0·775	0·399	0·202	0·102	0·051
	9	8·091	4·545	2·470	1·307	0·677	0·345	0·174	0·087	0·044
	10	7·846	4·345	2·327	1·215	0·623	0·315	0·158	0·079	0·040
	11	7·739	4·258	2·265	1·175	0·599	0·302	0·151	0·076	0·038
	12	7·739	4·258	2·265	1·175	0·599	0·302	0·151	0·076	0·038
13	4	15·210	10·756	7·395	4·916	3·151	1·933	1·134	0·630	0·336
	5	12·722	8·497	5·509	3·455	2·089	1·214	0·677	0·362	0·187
	6	11·017	7·003	4·320	2·580	1·489	0·829	0·446	0·232	0·118
	7	9·836	5·996	3·546	2·034	1·131	0·610	0·320	0·164	0·083
	8	9·022	5·314	3·036	1·686	0·911	0·480	0·247	0·125	0·063
	9	8·472	4·859	2·702	1·462	0·773	0·400	0·204	0·103	0·051
	10	8·118	4·569	2·490	1·322	0·688	0·352	0·178	0·089	0·045
	11	7·913	4·401	2·368	1·243	0·640	0·325	0·164	0·082	0·041
	12	7·822	4·327	2·315	1·207	0·618	0·313	0·157	0·079	0·039
	13	7·822	4·327	2·315	1·207	0·618	0·313	0·157	0·079	0·039
14	5	13·665	9·374	6·261	4·059	2·546	1·539	0·894	0·499	0·267
	6	11·811	7·714	4·902	3·024	1·806	1·042	0·580	0·312	0·163
	7	10·494	6·569	3·998	2·364	1·356	0·754	0·407	0·213	0·109
	8	9·557	5·770	3·386	1·932	1·073	0·579	0·305	0·157	0·080
	9	8·897	5·216	2·969	1·646	0·890	0·470	0·244	0·124	0·063
	10	8·443	4·838	2·690	1·457	0·772	0·401	0·205	0·104	0·052
	11	8·146	4·593	2·510	1·337	0·698	0·359	0·182	0·092	0·046

Table 1. (*Continued*)

