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OPTIMAL LOT SIZE DETERMINATION OF MULTISTAGE PRODUCTION SYSTEM

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0. INTRODUCTION

This paper deals with the optimization of total setup plus inventory cost of the multistage inventory-production systems by the lot sizes choice. A certain class of multistage single-product inventory-production systems in which the production stages are arranged in series and the production rates of the individual production stages are finite, fixed and generally different, is described in Section 1. The assumptions include an infinite horizon, constant continuous final product demand and time invariant lot sizes. Some special cases of systems of this class are those estimated by Thomas [8], Schussel [5], Taha and Skeith [7], Crowston, Wagner and Williams [2], Streck [6], Klapka [4] and others. These special cases are shown in Section 2. The problem of the cost optimization of such systems is formulated in Section 3 where the methods are also mentioned that the above authors have used to solve the individual systems. For large-scale systems it appears to be useful to have an apriori estimation of the minimal cost in the form of its analytically expressed lower bound. (An upper bound can be obtained in a trivial way.) In the present paper this lower bound is derived for the class of systems under consideration. For this purpose, first of all, in Section 4 a choice is made of a system from the above class, the optimal cost of which is minimal. The exact cost optimization of the system thus selected, presented in Section 5, is based on dynamic programming. Some elements of this solution are employed in Section 6 to derive a lower bound of the optimal cost of this system (Theorem 3). Another representation of this lower bound based on dynamic programming, which provides a clearer view of the proof of Theorem 3, is presented in Theorem 4 and in the Corollary. Some elements of the proof of Theorem 3 are employed in Theorem 5 to derive a simple formula for another lower bound, more or equally distant from the optimal cost. In the case of special conditions imposed on the production system, the said formula turns into a formula derived for this case by Crowston, Wagner and Williams [2]. Section 7 brings the results of some simple numerical examples of the exact cost optimization by the lot sizes choice and of the lower bounds of the optimal cost. The main results presented in this paper were reported by the author in the seminary [11] held on April 8, 1975, during his stay at the Università di Pisa.

1. SYSTEM DESCRIPTION

The system under consideration consists of production stages $s_1, s_2, \ldots, s_{m-1}$ $(m \geqq 2)$, a consumption stage s_m, intermediate stores $a_1, a_2, \ldots, a_{m-1}$, an input a_0 and an output a_m. Suppose the system processes a product the quantity of which is measurable by non-negative numbers. Each stage s_i $(i = 1, 2, \ldots, m-1)$ is characterized by a cumulative production $V_i(t)$ defined for time $t \in (-\infty, +\infty)$, the stage s_m being characterized by a cumulative demand $V_m(t)$ defined for $t \in (-\infty, +\infty)$. The meaning of the cumulative production and the cumulative demand is defined by the following two properties of the system:

1. The product flows through the system in such a way that during a time interval $[t, t+a]$ for $a \geqq 0$, $t \in (-\infty, +\infty)$ the quantity $V_i(t+a) - V_i(t)$ of the product flows away from a_{i-1} through s_i into a_i $(i = 1, 2, \ldots, m)$.

2. The quantity of the product in the store a_i $(i = 1, 2, \ldots, m-1)$ at an arbitrary time t is equal to $V_i(t) - V_{i+1}(t)$. No other product flows occur in the above system.

In this paper we consider the class of systems characterized by the cumulative production and the cumulative demand defined later in Definition 2 under the conditions defined in Definition 1. For such systems the production rates y_i $(i = 1, 2, \ldots$ $\ldots, m-1)$ of the individual stages and the consumption rate $y_m (0 < y_m < y_i < \infty$ for $i = 1, 2, \ldots, m-1)$ are given.

Definition 1. *Let R_m be the set of all vectors $\hat{x}_m = [x_1, x_2, \ldots, x_m]$ such that for given A, B, Δ $(0 < A < B < \infty, 0 < \Delta < B - A)$ it holds*

$$x_i \in X = [A, B] \cap \{\Delta, 2\Delta, 3\Delta, \ldots\} \quad (i = 1, 2, \ldots, m), \tag{1}$$

$$x_{i+1} = N_i x_i \quad (i = 1, 2, \ldots, m-1) \tag{2}$$

where

$$N_i \in S \cap P_i \cap T_i \tag{3}$$

for

$$S = \{1, 2, 3, \ldots; \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}, \quad P_{m-1} = T_{m-1} = \{1\}, \tag{4}$$
$$P_i = (0, \infty) \quad (i = 1, 2, \ldots, m-2),$$

T_i for $i = 1, 2, \ldots, m-2$ are given sets of nonnegative numbers chosen so that there exists at least one $\hat{x}_m$.

