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ON THE EXISTENCE OF A WEAK SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR THE EQUILIBRIUM OF A SHALLOW SHELL REINFORCED WITH STIFFENING RIBS

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The paper deals with the existence and the unicity of a weak solution of the boundary value problem for a shallow shell reinforced with stiffening ribs. The boundary value problem is formulated as a direct variational problem, hence we obtain a weak (or variational) solution of the problem (the corresponding bilinear form is not symmetric). The method of finite elements is used for numerical analysis of our problem.

1. FORMULATION OF THE PROBLEM

A shallow shell (Fig. 1) is a thin planar construction the tension of which is small compared with the radius of curvature of the middle surface. The theory of shallow shells is based on the assumption that the middle surface of the shell with its small

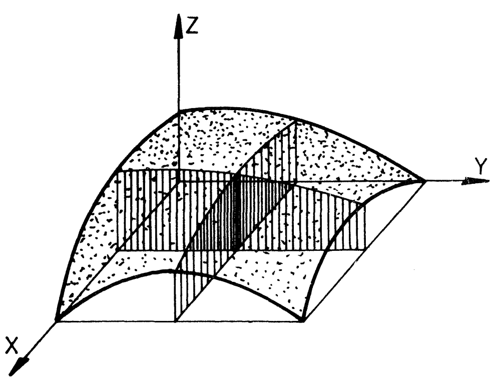

Fig. 1.

tension obeys the laws of Euclidean geometry in the plane with greatest values of projection of the whole surface.

We suppose that the middle surface of the shell can be expressed in Cartesian coordinates (x, y, z) by the equation $z = z(x, y)$. Then under the assumptions

$$\left|\frac{\partial z}{\partial x}\right|, \left|\frac{\partial z}{\partial y}\right| \ll 1$$

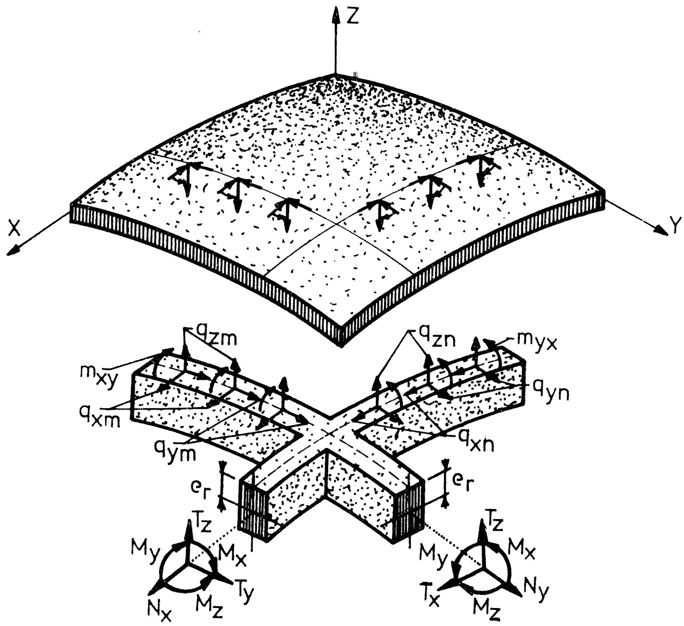

Fig. 2.

the middle surface of the shell obeys the laws of Euclidean geometry in its projection plane Ω. The stiffening ribs have generally the form of curved beams of a constant cross section. The reference axis of the rib is the line connecting the centres of all cross sections of the beam. We assume that the ribs are placed on the inner side of the shell ($e_r > 0$, Fig. 2). We denote by

$$(1.1) \qquad q_n = \langle q_{xn}, q_{yn}, q_{zn}, m_{yx} \rangle^{\mathrm{T}} ,$$
$$q_m = \langle q_{ym}, q_{xm}, q_{zm}, m_{xy} \rangle^{\mathrm{T}}$$

the vectors of the unknown contact forces on the boundary of the contact between the shell and the n-th rib (in the direction X) and the m-th rib (in the direction Y), respectively (Fig. 4; positively directed inner forces of the rib and of the shell as well as the components of the surface powers are shown in Figs. 2, 3, 4).

The equation for the equilibrium of the shell is of the operator form

$$[P]\{u\} = \{p\}, \tag{1.2}$$

where

$$\{p\} = \langle p_{NX}, p_{NY}, p_{NZ} \rangle^{\mathsf{T}}$$

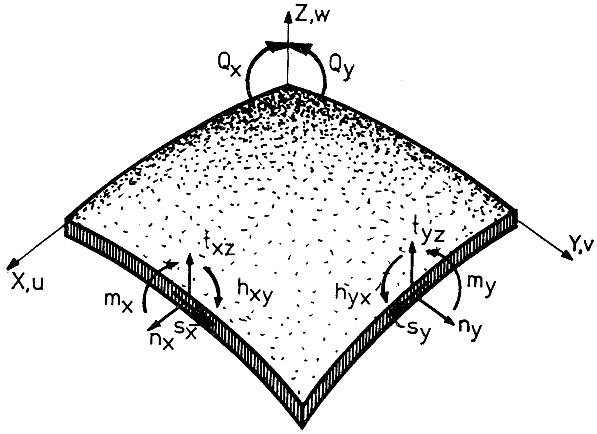

Fig. 3.

is the vector of the surface strains (Fig. 4),

$$\{u\} = \langle u, v, w \rangle^{\mathsf{T}}$$

is the displacement vector of the middle surface of the shell.

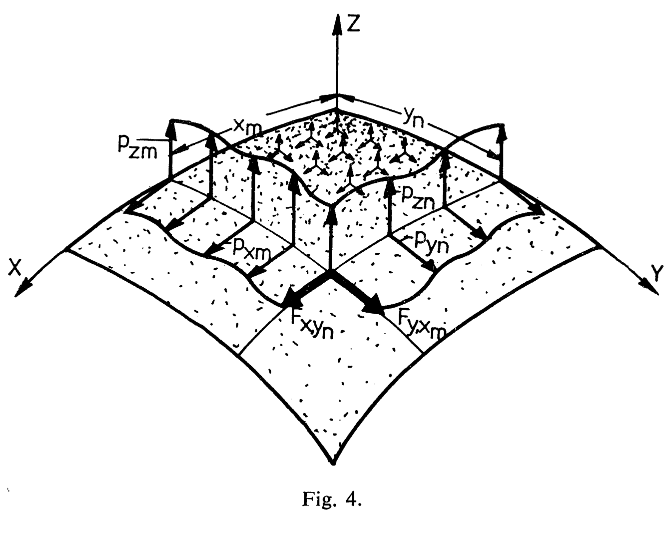

Fig. 4.