n	*m*	*k*								
		5	6	7	8	9	10	11	12	13
	12	7·972	4·450	2·405	1·267	0·655	0·334	0·169	0·085	0·042
	13	7·894	4·386	2·358	1·236	0·636	0·323	0·163	0·082	0·041
	14	7·894	4·386	2·358	1·236	0·636	0·323	0·163	0·082	0·041
15	6	12·605	8·438	5·510	3·501	2·160	1·290	0·745	0·415	0·223
	7	11·166	7·164	4·479	2·726	1·612	0·925	0·514	0·277	0·145
	8	10·117	6·255	3·766	2·208	1·260	0·700	0·378	0·199	0·103
	9	9·354	5·606	3·268	1·857	1·029	0·556	0·294	0·152	0·078
	10	8·807	5·145	2·921	1·617	0·875	0·463	0·241	0·123	0·062
	11	8·426	4·827	2·684	1·456	0·773	0·403	0·207	0·105	0·053
	12	8·173	4·617	2·529	1·351	0·707	0·365	0·186	0·094	0·047
	13	8·024	4·493	2·437	1·289	0·669	0·342	0·173	0·087	0·044
	14	7·957	4·438	2·396	1·262	0·652	0·332	0·168	0·084	0·042
	15	7·957	4·438	2·396	1·262	0·652	0·332	0·168	0·084	0·042
16	7	11·843	7·774	4·984	3·116	1·896	1·121	0·643	0·358	0·193
	8	10·692	6·761	4·172	2·511	1·472	0·840	0·466	0·252	0·133
	9	9·836	6·022	3·594	2·092	1·188	0·658	0·356	0·188	0·097
	10	9·203	5·482	3·180	1·800	0·995	0·538	0·285	0·148	0·076
	11	8·742	5·094	2·886	1·596	0·863	0·458	0·239	0·123	0·062
	12	8·417	4·822	2·682	1·456	0·774	0·405	0·208	0·106	0·054
	13	8·200	4·641	2·547	1·364	0·716	0·370	0·189	0·096	0·048
	14	8·070	4·532	2·467	1·310	0·682	0·350	0·178	0·090	0·045
	15	8·012	4·484	2·430	1·285	0·667	0·341	0·173	0·087	0·044
	16	8·012	4·484	2·430	1·285	0·667	0·341	0·173	0·087	0·044
17	8	11·278	7·284	4·599	2·837	1·707	1·001	0·571	0·317	0·171
	9	10·335	6·458	3·942	2·350	1·368	0·777	0·431	0·233	0·123
	10	9·622	5·844	3·462	2·004	1·133	0·627	0·339	0·179	0·093
	11	9·088	5·389	3·112	1·756	0·970	0·524	0·278	0·145	0·074
	12	8·695	5·056	2·860	1·580	0·855	0·454	0·237	0·122	0·062
	13	8·415	4·821	2·682	1·458	0·776	0·407	0·210	0·107	0·054
	14	8·226	4·663	2·564	1·376	0·725	0·376	0·192	0.098	0·049
	15	8·112	4·568	2·493	1·328	0·694	0·357	0·182	0.092	0·046
	16	8·060	4·524	2·461	1·306	0·680	0·349	0·177	0·089	0·045
	17	8·060	4·524	2·461	1·306	0·680	0·349	0·177	0·089	0·045
18	9	10·846	6·911	4·309	2·628	1·567	0·912	0·518	0·287	0·155
	10	10·059	6·225	3·764	2·227	1·288	0·729	0·403	0·218	0·115
	11	9·456	5·706	3·360	1·935	1·090	0·602	0·325	0·173	0·090
	12	8·999	5·316	3·060	1·722	0·949	0·513	0·273	0·142	0·073
	13	8·660	5·028	2·841	1·568	0·849	0·451	0·236	0·122	0·062

Table 1. (*Continued*)