Definition 2. *For each* $\hat{x}_m = [x_1, x_2, \ldots, x_m] \in R_m$, $t \in (-\infty, +\infty)$ *we define a sequence* $\{V_i(t)\}_{i=1}^m$ *of functions by*

$$(5) \qquad V_i(t) = [p_i]\, x_i + \min\left\{x_i\,, \quad \left(t - \frac{x_i}{y_m}[p_i] - \mu_i(\hat{x}_i)\right) y_i\right\}$$

where $\hat{x}_i = [x_1, x_2, \ldots, x_i]$, $p_i = (t - \mu_i(\hat{x}_i))\, y_m/x_i$, $[p_i]$ *is the greatest integer less than or equal to* p_i, $\mu_1(\hat{x}_1) = 0$, $\mu_i(\hat{x}_i)$ *for* $i = 2, 3, \ldots, m$ *is a given nonnegative function such that* $V_{i-1}(t) \geqq V_i(t)$ *for all* $t \in (-\infty, +\infty)$ *(no backlogging is admissible). Especially, let us denote by* $\mu_i^{(0)}(\hat{x}_i)$ *the* $\mu_i(\hat{x}_i)$ *that for each fixed* $\hat{x}_i$ *takes the smallest possible value.*

Note 1. The typical shape of function $V_i(t)$ considered in Definition 2 is depicted in Fig. 1. (5) implies that for the stage s_i ($i = 1, 2, \ldots, m - 1$) the time interval of its activity (during which the product flows through s_i with the production rate y_i) with the duration x_i/y_i alternates periodically with the time interval of its inactivity (when the product does not flow through s_i) with the duration $x_i((1/y_m) - (1/y_i))$. It is apparent from (5) that the consumption stage s_m is active continuously with the consumption rate y_m. (To allow the consumption to be discontinuous, the same description can be used e.g. if we interprete s_{m-1} as the consumption stage.)

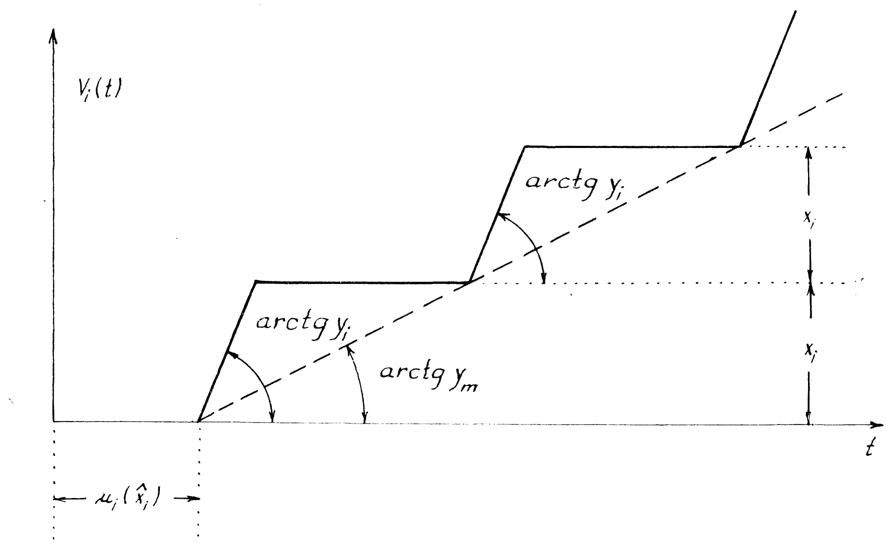

Fig. 1.

Note 2. The quantity x_i is called the lot size of the stage s_i. The condition (1) means that each lot size is an integer multiple of a given batch size Δ (given e.g. by the transportation technology). A and B are the minimal and maximal permissible lot sizes, respectively. The conditions $(2)-(4)$ imply that the lot size x_i is an integer multiple of x_{i-1} or x_{i+1}. The experience shows (cf. e.g. a note in [2] concerning the unpublished experiments of Jensen and Khan) that high average inventories result if $(2)-(4)$ are not satisfied. This concerns, for example, the system for which x_i/x_{i+1} is a rational number. A cost optimization of such a system is described in [12].

Note 3. It can be easily found that $\mu_i^{(0)}(\hat{x}_i)$ exists, that $\{V_i(t)\}_{i=1}^m$ is well defined and that the function $V_i(t) - V_{i+1}(t)$ $(i = 1, 2, \ldots, m-1)$ is periodic with the period

$$T(i) = \frac{1}{y_m} \max \{x_i, x_{i+1}\} . \tag{5a}$$

Note 4. An optimization of some systems with another type of cumulative production and cumulative demand, different from (5), has been estimated for $m = 2$ e.g. by Giannessi [3] and Manca [10] who involved some stochastic aspects, finite time horizon and time dependence of lot sizes.

The cost of the process studied in the present paper consists of a fixed charge per lot and a linear inventory carrying cost. Let C_{1i}, C_{2i} $(i = 1, 2, \ldots, m-1)$ be given finite constants for which $C_{1i} > 0$ $(i = 1, 2, \ldots, m-1)$, $C_{2,i+1} > C_{2i} > 0$ $(i = 1, 2, \ldots, m-2)$. The beginning of each activity interval of s_i $(i = 1, 2, \ldots, m-1)$ calls forth a rise of the setup cost C_{1i}, the presence of the product unit at the store a_i $(i = 1, 2, \ldots, m-1)$ per time unit calls forth the inventory carrying cost C_{2i}. Although the theory given in the present paper is developed for $y_i > y_m, C_{1i} > 0$, it is very easy to extend it to the case $y_i \geqq y_m, C_{i1} \geqq 0$. However, the validity of conditions $y_i \geqq y_m, C_{i1} \geqq 0$ may already now be assumed in Sections 3, 4 and 5.