The elements of the matrix $[P]$ are differential operators exactly expressed in [6], [7]. We do not express their explicit form here. The equations of statical compatibility for thin curved beams are expressed in the form

(1.3) $$[R]\{u_r\} = \{q_r\}$$

with

$$\{u_r\} = \langle u_r, v_r, w_r, \Theta_r \rangle^{\mathrm{T}} ,$$

where u_r, v_r, w_r express the displacement of the points of the axis of the rib and Θ_r expresses the rotation of the cross section of the rib. $\{q_r\}$ is the vector of generalized forces (1.1), $[R] = [K]$, where the elements of the matrix $[K]$ – differential operators – will be introduced in the end of this chapter. We suppose that the boundary of the contact between the rib and the shell is only on the intersection of the perpendicular plane of the rib and the middle surface of the shell. The perpendicular axis Z of the rib is the main central axis of the inertia, which is orthogonal to the "supporting" surface (r_1, r_2) of the shell. We formulate therefore the conditions of the contact in the following way:

1) statical

$$q_{xn} = -\Delta n_x , \quad q_{yn} = -\Delta s_x , \quad q_{zn} = -\Delta t_{xz} , \quad m_{yx} = -\Delta m_y$$

2) geometrical

$$u_r^* = u , \quad v_r^* = v , \quad w_r^* = w ,$$

$\Theta_r = \Theta_x$, (rotation of the orthogonal element of the shell)
where

$$u_r^* = u_r - e_r \Theta_r , \quad v_r^* = v_r - e_r \frac{\mathrm{d}w_r}{\mathrm{d}y} .$$

Using the stiffness matrix of the rib we express the unknown vector of the load, which acts on the surface of the contact between the shell and the n-th and the m-th rib, respectively, in the following way ([6], [7])

(1.5) $$\{q_n\} = [K_n]\{U\} , \quad \{q_m\} = [K_m]\{U\} , \quad \{U\} = (\{u\}, \Theta_x)$$

(1.6) $$[K_n] = \begin{bmatrix} b_{11} & 0 & 0 & b_{14} \\ 0 & b_{22} & b_{23} & 0 \\ 0 & b_{32} & b_{33} & 0 \\ b_{41} & 0 & 0 & b_{44} \end{bmatrix}$$

$$b_{11} = D_{rx} \frac{\mathrm{d}^4}{\mathrm{d}y^4} , \quad b_{22} = -B_r \frac{\mathrm{d}^2}{\mathrm{d}y^2} ,$$

$$b_{33} = D_{rx} \left[\frac{\mathrm{d}^4}{\mathrm{d}y^4} + \frac{1}{\varrho_r^2} \left(e_r \frac{\mathrm{d}^2}{\mathrm{d}y^2} + k_2 \right)^2 \right]$$

(1.7)
$$b_{44} = -\left(G_r + k_2 D_{rx} e_r - D_{rx} e_r^2 \frac{d^2}{dy^2}\right) \frac{d^2}{dy^2}$$

$$b_{14} = -b_{41} = D_{rx}\left(e_r \frac{d^2}{dy^2} - k^2\right) \frac{d^2}{dy^2} - k_2 G_r \frac{d^2}{dy^2}$$

$$b_{23} = -b_{32} = B_r\left[(k_2 \varrho_r^2 - e_r) \frac{d^2}{dy^2} - k_2 \frac{d}{dy}\right]$$

D_{rx}, B_r, G_r are the flexural, bending and torsional rigidities of the rib, ϱ_r is the cross section function (the radius of inertia of the cross section surface of the rib), k_2, k_1 are the main curvatures of the shell, e_r is the distance between the neutral axis of the cross section surface of the rib and the inner surface of the rib.

The stiffness matrix $[K_m]$ is of the same form with the operator d/dx instead of d/dy.

2. FUNDAMENTAL SPACES OF THE PROBLEM

For simplicity we assume the region Ω to be rectangular: $\Omega = (a, b) \times (c, d)$. We assume further that Ω is divided by the help of a division

$$a = x_0 < x_1 < \ldots < x_n = b, \; n \geqq 2$$

$$c = y_0 < y_1 < \ldots < y_m = c, \; m \geqq 2$$

$$\bar{\Omega} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \bar{R}_{ij}, \quad R_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j).$$

The stiffening ribs are defined by

$$I_{x_i} = x_i \times (c, d), \quad i = 1, \ldots, n-1$$

$$I_{y_j} = (a, b) \times y_j, \quad j = 1, \ldots, m-1.$$

Let $\mathscr{D}(\Omega)$ be the set of all arbitrarily differentiable functions with a compact support in Ω, $H_0^m(\Omega)$ the closure of $\mathscr{D}(\Omega)$ in the Sobolev space $H^m(\Omega)$, $V(\Omega) = H_0^1(\Omega) \times \times H_0^1(\Omega) \times H_0^2(\Omega)$. We shall define a weak solution of the problem (1.2), (1.3), (1.4) in the space

$$W(\Omega) = \Big\{\{z\} = \langle u, v, w\rangle \in V(\Omega);\, u_{y_j} \in H_0^1(a, b),\, u_{x_i} \in H_0^2(c, d),\, u_{y_j}(x_i) = u_{x_i}(y_j),$$

$$v_{y_j} \in H_0^2(a, b),\, v_{x_i} \in H_0^1(c, d),\, v_{y_j}(x_i) = v_{x_j}(y_j),\, w_{y_j} \in H_0^2(a, b),\, w_{x_i} \in H_0^2(c, d),$$

$$\left.\frac{\partial w}{\partial y}\right|_{y_j} \in H_0^2(a, b),\, \left.\frac{\partial w}{\partial x}\right|_{x_i} \in H_0^2(c, d),\, \left.\frac{\partial w}{\partial x}\right|_{x_i}(y_j) = \frac{\partial w_{y_j}}{\partial x}(x_i),\, \left.\frac{\partial w_{x_i}}{\partial y}\right|(y_j) = \left.\frac{\partial w}{\partial y}\right|_{y_j}(x_i),$$

$$\left.\frac{\partial}{\partial x}\frac{\partial w}{\partial y}\right|_{y_j}(x_i) = \left.\frac{\partial}{\partial y}\frac{\partial w}{\partial x}\right|_{x_i}(y_j)\Big\},$$

where $u_{y_j}(x)$, $u_{x_i}(y)$ are the traces of the function u on I_{y_j}, I_{x_i}, respectively.