n	m	k								
		5	6	7	8	9	10	11	12	13
	14	8·416	4·823	2·685	1·460	0·779	0·409	0·211	0·108	0·055
	15	8·250	4·683	2·580	1·388	0·733	0·381	0·195	0·099	0·050
	16	8·150	4·599	2·517	1·345	0·705	0·364	0·186	0·094	0·047
	17	8·104	4·561	2·488	1·325	0·692	0·356	0·181	0·092	0·046
	18	8·104	4·561	2·488	1·325	0·692	0·356	0·181	0·092	0·046
19	10	10·509	6·622	4·084	2·467	1·459	0·844	0·477	0·264	0·143
	11	9·842	6·042	3·625	2·130	1·226	0·691	0·381	0·206	0·109
	12	9·326	5·597	3·279	1·880	1·056	0·582	0·315	0·167	0·087
	13	8·930	5·259	3·019	1·695	0·933	0·504	0·268	0·140	0·072
	14	8·634	5·008	2·827	1·560	0·844	0·449	0·235	0·122	0·062
	15	8·419	4·827	2·689	1·464	0·782	0·411	0·213	0·109	0·055
	16	8·273	4·703	2·596	1·399	0·740	0·385	0·198	0·101	0·051
	17	8·184	4·628	2·539	1·360	0·715	0·370	0·189	0·096	0·048
	18	8·143	4·593	2·513	1·342	0·703	0·363	0·185	0·094	0·047
	19	8·143	4·593	2·513	1·342	0·703	0·363	0·185	0·094	0·047
20	11	10·242	6·393	3·907	2·341	1·374	0·791	0·446	0·246	0·133
	12	9·669	5·896	3·514	2·053	1·175	0·660	0·364	0·197	0·104
	13	9·221	5·510	3·214	1·836	1·029	0·566	0·306	0·162	0·085
	14	8·876	5·215	2·986	1·674	0·920	0·497	0·264	0·139	0·072
	15	8·615	4·993	2·816	1·554	0·841	0·448	0·235	0·122	0·062
	16	8·425	4·832	2·694	1·468	0·785	0·413	0·215	0·110	0·056
	17	8·295	4·722	2·610	1·409	0·747	0·390	0·201	0·103	0·052
	18	8·215	4·654	2·559	1·374	0·724	0·376	0·193	0·098	0·049
	19	8·178	4·623	2·535	1·357	0·713	0·369	0·189	0·096	0·048
	20	8·178	4·623	2·535	1·357	0·713	0·369	0·189	0·096	0·048
21	12	10·026	6·209	3·765	2·239	1·307	0·748	0·420	0·231	0·125
	13	9·529	5·777	3·424	1·990	1·135	0·635	0·349	0·189	0·100
	14	9·137	5·440	3·161	1·800	1·006	0·553	0·298	0·159	0·083
	15	8·833	5·179	2·960	1·656	0·910	0·492	0·261	0·137	0·071
	16	8·601	4·982	2·809	1·549	0·839	0·447	0·235	0·122	0·063
	17	8·432	4·838	2·699	1·472	0·788	0·415	0·216	0·111	0·057
	18	8·315	4·739	2·623	1·419	0·753	0·394	0·203	0·104	0·053
	19	8·243	4·678	2·577	1·387	0·732	0·381	0·196	0·100	0·050
	20	8·209	4·650	2·556	1·372	0·722	0·375	0·192	0·098	0·049
	21	8·209	4·650	2·556	1·372	0·722	0·375	0·192	0·098	0·049
22	13	9·850	6·058	3·649	2·156	1·252	0·713	0·399	0·220	0·119
	14	9·415	5·680	3·351	1·939	1·102	0·615	0·338	0·182	0·097
	15	9·069	5·382	3·119	1·771	0·988	0·542	0·292	0·155	0·082

Table 1. (*Continued*)

n	m	k								
		5	6	7	8	9	10	11	12	13
	16	8·798	5·151	2·939	1·643	0·902	0·487	0·259	0·136	0·071
	17	8·591	4·974	2·804	1·546	0·838	0·447	0·235	0·122	0·063
	18	8·439	4·845	2·705	1·476	0·791	0·418	0·218	0·112	0·057
	19	8·334	4·755	2·636	1·428	0·760	0·398	0·206	0·105	0·054
	20	8·269	4·700	2·594	1·399	0·740	0·386	0·199	0·101	0·051
	21	8·238	4·674	2·574	1·385	0·731	0·380	0·195	0·099	0·050
	22	8·238	4·674	2·574	1·385	0·731	0·380	0·195	0·099	0·050
23	14	9·705	5·934	3·553	2·088	1·206	0·685	0·382	0·210	0·113
	15	9·320	5·600	3·290	1·896	1·074	0·598	0·328	0·177	0·094
	16	9·012	5·335	3·083	1·747	0·972	0·533	0·287	0·153	0·080
	17	8·770	5·128	2·922	1·632	0·895	0·483	0·257	0·135	0·070
	18	8·585	4·969	2·800	1·545	0·837	0·446	0·235	0·122	0·063
	19	8·447	4·852	2·710	1·481	0·795	0·420	0·219	0·113	0·058
	20	8·352	4·771	2·648	1·437	0·765	0·402	0·208	0·107	0·054
	21	8·293	4·720	2·610	1·410	0·747	0·390	0·201	0·103	0·052
	22	8·265	4·697	2·592	1·397	0·739	0·385	0·198	0·101	0·051
	23	8·265	4·697	2·592	1·397	0·739	0·385	0·198	0·101	0·051
24	15	9·584	5·831	3·473	2·032	1·168	0·661	0·368	0·202	0·109
	16	9·241	5·533	3·239	1·861	1·051	0·584	0·320	0·172	0·092
	17	8·965	5·296	3·054	1·726	0·959	0·525	0·283	0·151	0·079
	18	8·748	5·109	2·909	1·623	0·889	0·480	0·256	0·134	0·070
	19	8·580	4·966	2·798	1·543	0·836	0·446	0·235	0·122	0·063
	20	8·456	4·859	2·716	1·485	0·798	0·422	0·220	0·114	0·058
	21	8·369	4·785	2·659	1·445	0·771	0·405	0·210	0·108	0·055
	22	8·314	4·739	2·624	1·420	0·754	0·395	0·204	0·104	0·053
	23	8·289	4·718	2·607	1·408	0·747	0·390	0·201	0·103	0·052
	24	8·289	4·718	2·607	1·408	0·747	0·390	0·201	0·103	0·052
25	16	9·482	5·743	3·406	1·984	1·137	0·641	0·356	0·195	0·105
	17	9·174	5·477	3·196	1·831	1·031	0·572	0·313	0·169	0·090
	18	8·926	5·263	3·029	1·710	0·949	0·519	0·279	0·149	0·078
	19	8·730	5·094	2·898	1·615	0·885	0·478	0·254	0·134	0·070
	20	8·577	4·964	2·797	1·543	0·836	0·447	0·235	0·123	0·063
	21	8·464	4·867	2·722	1·490	0·801	0·424	0·222	0·115	0·059
	22	8·384	4·799	2·670	1·453	0·776	0·408	0·212	0·109	0·056
	23	8·335	4·757	2·637	1·429	0·761	0·399	0·207	0·106	0·054
	24	8·312	4·737	2·622	1·419	0·753	0·394	0·204	0·104	0·053
	25	8·312	4·737	2·622	1·419	0·753	0·394	0·204	0·104	0·053