2. SOME SPECIAL CASES

The class of inventory-production systems just described includes systems which differ mutually by various definitions of $\mu_i(\hat{x}_i)$ $(i = 2, 3, \ldots, m)$ and T_i $(i = 1, 2, \ldots, m-2)$ and by special additional conditions imposed upon the system. Some inventory-production systems hitherto studied can be viewed as special cases of systems of the above class.

For example, in [8] a system has been investigated in which $T_i = (0, \infty)$ for $i = 1, 2, \ldots, m-2$ and

$$\mu_i(\hat{x}_i) = \sum_{k=2}^{i} \left[\frac{x_k}{y_{k-1}} + (x_k - x_{k-1}) \left(\frac{1}{y_m} - \frac{1}{y_{k-1}}\right) l(x_k - x_{k-1})\right] \quad (i = 2, 3, \ldots, m) \tag{6}$$

where

$$l(x) = \begin{cases} 0 & (x \leqq 0) \\ 1 & (x > 0) \end{cases} \tag{7}$$

((6) implies that the stage s_i ($i = 2, 3, \ldots, m$) starts its activity only when the quantity x_i of the product is present in the store a_{i-1}).

In [5] and [7] the case has been investigated where $T_i = (0, 1]$ for $i = 1, 2, \ldots, m-2$ and

$$\mu_i(\hat{x}_i) = \sum_{k=2}^{i} \frac{x_{k-1}}{y_{k-1}} \quad (i = 2, 3, \ldots, m). \tag{8}$$

((8) means that the product flowing away through s_{i-1} is not moved to s_i until the whole lot of size x_{i-1} is completed in the store a_{i-1}.)

References [2], [4], [6] deal with the case $\mu_i(\hat{x}_i) = \mu_i^{(0)}(\hat{x}_i)$ ($i = 2, 3, \ldots, m$) where the stage s_i ($i = 2, 3, \ldots, m$) starts as soon as possible. Simultaneously, for $i = 1, 2, \ldots, m-2$ it holds $T_i = (0, \infty)$ in [4] and $T_i = (0, 1]$ in [2]. In [6] a simple case is studied where $m \geqq 3$ and one integer b is given ($1 \leqq b \leqq m-2$) such that $T_i = \{1\}$ for all $i \neq b$, $T_b = (0, 1]$.

3. COST OPTIMIZATION PROBLEM

A problem the special cases of which were solved in [2], [4], [6], [7], [8], can be substantially formulated as follows. For given A, B, Δ, m, $\{y_i\}_{i=1}^{m}$, $\{C_{1i}\}_{i=1}^{m-1}$, $\{C_{2i}\}_{i=1}^{m-1}$, $\{T_i\}_{i=1}^{m-2}$, $\{\mu_i(.)\}_{i=2}^{m}$, find

$$F(\hat{x}_m^{(h)}, h(\hat{x}_m^{(h)})) = \min_{\hat{x}_m \in R_m} F(\hat{x}_m, h(\hat{x}_m)) \tag{9}$$

where

$$F(\hat{x}_m, h(\hat{x}_m)) = \sum_{i=1}^{m-1} \left[C_{1i} \frac{y_m}{x_i} + \frac{C_{2i}}{T(i)} \int_t^{t+T(i)} (V_i(\tau) - V_{i+1}(\tau))\, d\tau \right], \tag{10}$$

$$h(\hat{x}_m) = \{\mu_i(\hat{x}_i)\}_{i=2}^{m}, \tag{11}$$

where $\mu_i(\hat{x}_i)$ ($i = 2, 3, \ldots, m$) co-determines $V_i(t)$ according to Definition 2 and $T(i)$ is given by (5a). Because R_m is finite, $F(\hat{x}_m^{(h)}, h(\hat{x}_m^{(h)}))$ exists.

It follows from (1)–(4) that for each $\hat{x}_m \in R_m$ there exists a positive x such that x/x_i ($i = 1, 2, \ldots, m$) is an integer. Thus x/y_m is the period of each function $V_i(t) - V_{i+1}(t)$ ($i = 1, 2, \ldots, m-1$), therefore $(x/y_m) F(\hat{x}_m, h(\hat{x}_m))$ is the cost related to one period x/y_m of the process considered. Thus in (9)–(11) we deal with the minimization of the time averaged process cost related to the time unit by means of lot sizes determination. For individual systems this problem has been solved by the authors mentioned in Introduction. Thomas [8] and Schussel [5] have used heuristic approximate iterative procedures to this aim. Taha and Skeith [7] have solved the problem for the case $x_i/x_{m-1} \leqq y_i/y_m$ ($i = 1, 2, \ldots, m-1$) by substituting all $\hat{x}_m \in R_m$ into $F(\hat{x}_m, h(\hat{x}_m))$. The dynamic programming algorithm has been derived for the general case by Klapka [4]. Two years later Crowston, Wagner and Williams

[2] have derived a dynamic programming algorithm for the special case $y_i \geqq y_{i+1}$ $(i = 1, 2, \ldots, m - 1)$ (cf. Note 3 in Section 5). In Section 5 we present our above mentioned algorithm [4], some analytical expressions from which we shall employ when deriving a lower bound of the optimal cost in Section 6. Most of the above mentioned authors have replaced condition (1) by the condition $x_i > 0$ $(i = 1, 2, \ldots \ldots, m)$ without presenting any proof of existence of a solution of the corresponding cost optimization problem. An extension of the problem (1)–(5), (9), (10) to a multi-product sequencing case is presented in [4], an extension to the case of multistage assembly system is presented in [4a]. Other extensions and generalizations (nonlinear cost, backlogging etc.) are reviewed in [2].