We denote further

$$u(x_i, y_j) = u_{y_j}(x_i) = u_{x_i}(y_j)\,, \quad v(x_i, y_j) = v_{y_j}(x_i) = v_{x_i}(y_j)\,,$$

$$\frac{\partial^2 w}{\partial x\, \partial y}(x_i, y_j) = \frac{\partial}{\partial x}\frac{\partial w}{\partial y}\bigg|_{y_j}(x_i) = \frac{\partial}{\partial y}\frac{\partial w}{\partial x}\bigg|_{x_i}(y_j)\,.$$

Let $(.,.)_m$ be the scalar product in the space $H_0^m(\Omega)$. We define the scalar product in $W(\Omega)$ by

$$(2.1) \qquad ((\{z_1\}, \{z_2\})) = [\{z_1\}, \{z_2\}] + \langle\{z_1\}, \{z_2\}\rangle\,,$$

where

$$(2.2) \qquad [\{z_1\}, \{z_2\}] = (u_1, u_2)_1 + (v_1, v_2)_1 + (w_1, w_2)_2$$

$$(2.3)\ ^{*)} \qquad \langle\{z_1\}, \{z_2\}\rangle = \sum_{j=1}^{m-1} \int_a^b \Bigg[u'_{1y_j} u'_{2y_j} + v''_{1y_j} v''_{2y_j} + w''_{1y_j} w''_{2y_j} + \left(\frac{\partial w_1}{\partial y}\bigg|_{y_j}\right)'' \left(\frac{\partial w_2}{\partial y}\bigg|_{y_j}\right)'' \Bigg] \mathrm{d}x +$$

$$+ \sum_{i=1}^{n-1} \int_c^d \Bigg[\dot{u}_{1x_i} \dot{u}_{2x_i} + \dot{v}_{1x_i} \dot{v}_{2x_i} + \ddot{w}_{1x_i} \ddot{w}_{2x_i} + \left(\frac{\partial w_1}{\partial x}\bigg|_{x_i}\right)^{\cdot\cdot} \left(\frac{\partial w_2}{\partial x}\bigg|_{x_i}\right)^{\cdot\cdot} \Bigg] \mathrm{d}y\,.$$

We denote the norm in $W(\Omega)$ by

$$(2.4) \qquad |||\{z\}||| = ((\{z\}, \{z\}))^{1/2} = (\|\{z\}\|^2 + |\{z\}|^2)^{1/2}\,,$$

where

$$(2.5) \qquad \|\{z\}\|^2 = [\{z\}, \{z\}]\,,$$

$$(2.6) \qquad |\{z\}|^2 = \langle\{z\}, \{z\}\rangle\,.$$

Theorem 1. *The set $W(\Omega)$ with the scalar product* (2.1), (2.2), (2.3) *is a Hilbert space.*

Proof. It can be verified easily that $W(\Omega)$ is a linear space. The bilinear form (2.1) has all the properties of a scalar product. It remains to show that $W(\Omega)$ with the norm (2.4) is a Banach space. Let $\{z_k\} \in W(\Omega)$ be a Cauchy sequence in the norm (2.4) Then $\{z_k\} \to \{z\} \in V(\Omega) = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$. Using the imbedding theorems in Sobolev spaces ([8]) we obtain

$$u_{ky_j} \to u_{y_j}\,, \quad v_{ky_j} \to v_{y_j}\,, \quad w_{ky_j} \to w_{y_j}\,, \quad \frac{\partial w_k}{\partial y}\bigg|_{y_j} \to \frac{\partial w}{\partial y}\bigg|_{y_j}$$

*) We denote here $u' = \partial u/\partial x$ and $u^{\cdot} = \partial u/\partial y$.

for every $j = 1, \ldots, m - 1$ in the space $L_2(a, b)$ and

$$u_{kx_i} \to u_{x_i}, \quad v_{kx_i} \to v_{x_i}, \quad w_{kx_i} \to w_{x_i}, \quad \frac{\partial w_k}{\partial x}\Big|_{x_i} \to \frac{\partial w}{\partial x}\Big|_{x_i},$$

for every $i = 1, \ldots, n - 1$ in the space $L_2(c, d)$.

Let φ be an arbitrary function from $\mathscr{D}(\Omega)$. We have

$$\frac{\partial w_k}{\partial y}\Big|_{y_j} \in H_0^2(a, b)$$

for every $k = 1, 2, \ldots$ and hence

$$\int_a^b \left(\frac{\partial w_k}{\partial y}\Big|_{y_j}\right)'' \varphi(x)\,\mathrm{d}x = \int_a^b \frac{\partial w_k}{\partial y}\Big|_{y_j} \varphi''(x)\,\mathrm{d}x\,; \tag{2.7}$$

$(\partial w_k/\partial y|_{y_j})''$ is a Cauchy sequence in $L_2(a, b)$ and hence $(\partial w_k/\partial y|_{y_j})'' \to g(x)$ in the space $L_2(a, b)$.

It follows from (2.7) that

$$\int_a^b g(x)\,\varphi(x)\,\mathrm{d}x = \int_a^b \frac{\partial w}{\partial y}\Big|_{y_j} \varphi''(x)\,\mathrm{d}x$$

for every function $\varphi \in \mathscr{D}(a, b)$. Hence $\partial w/\partial y|_{y_j} \in H_0^2(a, b)$ and $(\partial w/\partial y|_{y_j})'' = g(x) \in L_2(a, b)$. The other properties of the function $\{z\}$ can be verified in a similar way. We show for instance that the function w satisfies the relations

$$\frac{\partial}{\partial x}\frac{\partial w}{\partial y}\Big|_{y_j}(x_i) = \frac{\partial}{\partial y}\frac{\partial w}{\partial x}\Big|_{x_i}(y_j)\,. \tag{2.8}$$

Using the assumption $\{z_k\} = \langle u_k, v_k, w_k\rangle \in W(\Omega)$ we obtain

$$\frac{\partial}{\partial x}\frac{\partial w_k}{\partial y}\Big|_{y_j}(x_i) = \frac{\partial}{\partial y}\frac{\partial w_k}{\partial x}\Big|_{x_i}(y_j)\,. \tag{2.9}$$

As we have shown above

$$\frac{\partial w_k}{\partial y}\Big|_{y_j} \to \frac{\partial w}{\partial y}\Big|_{y_j} \quad \text{in} \quad H_0^2(a, b)$$

and

$$\frac{\partial w_k}{\partial x}\Big|_{x_i} \to \frac{\partial w}{\partial x}\Big|_{x_i} \quad \text{in} \quad H_0^2(c, d)\,.$$

Using the imbedding theorem we obtain

$$\frac{\partial}{\partial x}\frac{\partial w_k}{\partial y}\Big|_{y_j}(x_i) \to \frac{\partial}{\partial x}\frac{\partial w}{\partial y}\Big|_{y_j}(x_i)$$

and

$$\frac{\partial}{\partial y}\frac{\partial w_k}{\partial x}\bigg|_{x_i}(y_j) \to \frac{\partial}{\partial y}\frac{\partial w}{\partial x}\bigg|_{x_i}(y_j)\,.$$

By virtue of the last relations and (2.9) we arrive at (2.8). Hence $\{z\} \in W(\Omega)$ and $W(\Omega)$ is a Hilbert space which is the assertion of the theorem.

The next theorem is important for the convergence of the finite element method in the space $W(\Omega)$. Let us denote

$$\mathscr{D}^3(\Omega) = \mathscr{D}(\Omega) \times \mathscr{D}(\Omega) \times \mathscr{D}(\Omega)\,.$$

Theorem 2. *The set $\mathscr{D}^3(\Omega)$ is dense in the space $W(\Omega)$ with the norm* (2.4).