Table 1. (*Continued*)

n	m	k								
		7	8	9	10	11	12	13	14	15
13	3	18·929	14·286	10·536	7·500	5·179	3·393	2·143	1·250	0·714
14	3	20·147	15·588	11·765	8·676	6·176	4·265	2·794	1·765	1·029
	4	16·307	11·830	8·366	5·752	3·824	2·451	1·503	0·882	0·490
15	3	21·324	16·789	12·990	9·804	7·230	5·147	3·554	2·328	1·471
	4	17·363	12·874	9·340	6·605	4·541	3·019	1·935	1·187	0·697
	5	14·590	10·249	7·030	4·696	3·044	1·909	1·155	0·671	0·374
16	3	22·394	17·957	14·138	10·939	8·256	6·089	4·334	2·993	1·961
	4	18·369	13·891	10·299	7·472	5·284	3·633	2·415	1·548	0·949
	5	15·490	11·116	7·809	5·357	3·578	2·320	1·455	0·880	0·511
	6	13·392	9·169	6·137	4·007	2·546	1·571	0·938	0·541	0·302
17	3	23·421	19·035	15·263	12·018	9·298	7·018	5·175	3·684	2·544
	4	19·332	14·871	11·245	8·338	6·048	4·277	2·941	1·955	1·253
	5	16·363	11·969	8·590	6·034	4·139	2·764	1·792	1·124	0·680
	6	14·167	9·899	6·777	4·536	2·962	1·882	1·161	0·693	0·400
	7	12·521	8·394	5·507	3·530	2·207	1·343	0·794	0·456	0·253
18	4	20·246	15·817	12·167	9·200	6·822	4·949	3·500	2·406	1·599
	5	17·207	12·806	9·367	6·722	4·722	3·239	2·164	1·403	0·880
	6	14·927	10·626	7·424	5·083	3·402	2·221	1·412	0·871	0·520
	7	13·195	9·019	6·044	3·965	2·542	1·589	0·967	0·572	0·328
	8	11·868	7·818	5·044	3·184	1·964	1·182	0·693	0·396	0·220
19	5	18·022	13·622	10·138	7·416	5·322	3·738	2·564	1·713	1·110
	6	15·670	11·345	8·076	5·643	3·863	2·586	1·688	1·073	0·662
	7	13·861	9·645	6·591	4·417	2·897	1·857	1·161	0·707	0·418
	8	12·458	8·359	5·502	3·549	2·241	1·382	0·832	0·488	0·278
	9	11·364	7·377	4·692	2·924	1·783	1·063	0·619	0·352	0·195
20	6	16·394	12·054	8·727	6·212	4·340	2·971	1·989	1·299	0·825
	7	14·518	10·270	7·145	4·882	3·272	2·146	1·376	0·860	0·524
	8	13·047	8·905	5·971	3·930	2·535	1·600	0·987	0·594	0·348
	9	11·886	7·851	5·089	3·236	2·017	1·230	0·733	0·427	0·242
	10	10·968	7·032	4·419	2·724	1·645	0·973	0·563	0·318	0·176
21	7	15·163	10·890	7·702	5·358	3·662	2·454	1·610	1·032	0·645
	8	13·630	9·452	6·449	4·324	2·846	1·836	1·159	0·715	0·430
	9	12·409	8·331	5·497	3·563	2·265	1·412	0·861	0·513	0·299
	10	11·433	7·451	4·767	2·994	1·845	1·114	0·659	0·381	0·216