Note. It can be easily found that a change of any $\mu_i(.)$ by adding a positive constant does not influence $\hat{x}_m^{(h)}$. The fact that constant delays do not alter optimal policies of lot sizes is also mentioned in [2], [7] and [9].

4. MINIMAL SYSTEM CHOICE

Let us find a system inside the class described in Section 1, the optimal cost of which is minimal. It is easy to find (by considering the magnitude of the set R_m) that such a system pertains to a set of the systems for which $T_i = (0, \infty)$ $(i = 1, 2, \ldots \ldots, m - 2)$. It remains thus to make a choice of a sequence $h(\hat{x}_m)$ minimizing the optimal cost. To this aim we present the following theorem.

Theorem 1. *Let M be the set of all $h(.)$ defined for a given m by (11). Let $H(\hat{x}_m) = \{\mu_i^{(0)}(\hat{x}_i)\}_{i=2}^m$ where $\mu_i^{(0)}(\hat{x}_i)$ is defined in Definition 2. Then the functional $F(\hat{x}_m^{(h)}, h(\hat{x}_m^{(h)}))$ defined in (9) satisfies $\min_{h \in M} F(\hat{x}_m^{(h)}, h(\hat{x}_m^{(h)})) = F(\hat{x}_m^{(H)}, H(\hat{x}_m^{(H)}))$.*

Proof.

(i) Definition 2 implies that $H(.) \in M$.

(ii) For each $\hat{x}_m \in R_m$, if we denote $\hat{x}_m = [\hat{x}_i, x_{i+1}, x_{i+2}, \ldots, x_m] = [\hat{x}_{i+1}, x_{i+2}, \ldots \ldots, x_m]$, it can be easily derived from Definition 2 that

$$(12) \qquad \mu_{i+1}^{(0)}(\hat{x}_{i+1}) - \mu_i^{(0)}(\hat{x}_i) \leqq \mu_{i+1}(\hat{x}_{i+1}) - \mu_i(\hat{x}_i) \quad \text{for} \quad i = 1, 2, \ldots, m - 1 \,.$$

(iii) From (10) it follows that $F(\hat{x}_m, h(\hat{x}_m)) - F(\hat{x}_m, H(\hat{x}_m)) = y_m \sum_{i=1}^{m-1} C_{2i} \,.$

$$. \left(\mu_{i+1}(\hat{x}_{i+1}) - \mu_{i+1}^{(0)}(\hat{x}_{i+1}) - \mu_i(\hat{x}_i) + \mu_i^{(0)}(\hat{x}_i)\right).$$

(iv) Hence with respect to (12) we obtain

$$(13) \qquad F(\hat{x}_m^{(h)}, H(\hat{x}_m^{(h)})) \leqq F(\hat{x}_m^{(h)}, h(\hat{x}_m^{(h)})) \,.$$

(v) (9) implies

$$(14) \qquad F(\hat{x}_m^{(H)}, H(\hat{x}_m^{(H)})) \leqq F(\hat{x}_m^{(h)}, H(\hat{x}_m^{(h)})) \,.$$

Theorem 1 follows from (13) and (14).

To obtain a lower bound of the optimal cost for the class of systems under consideration, it is now sufficient to find a lower bound of the optimal cost of the system for which $T_i = (0, \infty)$, $h(.) = H(.)$.

Let us now denote

$$E = F(\hat{x}_m^{(H)}, H(\hat{x}_m^{(H)})) \tag{15}$$

for $T_i = (0, \infty)$ $(i = 1, 2, \ldots, m - 2)$. The following section deals with the determination of E, $\hat{x}_m^{(H)}$.

5. DYNAMIC PROGRAMMING SOLUTION

This section is an extension of a part of the Research Report [4] of the author. Here, as well as in Section 6, the notation introduced in the previous sections is used except that $\hat{x}_i^{(H)}$ is denoted by $\hat{x}_i$. Using the following definition we shall prove Theorem 2 that gives an algorithm for the determination of the optimal cost E and of the optimal lot sizes policy $\{x_i\}_{i=1}^{m-1}$.