We verify first the following lemma:

Lemma 1. *Let us denote*

$$\mathring{W}(\Omega) = \left\{\{z\} \in W(\Omega)\,;\quad u_{x_i} = v_{x_i} = w_{x_i} = \frac{\partial w}{\partial x}\bigg|_{x_i} = u_{y_j} = v_{y_j} = w_{y_j} = \frac{\partial w}{\partial y}\bigg|_{y_j} \equiv 0\right\},$$

$$\mathscr{D}_W(\Omega) = \mathscr{D}^3(\Omega) \cap \mathring{W}(\Omega)\,.$$

The set $\mathscr{D}_W(\Omega)$ is dense in $\mathring{W}(\Omega)$ in the norm (2.4).

Proof. As the function $\{z\} \in \mathring{W}(\Omega)$ satisfies the homogeneous conditions on the "ribs" I_{x_i}, I_{y_j} we obtain

$$\{z\} = \langle u, v, w\rangle \in \mathring{W}(\Omega) \Rightarrow u \in H_0^1(R_{ij})\,,\quad v \in H_0^1(R_{ij})\,,\quad w \in H_0^2(R_{ij})$$

for every $i = 1, \dots, n - 1$ and $j = 1, \dots, m - 1$.

Therefore there exist sequences $\{z_k^{ij}\} \in \mathscr{D}^3(R_{ij})$ satisfying $\{z_k^{ij}\} \to \{z\}$ in the spaces $V(R_{ij})$. We define such a sequence $\{z_k\} \in \mathscr{D}_W(\Omega)$ that $\{z_k\}|R_{ij} = \{z_k^{ij}\}$. As $\bar{\Omega} = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}\bar{R}_{ij}$, we obtain that $\{z_k\} \to \{z\}$ in the space $V(\Omega)$ with the norm (2.5) which is in $W(\Omega)$ identical with the norm (2.4). This completes the proof.

Proof of Theorem 2. Let $\varphi_{x_i}(x) \in \mathscr{D}(a, b)$, $\varphi_{y_j}(y) \in \mathscr{D}(c, d)$ be test functions satisfying

$$\begin{aligned}
&\varphi_{x_i}(x) = 1\,,\quad x \in (x_i - \alpha_i, x_i + \alpha_i)\,,\quad \alpha_i < \min\{x_i - x_{i-1}, x_{i+1} - x_i\}\,,\\
&\varphi_{y_j}(y) = 1\,,\quad y \in (y_j - \beta_j, y_j + \beta_j)\,,\quad \beta_j < \min\{y_j - y_{j-1}, y_{j+1} - y_j\}\,,\\
&0 \leqq \varphi_{x_i}(x) \leqq 1\,,\quad x \in (a, b)\,;\quad 0 \leqq \varphi_{y_j}(y) \leqq 1\,,\quad y \in (c, d)\,,\\
&i = 1, \dots, n - 1\,;\quad j = 1, \dots, m - 1\,.
\end{aligned}$$

We express the function $\{z\} = \langle u, v, w\rangle \in W(\Omega)$ in the form

(2.10) $\qquad \{z\} = \{z_0\} + \{Z\}\,,\quad \{z_0\} = \langle u_0, v_0, w_0\rangle\,,\quad \{Z\} = \langle U, V, W\rangle\,,$

where

$$(2.11)\quad U = \sum_{i=1}^{n-1} u_{x_i}(y)\,\varphi_{x_i}(x) + \sum_{j=1}^{m-1} u_{y_j}(x)\,\varphi_{y_j}(y) - \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u(x_i, y_j)\,\varphi_{x_i}(x)\,\varphi_{y_j}(y)\,,$$

$$V = \sum_{i=1}^{n-1} v_{x_i}(y)\,\varphi_{x_i}(x) + \sum_{j=1}^{m-1} v_{y_j}(x)\,\varphi_{y_j}(y) - \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} v(x_i, y_j)\,\varphi_{x_i}(x)\,\varphi_{y_j}(y)\,,$$

$$W = \sum_{i=1}^{n-1} w_{x_i}(y)\,\varphi_{x_i}(x) + \sum_{j=1}^{m-1} w_{y_j}(x)\,\varphi_{y_j}(y) - \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} w(x_i, y_j)\,\varphi_{x_i}(x)\,\varphi_{y_j}(y) +$$

$$+ \sum_{i=1}^{n-1} \partial w/\partial x|_{x_i}^{(y)}(y)\,(x - x_i)\,\varphi_{x_i}(x) + \sum_{j=1}^{m-1} \partial w/\partial y|_{y_j}(x)\,(y - y_j)\,\varphi_{y_j}(y) -$$

$$- \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} [\partial w/\partial x|_{x_i}^{(y)}(y_j)\,(x - x_i) + \partial w/\partial y|_{y_j}(x_i)\,(y - y_j) +$$

$$+ \partial^2 w/\partial x\,\partial y\,(x_i, y_j)\,(x - x_i)\,(y - y_j)]\,\varphi_{x_i}(x)\,\varphi_{y_j}(y)\,.$$

According to the properties of the function $\{z\}$ we have

$$\{Z\} \in W(\Omega)\,, \quad \{z_0\} \in \mathring{W}(\Omega)\,.$$

In virtue of Lemma 1 there exists a sequence

$$\{z_{0_k}\} \in \mathscr{D}_W(\Omega) = \mathscr{D}^3(\Omega) \cap \mathring{W}(\Omega)$$

which satisfies

$$(2.12)\qquad \lim_{k\to\infty} |||\{z_{0_k}\} - \{z_0\}||| = 0\,.$$

There exist sequences $u_k^{x_i}, v_k^{x_i}, \tilde{w}_k^{x_i}, w_k^{x_i} \in \mathscr{D}(c, d)$, $u_k^{y_j}, v_k^{y_j}, w_k^{y_j}, \tilde{w}_k^{y_j} \in \mathscr{D}(a, b)$ which satisfy the relations

$$(2.13)\qquad u_k^{x_i} \to u_{x_i}\,, \quad w_k^{x_i} \to w_{x_i}\,, \quad \tilde{w}_k^{x_i} \to \frac{\partial w}{\partial x}\bigg|_{x_i} \quad \text{in} \quad H_0^2(c, d)\,,$$

$$v_k^{x_i} \to v_{x_i} \quad \text{in} \quad H_0^1(c, d)\,, \quad u_k^{y_j} \to u_{y_j} \quad \text{in} \quad H_0^1(a, b)\,,$$

$$v_k^{y_j} \to v_{y_j}\,, \quad w_k^{y_j} \to w_{y_j}\,, \quad \tilde{w}_k^{y_j} \to \frac{\partial w}{\partial y}\bigg|_{y_j} \quad \text{in} \quad H_0^2(a, b)\,.$$

Using the imbedding theorems we obtain

$$(2.14)\qquad u_k^{x_i}(y_j) \to u(x_i, y_j)\,, \quad u_k^{y_j}(x_i) \to u(x_i, y_j)\,,$$

$$v_k^{x_i}(y_j) \to v(x_i, y_j)\,, \quad v_k^{y_j}(x_i) \to v(x_i, y_j)\,,$$

$$w_k^{x_i}(y_j) \to w(x_i, y_j)\,, \quad w_k^{y_j}(x_i) \to w(x_i, y_j)\,,$$

$$\tilde{w}_k^{x_i}(y_j) \to \frac{\partial w}{\partial x}\bigg|_{x_i}(y_j)\,, \quad \tilde{w}_k^{y_j}(x_i) \to \frac{\partial w}{\partial y}\bigg|_{y_j}(x_i)\,,$$

$$\frac{\partial \tilde{w}_k^{x_i}}{\partial y}(y_j) \to \frac{\partial^2 w}{\partial x \partial y}(x_i, y_j)\,, \frac{\partial \tilde{w}_k^{y_j}}{\partial x}(x_i) \to \frac{\partial^2 w}{\partial x \partial y}(x_i, y_j)\,.$$