Table 1. (*Continued*)

n	m	k: 7	8	9	10	11	12	13	14	15
	11	10·651	6·757	4·203	2·565	1·536	0·902	0·519	0·292	0·161
22	8	14·208	10·000	6·932	4·729	3·171	2·087	1·346	0·850	0·524
	9	12·932	8·816	5·914	3·901	2·528	1·608	1·002	0·611	0·364
	10	11·901	7·878	5·125	3·278	2·059	1·268	0·766	0·453	0·262
	11	11·068	7·130	4·510	2·803	1·710	1·024	0·601	0·346	0·195
	12	10·394	6·534	4·028	2·438	1·449	0·846	0·484	0·272	0·150
23	9	13·450	9·302	6·337	4·251	2·804	1·817	1·156	0·720	0·439
	10	12·371	8·309	5·493	3·573	2·284	1·435	0·884	0·534	0·316
	11	11·490	7·512	4·828	3·052	1·896	1·157	0·693	0·407	0·234
	12	10·770	6·869	4·302	2·648	1·602	0·952	0·556	0·318	0·179
	13	10·182	6·351	3·885	2·334	1·378	0·800	0·456	0·255	0·140
24	10	12·840	8·744	5·867	3·877	2·522	1·613	1·013	0·624	0·377
	11	11·914	7·899	5·155	3·311	2·093	1·300	0·793	0·475	0·279
	12	11·151	7·213	4·586	2·869	1·766	1·068	0·635	0·370	0·212
	13	10·523	6·654	4·131	2·522	1·514	0·894	0·519	0·296	0·166
	14	10·006	6·199	3·766	2·248	1·320	0·762	0·433	0·242	0·133
25	11	12·339	8·291	5·489	3·580	2·300	1·453	0·903	0·551	0·330
	12	11·536	7·562	4·879	3·100	1·939	1·193	0·722	0·429	0·250
	13	10·870	6·965	4·387	2·720	1·660	0·996	0·588	0·341	0·195
	14	10·316	6·474	3·989	2·417	1·442	0·846	0·488	0·277	0·155
	15	9·859	6·072	3·666	2·176	1·271	0·731	0·413	0·230	0·126

n	m	k: 9	10	11	12	13	14	15	16	17
18	3	24·361	20·075	16·316	13·083	10·301	7·970	6·015	4·436	3·158
19	3	25·260	21·039	17·338	14·091	11·299	8·896	6·883	5·195	3·831
	4	21·118	16·725	13·066	10·051	7·600	5·635	4·088	2·891	1·988
20	3	26·087	21·965	18·295	15·076	12·253	9·825	7·736	5·985	4·517
	4	21·946	17·598	13·938	10·888	8·376	6·334	4·696	3·407	2·409
	5	18·807	14·417	10·898	8·110	5·933	4·257	2·991	2·052	1·370
21	3	26·877	22·826	19·219	16·008	13·192	10·721	8·597	6·769	5·237
	4	22·735	18·435	14·783	11·708	9·146	7·036	5·320	3·945	2·862