Definition 3. *$L_i(x_i)$ is a set of vectors defined for each $i \in \{1, 2, \ldots, m - 1\}$, $x_i \in X$, so that $\hat{K}_i = [K_i, K_{i+1}, \ldots, K_{m-1}] \in L_i(x_i)$ iff $K_n \in S_n$ for each $n \in \{i, i + 1,, \ldots, m - 1\}$ where $S_n = S \cap P_n \cap [A/x_n, B/x_n] \cap \{\Delta/x_n, 2\Delta/x_n, \ldots\}$, $x_{n+1} = K_n x_n$, m is fixed.*

Theorem 2. *For a sequence of functions $\{f_{m-i+1}(x_i)\}$ $(i = 1, 2, \ldots, m - 1)$ defined for $x_i \in X$ by*

$$f_{m-i+1}(x_i) = \min_{\hat{K}_i \in L_i(x_i)} \sum_{n=i}^{m-1} g_n(x_n, K_n) \tag{16}$$

where

$$g_n(x_n, K_n) = \frac{C_{1n}}{x_n} + x_n C_{2n} \operatorname{sgn}(1 - K_n) [B_n - K_n B_{n+1} + D_n \min\{1, K_n\} \, l(D_n \operatorname{sgn}(1 - K_n))] , \tag{17}$$

$$\operatorname{sgn}(x) = \begin{cases} -1 & (x < 0) \\ +1 & (x \geqq 0) \end{cases}, \tag{18}$$

$$B_n = \frac{1}{2}\left(\frac{1}{y_m} - \frac{1}{y_n}\right) \geqq 0 \quad (n = 1, 2, \ldots, m) , \tag{19}$$

$$D_n = 2(B_{n+1} - B_n) \quad (n = 1, 2, \ldots, m - 1) \tag{20}$$

and where $l(x)$ is defined in (7), the relations

$$f_{m-i+1}(x_i) = \min_{K_i \in S_i} [g_i(x_i, K_i) + f_{m-i}(K_i x_i)] \quad (i = 1, 2, \ldots, m - 2) , \tag{21}$$

$$f_2(x_{m-1}) = \frac{C_{1,m-1}}{x_{m-1}} + C_{2,m-1} B_{m-1} x_{m-1} \tag{22}$$

and

(23) $$y_m \min_{x_1 \in X} f_m(x_1) = E$$

hold.

Proof of (23). With respect to (17) and (16), it follows from (5), (9), (10), (15) and from Theorem 1 that

(24) $$E = y_m \min_{\hat{x}_m \in R_m} \sum_{i=1}^{m-1} g_i(x_i, x_{i+1}/x_i) = y_m \min_{x_1 \in X} \min_{\hat{K}_1 \in L_1(x_1)} \sum_{i=1}^{m-1} g_i(x_i, K_i) =$$
$$= y_m \min_{x_1 \in X} f_m(x_1) .$$

Proof of (21) and (22) follows from (16)–(20) by the Bellman principle of optimality [1].

Note 1. Computational efficiency. The time required for the calculation of E, $\{x_i\}_{i=1}^{m-1}$ by the algorithm (16)–(23) on the computer DATASAAB D 21 is approximately

$$(0{\cdot}002((B - A)/\Delta)^2 + 0{\cdot}07(B - A)/\Delta)(m - 2) \quad \text{seconds.}$$

Note 2. Extension. In our Research Report [4a] we have extended the algorithm (16)–(23) to the case of a multistage assembly system where each production stage may have many predecessors but only a single successor. The optimal cost determinations for systems of this type have been hitherto solved in [5] by a heuristic approximate iterative procedure and in [2] in an exact way under the restriction that the production rate of a current stage must be greater than or equal to the production rate of its successor stage and that the lot size of a current stage must be greater than or equal to the lot size of its successor stage, and for the case of infinite production rates.

Note 3. In the special case $y_i \geqq y_{i+1}$ $(i = 1, 2, \ldots, m - 1)$, $T_i = (0, 1]$ $(i = 1, 2, \ldots, m - 2)$, studied by Crowston, Wagner and Williams [2], the system (21), (22) can be transformed into an equivalent form

$$f_{m-i+1}(x_i) = h_i(x_i) + \min_{x_{i+1} \in X \cap \{x_i, x_i/2, x_i/3, \ldots\}} f_{m-i}(x_{i+1}) \quad (i = 1, 2, \ldots, m - 2)$$

where

$$h_i(x_i) = \begin{cases} \dfrac{C_{1i}}{x_i} + \dfrac{x_i}{2}(C_{2i} - C_{2,i-1})\left(\dfrac{1}{y_m} - \dfrac{1}{y_i}\right) & (i = 2, 3, \ldots, m - 1) \\ \dfrac{C_{1i}}{x_i} + \dfrac{x_i}{2} C_{2i}\left(\dfrac{1}{y_m} - \dfrac{1}{y_i}\right) & (i = 1), \end{cases}$$

$$f_2(x_{m-1}) = h_{m-1}(x_{m-1}) .$$

In this form, the system has been derived in [2].