We define now the function $\{Z_k\} = \langle U_k, V_k, W_k \rangle$ in the following way:

$$\begin{aligned}
U_k &= \sum_{i=1}^{n-1} u_k^{x_i}(y)\, \varphi_{x_i}(y) + \sum_{j=1}^{m-1} u_k^{y_j}(x)\, \varphi_{y_j}(y) - \\
&\quad - \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} u_k^{x_i}(y_j)\, \varphi_{x_i}(x)\, \varphi_{y_j}(y)\,, \\
V_k &= \sum_{i=1}^{n-1} v_k^{x_i}(y)\, \varphi_{x_i}(x) + \sum_{j=1}^{m-1} v_k^{y_j}(x)\, \varphi_{y_j}(y) - \\
&\quad - \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} v_k^{x_i}(y_j)\, \varphi_{x_i}(x)\, \varphi_{y_j}(y)\,, \\
W_k &= \sum_{m-1}^{n-1} w_k^{x_i}(y)\, \varphi_{x_i}(x) + \sum_{j=1}^{m-1} w_k^{y_j}(x)\, \varphi_{y_j}(y) - \\
&\quad - \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} w_k^{x_i}(y_j)\, \varphi_{x_i}(x)\, \varphi_{y_j}(y) + \\
&\quad + \sum_{i=1}^{n-1} \tilde{w}_k^{x_i}(y)\,(x - x_i)\, \varphi_{x_i}(x) + \sum_{j=1}^{m-1} \tilde{w}_k^{y_j}(x)\,(y - y_j)\, \varphi_{y_j}(y) - \\
&\quad - \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} [\tilde{w}_k^{x_i}(y_j)\,(x - x_i) + \tilde{w}_k^{y_j}(x_i)\,(y - y_j) - \\
&\quad - \frac{\partial \tilde{w}_k^{x_i}}{\partial y}(y_j)\,(x - x_i)\,(y - y_j)]\varphi_{x_i}(x)\, \varphi_{y_j}(y)\,.
\end{aligned}$$

Let us further introduce the sequence $\{z_k\} = \{z_{0_k}\} + \{Z_k\}$. It can be seen easily that $\{z_k\} \in \mathscr{D}^3(\Omega)$.

Using (2.13), (2.14) and the properties of the function $\{z\} \in W(\Omega)$ we obtain in the same way as in [3]

$$\lim_{k \to \infty} |||\{Z_k\} - \{Z\}||| = 0\,. \tag{2.15}$$

Combining (2.12), (2.15) we have

$$\lim_{k \to \infty} |||\{z_k\} - \{z\}||| = 0\,, \quad \{z_k\} \in \mathscr{D}^3(\Omega) \tag{2.16}$$

and the proof is complete.

3. WEAK SOLUTION OF THE BOUNDARY VALUE PROBLEM

We introduce the following notation. Let $\{z\} = \langle u, v, w \rangle \in W(\Omega)$. We denote

$$\{\varepsilon(\{z\})\}^{\mathsf{T}} = \langle \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle\,, \quad \{\varkappa(\{z\})\}^{\mathsf{T}} = \langle \varkappa^1, \varkappa^2, \varkappa^3 \rangle \tag{3.1}$$

where

$$\varepsilon^1 = \frac{\partial u}{\partial x} + k_1 w\,; \quad \varkappa^1 = \frac{\partial^2 w}{\partial x^2}$$

(3.2)
$$\varepsilon^2 = \frac{\partial v}{\partial y} + k_2 w\,; \quad \varkappa^2 = \frac{\partial^2 w}{\partial y^2}$$

$$\varepsilon^3 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\,; \quad \varkappa^3 = 2\,\frac{\partial^2 w}{\partial x \partial y}\,.$$

The stiffness matrix is defined by

(3.3)
$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \dfrac{1-\mu}{2} \end{bmatrix}_{(3,3)},$$

$$0 < \mu < 1\,.$$

Let us denote vectors and matrices by

(3.4)
$$\{\varepsilon_{x_i}\}^{\mathsf{T}} = \Big\langle \dot{u}_{x_i}, \ddot{u}_{x_i}, \dot{v}_{x_i}, \ddot{w}_{x_i}, e_{x_i}\ddot{w}_{x_i} + k_2 w_{x_i}, \left(\frac{\partial w}{\partial x}\Big|_{x_i} - k_2\, u_{x_i}\right)^{\cdot}, \left(\frac{\partial w}{\partial x}\Big|_{x_i} - k_2\, u_{x_i}\right)^{\cdot\cdot}\Big\rangle$$

(3.5)
$$[K_{x_i}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & k_2(D_{x_i} + G_{x_i}) & 0 \\ 0 & D_{x_i} & 0 & 0 & 0 & 0 & D_{x_i}e_{x_i} \\ 0 & 0 & B_{x_i} & -B_{x_i}k_2\varrho^2_{x_i} & -B_{x_i} & 0 & 0 \\ 0 & 0 & -B_{x_i}k_2\varrho^2_{x_i} & D_{x_i} & 0 & 0 & 0 \\ 0 & 0 & -B_{x_i} & 0 & D_{x_i}\varrho^{-2}_{x_i} & 0 & 0 \\ -k_2(D_{x_i} + G_{x_i}) & 0 & 0 & 0 & 0 & G_{x_i} + k_2 D_{x_i}e_{x_i} & 0 \\ 0 & -D_{x_i}e_{x_i} & 0 & 0 & 0 & 0 & D_{x_i}e^2_{x_i} \end{bmatrix}_{(7,7)}$$

for $i = 1, \ldots, n-1$,

(3.6)
$$\{\varepsilon_{y_j}\}^{\mathsf{T}} = \Big\langle v'_{y_j}, v''_{y_j}, u'_{y_j}, w''_{y_j}, e_{y_j}w''_{y_j} + k_1 w_{y_j}, \left(\frac{\partial w}{\partial y}\Big|_{y_j} - k_1 v_{y_j}\right)' \left(\frac{\partial w}{\partial y}\Big|_{y_j} - k_1 v_{y_j}\right)''\Big\rangle$$

(3.7)

$$[K_{y_j}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & k_1(D_{y_j} + G_{y_j}) & 0 \\ 0 & D_{y_j} & 0 & 0 & 0 & 0 & D_{y_j} e_{y_j} \\ 0 & 0 & B_{y_j} & -B_{y_j} k_1 \varrho_{y_j}^2 & -B_{y_j} & 0 & 0 \\ 0 & 0 & -B_{y_j} k_1 \varrho_{y_j}^2 & D_{y_j} & 0 & 0 & 0 \\ 0 & 0 & -B_{y_j} & 0 & D_{y_j} \varrho_{y_j}^{-2} & 0 & 0 \\ -k_1(D_{y_j} + G_{y_j}) & 0 & 0 & 0 & 0 & G_{y_j} + k_1 D_{y_j} e_{y_j} & 0 \\ 0 & -D_{y_j} e_{y_j} & 0 & 0 & 0 & 0 & D_{y_j} e_{y_j}^2 \end{bmatrix}_{(7,7)}$$

for $j = 1, \ldots, m - 1$.