Table 1. (*Continued*)

n	m	k								
		9	10	11	12	13	14	15	16	17
	5	19·562	15·190	11·645	8·802	6·551	4·792	3·439	2·416	1·657
	6	17·098	12·751	9·375	6·788	4·832	3·376	2·311	1·547	1·010
22	3	27·609	23·652	20·087	16·913	14·087	11·609	9·435	7·565	5·957
	4	23·485	19·237	15·599	12·508	9·906	7·739	5·953	4·502	3·338
	5	20·289	15·940	12·377	9·488	7·172	5·338	3·904	2·802	1·968
	6	17·781	13·434	10·019	7·366	5·333	3·796	2·652	1·816	1·215
	7	15·795	11·504	8·261	5·843	4·065	2·778	1·861	1·221	0·783
23	3	28·308	24·423	20·923	17·769	14·962	12·462	10·269	8·346	6·692
	4	24·199	20·006	16·387	13·288	10·655	8·439	6·593	5·071	3·835
	5	20·987	16·666	13·093	10·167	7·794	5·891	4·384	3·207	2·302
	6	18·445	14·103	10·655	7·946	5·842	4·230	3·011	2·104	1·440
	7	16·413	12·110	8·820	6·334	4·480	3·117	2·130	1·427	0·936
	8	14·777	10·544	7·419	5·143	3·509	2·353	1·548	0·999	0·631
24	3	28·957	25·162	21·709	18·598	15·795	13·299	11·077	9·128	7·419
	4	24·879	20·742	17·148	14·046	11·389	9·133	7·233	5·651	4·347
	5	21·658	17·369	13·792	10·836	8·414	6·450	4·876	3·628	2·654
	6	19·087	14·756	11·282	8·524	6·357	4·674	3·384	2·409	1·683
	7	17·017	12·708	9·376	6·828	4·904	3·468	2·413	1·649	1·105
	8	15·337	11·085	7·908	5·564	3·857	2·632	1·765	1·161	0·749
	9	13·965	9·788	6·766	4·609	3·091	2·039	1·322	0·840	0·524
25	3	29·579	25·855	22·466	19·383	16·606	14·103	11·874	9·890	8·150
	4	25·527	21·448	17·881	14·783	12·109	9·819	7·873	6·236	4·871
	5	22·303	18·048	14·474	11·494	9·030	7·012	5·375	4·063	3·024
	6	19·710	15·393	11·900	9·099	6·874	5·126	3·769	2·729	1·942
	7	17·606	13·295	9·928	7·325	5·335	3·831	2·710	1·885	1·288
	8	15·887	11·621	8·398	5·991	4·216	2·922	1·994	1·337	0·880
	9	14·474	10·273	7·198	4·975	3·389	2·273	1·500	0·972	0·618
	10	13·307	9·181	6·247	4·191	2·770	1·801	1·152	0·723	0·446

$m, n \to \infty$	k								
$\frac{m}{m+n}$	5	6	7	8	9	10	11	12	13
0·25	16·495	12·387	9·294	6·972	5·229	3·922	2·941	2·206	1·655
0·30	13·557	9·534	6·687	4·685	3·281	2·297	1·608	1·126	0·788
0·35	11·419	7·528	4·930	3·217	2·095	1·364	0·887	0·577	0·375
0·40	9·969	6·204	3·811	2·322	1·408	0·850	0·512	0·308	0·185
0·45	9·129	5·452	3·193	1·843	1·053	0·597	0·336	0·188	0·105
0·50	8·854	5·208	2·995	1·693	0·944	0·521	0·285	0·155	0·083