6. LOWER BOUND OF OPTIMAL COST

From the view-point of the computational efficiency, for large $(B - A)/\Delta$ it is useful to have an apriori estimation of E in the form of its analytically expressed lower bound the computational time of which does not depend on $(B - A)/\Delta$. (The upper bound can be trivially obtained, e.g. by calculating E from (16)–(23) while replacing each S_i $(i = 1, 2, \ldots, m - 2)$ by $\{1\}$ or by another subset of S_i. An analogous rule can be obtained for the upper bound of $F(\hat{x}_m^{(h)}, h(\hat{x}_m^{(h)}))$. The upper bound can be also obtained by involving a greater Δ than in the case of an exact solution.) In the case of finite production rates, a lower bound of the optimal cost has been hitherto derived for the special case $y_i \geqq y_{i+1}$ $(i = 1, 2, \ldots, m - 1)$, $T_i = (0, 1]$ $(i = 1, 2, \ldots, m - 2)$, $h(.) = H(.)$ only, where it is obvious from specializing (23), (16) that the optimal cost can be written in the form $y_m \min_{\hat{x}_m \in R_m} . \{(C_{11}/x_1) + x_1(C_{21}/2)((1/y_m) - (1/y_1)) + \sum_{i=1}^{m-2} [(C_{1,i+1}/x_{i+1}) + (x_{i+1}/2)(C_{2,i+1} - C_{2i})((1/y_m) - (1/y_{i+1}))]\}$. Crowston, Wagner and Williams have found a lower bound of this cost (see [2], Section VII) by simple minimization of the individual terms, assuming no interdependence between the successive terms. Their lower bound that they have employed in a special way to improve the computational efficiency of the dynamic programming algorithm is a special case of our lower bound given below in Theorem 5. An additional improvement of these computational refinements could be materialized through some increase of the lower bound. In the present section in Theorem 3 we shall derive a lower bound related to the general system, which in the above mentioned special case is greater than or equal to the lower bound derived by Crowston, Wagner and Williams. The property just mentioned follows directly from Theorem 5. With respect to the convexity of the function $g_i(x_i, K_i)$ in a connected region, our results make it possible to employ the basis of the computational efficiency improvement idea of Crowston, Wagner and Williams also in the case of a general system, as can be easily found.

Note. To this aim, we can begin with the determination of an upper bound $E_{m-i+1}(x_i)$ of $f_{m-i+1}(x_i)$ we have demonstrated in [4a]. It consists, substantially, in a heuristic reduction of the elements of the sets S_i. It appears that $E_{m-i+1}(x_i)$ – (lower bound of optimal cost of the subsystem involving the stages $s_{i+1}, s_{i+2}, \ldots, s_m$ and the stores $a_{i+1}, a_{i+2}, \ldots, a_{m-1}$) = upper bound of $g_i(x_i, K_i)$. The knowledge of an upper bound of $g_i(x_i, K_i)$ leads to another reduction of S_i in such a way that the reduced S_i contains the optimal K_i.

Theorem 3. *We consider the function*

$$E_1 = y_m \sum_{n=0}^{m-2} A_n , \tag{25}$$

where

$$(26)\qquad A_n = \begin{cases} 2\sqrt{(\xi_n^{(1)}\eta_n^{(1)})} & ((\tau_n^{(2)} \leqq \gamma_n^{(1)} \wedge \lambda_n^{(1)} = \lambda_n^{(2)}) \vee n = 0) \\ 2\sqrt{(\xi_n^{(2)}\eta_n^{(2)})} & ((\gamma_n^{(2)} < \tau_n^{(1)} \wedge \lambda_n^{(1)} = \lambda_n^{(2)}) \vee \lambda_n^{(1)} \neq \lambda_n^{(2)}) \\ 0 & \text{(otherwise)} \end{cases}$$

while

$$(27)\qquad \tau_n^{(\nu)} = \begin{cases} \gamma_{n-1}^{(\nu)} & (\lambda_{n-1}^{(1)} = \lambda_{n-1}^{(2)}), \\ \gamma_{n-1}^{(2)} & (\lambda_{n-1}^{(1)} \neq \lambda_{n-1}^{(2)}) \end{cases} \quad \nu = 1, 2\,; \quad n \geqq 1\,.$$

These quantities are defined by the relations

$$(28)\qquad \gamma_n^{(\nu)} = (\xi_n^{(\nu)}/\eta_n^{(\nu)})^{1/2}\,; \quad \nu = 1, 2\,; \quad n = 0, 1, 2, \ldots, m-2\,,$$

$$(29)\qquad \lambda_0^{(1)} = \lambda_0^{(2)},\ \lambda_{m-2}^{(1)} = \lambda_{m-2}^{(2)} \quad (\lambda_0^{(1)} \textit{ and } \lambda_{m-2}^{(1)} \textit{ are arbitrary constants})\,,$$

$$(30)\qquad \xi_{m-2}^{(1)} = \xi_{m-2}^{(2)} = C_{1,m-1}\,, \quad \eta_{m-2}^{(1)} = \alpha_{m-2}^{(1)}\,, \quad \eta_{m-2}^{(2)} = \alpha_{m-2}^{(2)}\,,$$

$$(31)\qquad \alpha_n^{(1)} = B_{n+1}(C_{2n} + C_{2,n+1})\,, \quad \alpha_n^{(2)} = C_{2,n+1}B_{n+1} + C_{2n}(D_n\, l(D_n) - B_{n+1})$$
$$(n = 1, 2, \ldots, m-2)\,,$$