We introduce now the bilinear forms belonging to the problem (1.2), (1.3), (1.4). Assume that $\{z_1\}, \{z_2\} \in W(\Omega)$. Then

$$A(\{z_1\}, \{z_2\}) = h \int_\Omega \{\varepsilon_1\}^{\mathrm{T}} [D] \{\varepsilon_2\} \, \mathrm{d}\Omega + \frac{h^3}{12} \int_\Omega \{\varkappa_1\}^{\mathrm{T}} [D] \{\varkappa_2\} \, \mathrm{d}\Omega$$

(3.8)

$$b_{x_i}(\{z_1\}, \{z_2\}) = \int_{Ix_i} \{\varepsilon_{1x_i}\}^{\mathrm{T}} [K_{x_i}] \{\varepsilon_{2x_i}\} \, \mathrm{d}y \,, \quad i = 1, \ldots, n - 1 \,,$$

(3.9) $$b_{y_j}(\{z_1\}, \{z_2\}) = \int_{Iy_j} \{\varepsilon_{1y_j}\}^{\mathrm{T}} [K_{y_j}] \{\varepsilon_{2y_j}\} \, \mathrm{d}x \,, \quad j = 1, \ldots, m - 1 \,,$$

(3.10) $$b(\{z_1\}, \{z_2\}) = \sum_{i=1}^{n-1} b_{x_i}(\{z_1\}, \{z_2\}) + \sum_{j=1}^{m-1} b_{y_j}(\{z_1\}, \{z_2\}) \,,$$

(3.11) $$a(\{z_1\}, \{z_2\}) = A(\{z_1\}, \{z_2\}) + b(\{z_1\}, \{z_2\}) \,.$$

Let $\{p\} = \langle p_1, p_2, p_3 \rangle^{\mathrm{T}} \in [L_2(\Omega)]^3$. We define a linear bounded functional $l \in W(\Omega)^*$ by

(3.12)

$$l(\{z\}) = \int_\Omega \{p\}^{\mathrm{T}} \{z\} \, \mathrm{d}\Omega + \sum_{i=1}^{n-1} \sum_{j-1}^{m-1} u_{x_i}(y_j) + u_{y_j}(x_i) + v_{x_i}(y_j) + v_{y_j}(x_i) + w_{x_i}(y_j) + w_{y_j}(x_i) + \left(\frac{\partial w}{\partial x} \bigg|_{x_i} - k_2 u_{x_i} \right)(y_j) + \left(\frac{\partial w}{\partial y} \bigg|_{y_j} - k_1 v_{y_j} \right)(x_i) \,, \quad \{z\} \in W(\Omega) \,.$$

Definition 1. *Let $\{p\} \in L_2(\Omega)^3$. A vector-function $\{z_0\} \in W(\Omega)$ is a weak solution of the problem* (1.2), (1.3), (1.4) *if it is a solution of the equation*

(3.13) $$a(\{z_0\}, \{z\}) = l(\{z\}) \quad \text{(equation of virtual work)}$$

for every vector-function $\{z\} = \langle u, v, w \rangle \in W(\Omega)$, where $l \in W(\Omega)^*$ is defined in (3.12).

We shall show that the problem (1.2), (1.3), (1.4) has a unique weak solution. According to the Lax-Milgram theorem it is sufficient to show that the bilinear form $a(\{z_1\}, \{z_2\})$ is coercive, i.e.

$$a(\{z\}, \{z\}) \geqq \alpha |||\{z\}|||^2 , \quad \forall\{z\} \in W(\Omega) . \tag{3.14}$$

We verify first that the bilinear form $A(\{z_1\}, \{z_2\})$ is coercive on the space $V(\Omega) \supset \supset W(\Omega)$. In virtue of the positive definiteness of the matrix $[D]$ we obtain

$$A(\{z\}, \{z\}) \geqq \alpha \int_\Omega (|\{\varepsilon\}|^2 + |\{\varkappa\}|^2) \, d\Omega =$$

$$= \alpha \int_\Omega \left[\left(\frac{\partial u}{\partial x} + k_1 w \right)^2 + \left(\frac{\partial v}{\partial y} + k_2 w \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \right. \tag{3.15}$$

$$\left. + \left(\frac{\partial^2 w}{\partial x^2} \right) + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4 \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right] d\Omega , \quad \alpha > 0 , \quad \forall z \in V(\Omega) .$$

We shall use in our further considerations the results from [2] about the inequalities of Korn's type, employing the same notation as in [2]. We introduce therefore the operators

$$N_1\{z\} = \varepsilon^1 , \quad N_2\{z\} = \varepsilon^2 , \quad N_3\{z\} = \varepsilon^3 , \tag{3.16}$$

$$N_4\{z\} = \varkappa^1 , \quad N_5\{z\} = \varkappa^2 , \quad N_6\{z\} = \varkappa^3 .$$

The components u, v, w are denoted by

$$u = u_1 , \quad v = u_2 , \quad w = u_3 . \tag{3.17}$$

The inequality (3.15) has now the form

$$A(\{z\}, \{z\}) \geqq \alpha \sum_{l=1}^{6} \|N_l\{z\}\|^2_{L_2(\Omega)} . \tag{3.18}$$

The operators N_l have the form

$$N_l(\{z\}) = \sum_{s=1}^{3} \sum_{|\alpha| \leqq k_s} n_{l_{s\alpha}} D^\alpha u_s . \tag{3.19}$$

We define the components of the matrix $N(\{\xi\})_{3\times 6}$ by

$$N_{l_s}(\{\xi\}) = \sum_{|\alpha| = k_s} n_{l_{s\alpha}} \xi_1^{\alpha_1} \xi_2^{\alpha_2} , \quad |\alpha| = \alpha_1 + \alpha_2 . \tag{3.20}$$

We have in our case

$$N(\{\xi\}) = \begin{bmatrix} \xi_1 & 0 & \xi_2 & 0 & 0 & 0 \\ 0 & \xi_2 & \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1^2 & \xi_2^2 & 2\xi_1\xi_2 \end{bmatrix}_{(3,6)} . \tag{3.21}$$

If $\{\xi\} \neq \{0\}$ then *rank* $(N(\{\xi\})) = 3$ and by Theorem 3.1 from [2] we obtain that the system $\{N_l\}$ is coercive on $V(\Omega)$, i.e.