$$(32)\qquad \underset{(n=1,2,\ldots,m-3)}{[\xi_n^{(\nu)}, \eta_n^{(\nu)}, \lambda_n^{(\nu)}]} = \begin{cases} [C_{1,n+1}, \delta_n^{(\nu)}, 1] & \left(\dfrac{C_{1,n+1}}{\delta_n^{(\nu)}} \leqq (\gamma_{n+1}^{(1)})^2 \wedge \delta_n^{(\nu)} > 0 \wedge \lambda_{n+1}^{(1)} = \lambda_{n+1}^{(2)}\right) \\ [C_{1,n+1} + \xi_{n+1}^{(2)}, \alpha_n^{(\nu)} + \eta_{n+1}^{(2)}, 2] & \left((\gamma_{n+1}^{(1)})^2 < \dfrac{C_{1,n+1} + \xi_{n+1}^{(2)}}{\alpha_n^{(\nu)} + \eta_{n+1}^{(2)}} \leqq \right. \\ & \qquad \left. \leqq (\gamma_{n+1}^{(2)})^2 \wedge \lambda_{n+1}^{(1)} = \lambda_{n+1}^{(2)}\right) \\ [C_{1,n+1}, \alpha_n^{(\nu)}, 3] & \text{(otherwise)} \end{cases},$$

$$(33)\qquad \begin{aligned}&[\xi_0^{(1)}, \eta_0^{(1)}] = \\ &= [\xi_0^{(2)}, \eta_0^{(2)}] = \end{aligned} \begin{cases} [C_{11}, \vartheta_1] & \left(\dfrac{C_{11}}{\vartheta_1} \leqq (\gamma_1^{(1)})^2 \wedge \vartheta_1 > 0 \wedge \lambda_1^{(1)} = \lambda_1^{(2)}\right) \\ [C_{11} + \xi_1^{(2)}, \vartheta_1 + \eta_1^{(1)}] & \left((\gamma_1^{(1)})^2 < \dfrac{C_{11} + \xi_1^{(2)}}{\vartheta_1 + \eta_1^{(1)}} \leqq (\gamma_1^{(2)})^2 \wedge \right. \\ & \qquad \left. \wedge \lambda_1^{(1)} = \lambda_1^{(2)}\right) \\ [C_{11}, C_{21}B_1] & \text{(otherwise)} \end{cases}$$

where

$$(34)\qquad \begin{aligned} \delta_n^{(1)} &= C_{2n}B_{n+1} - C_{2,n+1}(B_{n+1} + D_{n+1}\, l(-D_{n+1})) \\ \delta_n^{(2)} &= C_{2n}(D_n\, l(D_n) - B_{n+1}) - C_{2,n+1}(B_{n+1} + D_{n+1}\, l(-D_{n+1})) \end{aligned}$$
$$(n = 1, 2, \ldots, m-3)\,,$$

$$(35)\qquad \vartheta_n = -C_{2n}[B_n + D_n\, l(-D_n)] \quad (n = 1, 2, \ldots, m-2)\,.$$

The inequality $E_1 \leqq E$ holds for this function.

Proof. A_n, $\xi_n^{(\nu)}$, $\eta_n^{(\nu)}$ $(n = 0, 1, \ldots, m-3, \nu = 1, 2)$ and $\lambda_n^{(\nu)}$ $(n = 1, 2, \ldots, m-3;$ $\nu = 1, 2)$ are defined uniquely, which is easy to show. (28)–(35) imply $0 < \gamma_n^{(1)} \leqq \gamma_n^{(2)}$ for $n = 0, 1, \ldots, m-2$. Using (22) for all $x_{m-1} > 0$, let us denote

$$\Phi_2(x_{m-1}) = f_2(x_{m-1}) \tag{36}$$

and define recursively for $i = m-2,\ m-3, \ldots, 1,\ x_i > 0$ the functions $\varphi_{m-i+1}(x_i)$, $\Phi_{m-i+1}(x_i)$ by the expressions

$$\varphi_{m-i+1}(x_i) = \min_{k_i \in \sigma} \left[g_i(x_i, k_i) + \Phi_{m-i}(k_i\, x_i) \right], \tag{37}$$

$$\Phi_{m-i+1}(x_i) = \begin{cases} \varphi_{m-i+1}(x_i) & (\lambda_i^{(1)} = \lambda_i^{(2)}) \\ f_2(x_i) + \sum\limits_{n=i}^{m-2} A_n & (\lambda_i^{(1)} \neq \lambda_i^{(2)}), \end{cases} \tag{38}$$

where $\sigma = (0, \infty)$, $g_i(x_i, k_i)$ is defined by (17) for all $x_i > 0$, $k_i \in \sigma$. We can justify inductively that the function $g_i(x_i, k_i) + \Phi_{m-i}(k_i x_i)$ $(i = m-2, m-3, \ldots, 1)$ is strictly convex in σ, has a continuous first derivative and a unique minimum. Consequently, the system of Eqs. (37), (38) $(i = 1, 2, \ldots, m-2)$ has a unique solution which is