$$(3.22) \qquad \sum_{l=1}^{6} \|N_l(\{z\})\|^2_{L_2(\Omega)} + \|\{z\}\|^2_{[L_2(\Omega)]^3} \geqq c\|\{z\}\|^2_{V(\Omega)}\,, \quad c > 0\,.$$

Our further considerations will be the same as in the proof of Theorem 2.1 in [2]. Let us denote

$$P_1 = \{\{z\} \in V(\Omega)\,, \quad \sum_{l=1}^{6} \|N_l\{z\}\,{}^2_{L_2(\Omega)}\} = 0\,.$$

Let $\{z\} \in P_1$. Using the fact that $w \in H_0^2(\Omega)$ we obtain $w = 0$. Using the form of ε^1 and ε^2 in (3.2) we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0\,.$$

We assume $u, v \in H_0^1(\Omega)$. Using the Fridrichs inequality ([8]) we obtain $u = v = 0$ and hence $\{z\} = \{0\}$. We have now the relation

$$(3.23) \qquad P_1 = \{0\}\,.$$

Let us assume that the bilinear form $A(.,.)$ is not coercive in $V(\Omega)$. Then there exists a sequence $\{z_k\} \in V(\Omega)$ satisfying the relations

$$(3.24) \qquad \|\{z_k\}\|_{V(\Omega)} = 1\,,$$

$$(3.25) \qquad A(\{z_k\}, \{z_k\}) < \frac{1}{k}\,, \quad k = 1, 2, \ldots\,.$$

We can choose a subsequence (we do not change the notation) $\{z_k\}$ such that $\{z_k\} \rightharpoonup \{z\}$ in $V(\Omega)$, $\{z_k\} \to \{z\}$ in $[L_2(\Omega)]^3$ and $N_l(\{z_k\}) \rightharpoonup N_l(\{z_k\})$ in $L_2(\Omega)$. Combining (3.18) and (3.25) we arrive at

$$(3.26) \qquad \|N_l(\{z_k\})\|^2_{L_2(\Omega)} \leqq \frac{1}{\alpha k}\,, \quad l = 1, \ldots, 6\,, \quad k = 1, 2, \ldots\,.$$

The weak convergence of $N_l(\{z_k\})$ implies that

$$\|N_l\{z\}\| \leqq \liminf_{k\to\infty} N_l(\{z_k\}) = 0\,.$$

Combining (3.23) and the last relation we have $\{z\} = \{0\}$. Using the inequality (3.22) we obtain $\{z_k\} \to \{0\}$ in $V(\Omega)$ which contradicts (3.24). This means that the bilinear form $A(.,.)$ is coercive, i.e.

$$(3.27) \qquad A(\{z\}, \{z\}) \geqq \alpha_1\|\{z\}\|^2\,, \quad \alpha_1 > 0\,, \quad \forall\{z\} \in V(\Omega)\,.$$

If we verify the inequality

$$(3.28) \qquad b(\{z\}), \{z\}) \geqq \alpha_2|\{z\}|^2\,, \quad \alpha_2 > 0\,, \quad \forall\{z\} \in W(\Omega)$$

then after combining (3.11), (3.27), (3.28) we obtain the coerciveness of the bilinear form $a(\{z_1\}, \{z_2\})$. With regard to (2.2), (2.6), (3.10) it is sufficient to verify the inequalities

$$(3.29) \qquad b_{x_i}(\{z\}, \{z\}) \geqq \beta_{x_i} \int_c^d \left[\ddot{u}_{x_i}^2 + \dot{v}_{x_i}^2 + \ddot{w}_{x_i}^2 + \left(\left. \frac{\partial w}{\partial x} \right|_{x_i} \right)^{\cdot\cdot 2} \right] \mathrm{d}y \,,$$

$$\beta_{x_i} > 0 \,, \quad i = 1, \ldots, n-1 \,, \quad \forall \{z\} \in W(\Omega) \,,$$

$$(3.30) \qquad b_{y_j}(\{z\}, \{z\}) \geqq \beta_{y_j} \int_a^b \left[u_{y_j}'^2 + v_{y_j}''^2 + w_{y_j}''^2 + \left(\left. \frac{\partial w}{\partial y} \right|_{y_j} \right)''^2 \right] \mathrm{d}x \,,$$

$$\beta_{y_j} > 0 \,, \quad j = 1, \ldots, m-1 \,, \quad \forall \{z\} \in W(\Omega) \,.$$

We verify that the inequality (3.29) holds. Using (3.4), (3.5), (3.9) we express $b_{x_i}(\{z\}, \{z\})$ in the form

$$(3.31) \qquad b_{x_i}(\{z\}, \{z\}) = \int_c^d \{\hat{\varepsilon}_{x_i}\}^{\mathrm{T}} [K_{x_i}] \{\hat{\varepsilon}_{x_i}\} \, \mathrm{d}y \,,$$

where

$$(3.32) \qquad \{\hat{\varepsilon}_{x_i}\}^{\mathrm{T}} = \left\langle \ddot{u}_{x_i}, \dot{v}_{x_i}, \ddot{w}_{x_i}, e_{x_i} \ddot{w}_{x_i} + k_2 w_{x_i}, \right.$$
$$\left. \left(\left. \frac{\partial w}{\partial x} \right|_{x_i} - k_2 u_{x_i} \right)^{\cdot} ; \left(\left. \frac{\partial w}{\partial x} \right|_{x_i} - k_2 u_{x_i} \right)^{\cdot\cdot} \right\rangle ,$$

$$(3.33) \quad [\hat{K}_{x_i}] = \begin{bmatrix} D_{x_i} & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{x_i} & -B_{x_i} k_2 \varrho_{x_i}^2 & -B_{x_i} & 0 & 0 \\ 0 & -B_{x_i} k_2 \varrho_{x_i}^2 & D_{x_i} & 0 & 0 & 0 \\ 0 & -B_{x_i} & 0 & D_{x_i} \varrho_{x_i}^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{x_i} + k_2 D_{x_i} e_{x_i} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{x_i} e_{x_i}^2 \end{bmatrix}_{(6,6)}$$

The quantity ϱ_{x_i} is sufficiently small compared with the other quantities in the matrix. All the main minors are then positive and hence the matrix $[\hat{K}_{x_i}]$ is symmetric and positive definite. From (3.31) we obtain now the relation

$$(3.34) \qquad b_{x_i}(\{z\}, \{z\}) \geqq \hat{\beta}_{x_i} \int_c^d |\{\hat{\varepsilon}_{x_i}\}|^2 \, \mathrm{d}y \,, \quad \hat{\beta}_{x_i} > 0 \,.$$

Using (3.32) and the Fridrichs inequality applied to $\ddot{u}_{x_i}$ we obtain the inequality

$$(3.35) \qquad \int_c^d |\{\hat{\varepsilon}_{x_i}\}|^2 \, \mathrm{d}y \geqq \gamma_{x_i} \int_c^d [u_{x_i}^2 + \dot{u}_{x_i}^2 + \ddot{u}_{x_i}^2 + \dot{v}_{x_i}^2 +$$
$$+ \ddot{w}_{x_i} + \left(\left. \frac{\partial w}{\partial x} \right|_{x_i} - k_2(y) \, u_{x_i} \right)^{\cdot\cdot 2} \mathrm{d}y \,, \quad \gamma_{x_i} > 0 \,.$$

We assume that $k_2 \in C^2([c, d])$. The expression in the integral on the right hand side of the inequality (3.35) is of the form

$$B(\{\xi\}, y) = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + \\ + (\xi_6 - \ddot{k}_2(y)\,\xi_1 - 2\dot{k}_2(y)\,\xi_2 - k_2(y)\,\xi_3)^2 .$$

$B(\{\xi\}, y)$ is for every $y \in [c, d]$ a positive definite quadratic form. Hence $B(\{\xi\}, y)$ is a positive and continuous function on the compact set $S_6 \times [c, d]$, where S_6 is the unit circle in R^6. $B(\{\xi\}, y)$ has then its minimum $\mu > 0$ on $S_6 \times [c, d]$, which implies

$$B(\{\xi\}, y) \geqq \mu(\xi_1^2 + \xi_2^2 + \ldots + \xi_6^2) \tag{3.36}$$

for every $\{\xi\} \in R^6$ and $y \in [c, d]$.

After inserting (3.36) in (3.35) we obtain

$$\int_c^d |\{\hat{\varepsilon}_{x_i}\}|^2 \, \mathrm{d}y \geqq \gamma_{x_i}\mu \int_c^d \left[\ddot{u}_{x_i}^2 + \dot{v}_{x_i}^2 + \ddot{w}_{x_i}^2 + \left(\frac{\partial w}{\partial x}\bigg|_{x_i}\right)^{\cdot\cdot 2} \right] \mathrm{d}y \,,$$

where we have omitted the positive quantities $u_{x_i}^2, \dot{u}_{x_i}^2$. Comparing the last inequality with the right hand side of (3.34) we obtain the inequality (3.29) with the constant $\beta_{x_i} = \hat{\beta}_{x_i}\gamma_{x_i}\mu$. In the same way we can verify (3.30). (3.29), (3.30) imply the inequality (3.28)

Combining the relations (3.11), (3.27), (3.28) we obtain (3.14) which implies the positive definiteness of the form $a(\{z_1\}, \{z_2\})$ on the space $W(\Omega)$. According to the theorem of Lax-Milgram there exists a unique weak solution of the problem (1.2), (1.3), (1.4). This is expressed in the following theorem.

Theorem 3. *Let $l(\{.\})$ be a linear bounded functional on $W(\Omega)$. Let $k_1 \in C^2([a, b])$, $k_2 \in C^2([c, d])$, $\varrho_{x_i} \gg D_{x_i}, B_{x_i}, G_{x_i}, k_2, \varrho_{y_j} \gg D_{y_j}, B_{y_j}, G_{y_j}, k_1$. Then there exists a unique vector-function $\{z_0\} \in W(\Omega)$ satisfying the identity*

$$a(\{z_0\}, \{z\}) = l(\{z\}) \tag{3.37}$$

for every vector-function $\{z\} \in W(\Omega)$.

4. APPLICATION OF THE FINITE ELEMENT METHOD

Let $V_{h_k}(\Omega) \supset W(\Omega)$ be a sequence of finite dimensional subspaces of the space $W(\Omega)$ satisfying the relation

$$\lim_{k \to \infty} \operatorname{dist}(\{z\}, V_{h_k}(\Omega)) = 0 \tag{4.1}$$

for every function $\{z\} \in W(\Omega)$. A function $\{z_k\} \in V_{h_k}(\Omega)$ is called a Galerkin approximation of the solution of (3.37) if $\{z_k\}$ is a solution of the identity

$$a(\{z_k\}, \{z\}) = l(\{z\}) \,, \quad \forall \{z\} \in V_{h_k}(\Omega) \,. \tag{4.2}$$

As a consequence of coercivity of the bilinear form $a(\{z^1\}, \{z^2\})$ there exists a unique sequence of Galerkin approximations which is convergent ([8]) i.e.

$$\lim_{k \to \infty} |||\{z_k\} - \{z_0\}||| = 0 , \tag{4.3}$$

where $\{z_0\} \in W(\Omega)$ is a solution of (3.37).

A useful method for approximate solution of the problem (3.37) is the method of finite elements, which is considered one type of the Galerkin method.

Let $h_k > 0$, $\lim_{k \to \infty} h_k = 0$. We consider a sequence of rectangles $\{K_i^{(k)}\}_{i=1}^{N(k)}$ which have the following properties:

i) $\bar{\Omega} = \bigcup_{i=1}^{N(k)} K_i^{(k)}$,

ii) $K_i^{(k)} \cap K_j^{(k)} = \emptyset , \quad i \neq j$,

iii) $\operatorname{diam} K_i^{(k)} \leqq h_k , \quad i = 1, \ldots, N(k)$,

iv) $K_l^{(k)} \cap I_{x_i} = \emptyset , \quad K_l^{(k)} \cap I_{y_j} = \emptyset , \quad i = 1, \ldots, n - 1, \; j = 1, \ldots, m - 1$.

The last property means that the ribs I_{x_i}, I_{y_j} coincide with the sides of the rectangles $K_l^{(k)}$. We call a division of the set Ω with the above properties a regular division.

We consider the approximate solution $\{z_k\} = \langle u_k, v_k, w_k \rangle \in V_{h_k}(\Omega)$ on every rectangle $K_l^{(k)}$ in the form

$$\begin{aligned} u_k &= \sum_{0 \leqq i,j \leqq 3} a_{ij}^{(l)} x^i y^j , \\ v_k &= \sum_{0 \leqq i,j \leqq 3} b_{ij}^{(l)} x^i y^j , \\ w_k &= \sum_{0 \leqq i,j \leqq 3} c_{ij}^{(l)} x^i y^j \end{aligned} \tag{4.4}$$

with such coefficients $a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}$ that $u_k, v_k, w_k \in C^1(\bar{\Omega})$ (the condition of conformity). Using the properties of the Hermit interpolating polynomials ([12]) we obtain the inclusion $V_{h_k}(\Omega) \subset W(\Omega)$. By Theorem 2 we can verify in the same way as in ([3]) that the method of finite elements with bicubic polynomials (4.4) is convergent in our case.

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Súhrn

O EXISTENCII SLABÉHO RIEŠENIA OKRAJOVEJ ÚLOHY ROVNOVÁHY PLOCHEJ ŠKRUPINY ZOSILNENEJ TUHOSTNÝMI REBRAMI

Igor Bock, Ján Lovíšek

V tejto práci sa priamou variačnou metódou dokazuje existencia a jednoznačnosť slabého riešenia okrajovej úlohy pre plochú škrupinu zosilnenú tuhostnými rebrami. Okrajová úloha sa rieši na priestore $W(\Omega) \subset H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$, na ktorom je odpovedajúca bilineárna forma koercitívna. Pre numerické riešenie sa navrhuje metóda konečných prvkov. Približné riešenia konvergujú k slabému riešeniu v priestore $W(\Omega)$.

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