$$\begin{aligned} \varphi_{m-i+1}(x_i) = {} & \frac{C_{1i} + \left(l(x_i - \gamma_i^{(1)}) - l(x_i - \gamma_i^{(2)})\right) \xi_i(x_i)}{x_i} + x_i\{\vartheta_i + \\ & + [\eta_i(x_i) - \vartheta_i + C_{2i}B_i]\, l(x_i - \gamma_i^{(1)}) - \eta_i(x_i)\, l(x_i - \gamma_i^{(2)})\} + \sum_{n=i}^{i+j-2} A_n^*(x_i) + \\ & + \sum_{n=i+j-1}^{m-2} A_n, \end{aligned} \tag{39}$$

$$\begin{aligned} \Phi_{m-i+1}(x_i) = {} & \frac{C_{1i} + \left(l(\omega_i - \gamma_i^{(1)}) - l(\omega_i - \gamma_i^{(2)})\right) \xi_i^{(2)}}{x_i} + \\ & + x_i\{\vartheta_i + [\eta_i^{(2)} - \vartheta_i + C_{2i}B_i]\, l(\omega_i - \gamma_i^{(1)}) - \eta_i^{(2)}\, l(\omega_i - \gamma_i^{(2)})\} + \\ & + A_i^*(\omega_i) + \sum_{n=i+1}^{m-2} A_n, \end{aligned} \tag{40}$$

$$k_i = \begin{cases} \gamma_i^{(1)}/x_i & (x_i \leqq \gamma_i^{(1)}) \\ 1 & (\gamma_i^{(1)} < x_i \leqq \gamma_i^{(2)}), \\ \gamma_i^{(2)}/x_i & (\gamma_i^{(2)} < x_i), \end{cases} \tag{41}$$

where

$$j = \begin{cases} 2 & (i = m-2), \\ 3 & (i < m-2), \end{cases} \tag{42}$$

(43)

$$\underset{(n=i,i+1,\dots,m-3)}{\xi_n(x_i)} = \begin{cases} C_{1,n+1} & (x_i \leqq \gamma_{n+1}^{(1)}) \\ C_{1,n+1} + \xi_{n+1}(x_i) & (\gamma_{n+1}^{(1)} < x_i \leqq \gamma_{n+1}^{(2)}) \\ C_{1,n+1} & (\gamma_{n+1}^{(2)} < x_i) \end{cases} \left.\begin{matrix} \delta_n^{(2)} \\ \alpha_n^{(2)} + \eta_{n+1}(x_i) \\ \alpha_n^{(2)} \end{matrix}\right\} =$$

$$= \eta_n(x_i)\,,$$

$$\xi_{m-2}(x_i) = C_{1,m-1}\,, \quad \eta_{m-2}(x_i) = \alpha_{m-2}^{(2)}\,,$$

and

(44) $A_n^*(x_i) =$

$$= \begin{cases} 2\sqrt{(\xi_n^{(1)}\eta_n^{(1)})} & ((x_i \leqq \gamma_n^{(1)} \wedge (\neg(\gamma_n^{(1)} < \gamma_{n-1}^{(1)} \wedge n \neq i))) \vee n \neq i \wedge \gamma_{n-1}^{(2)} \leqq \gamma_n^{(1)}) \\ 2\sqrt{(\xi_n^{(2)}\eta_n^{(2)})} & ((\gamma_n^{(2)} < x_i \wedge (\neg(\gamma_{n-1}^{(2)} \leqq \gamma_n^{(2)} \wedge n \neq i))) \vee n \neq i \wedge \gamma_n^{(2)} < \gamma_{n-1}^{(1)}) \\ 0 & (\text{otherwise})\,, \end{cases}$$

(45)
$$\omega_i = \begin{cases} x_i & (\lambda_i^{(1)} = \lambda_i^{(2)}) \\ +\infty & (\lambda_i^{(1)} \neq \lambda_i^{(2)})\,. \end{cases}$$

It follows from (39) that the function $\varphi_{m-i+1}(x_i)$ is for $x_i > 0$ strictly convex and has a continuous first derivative for which

(46)
$$\left[\frac{d\varphi_{m-i+1}}{dx_i}\right]_{x_i > \gamma_i^{(2)}} = C_{2i}B_i - \frac{C_{1i}}{x_i^2}\,, \quad \left[\frac{d\varphi_{m-i+1}}{dx_i}\right]_{x_i \leqq \gamma_i^{(2)}} \leqq C_{2i}B_i - \frac{C_{1i}}{x_i^2}\,.$$

From (39), (40) and (46) we obtain

(47)
$$[\Phi_{m-i+1}(x_i)]_{x_i \geqq \gamma_i^{(2)}} = \varphi_{m-i+1}(x_i)\,, \quad \left[\frac{d\Phi_{m-i+1}}{dx_i}\right]_{x_i < \gamma_i^{(2)}} \geqq \left[\frac{d\varphi_{m-i+1}}{dx_i}\right]_{x_i < \gamma_i^{(2)}}.$$

From (47) it follows

(48)
$$\Phi_{m-i+1}(x_i) \leqq \varphi_{m-i+1}(x_i) \quad (x_i > 0)\,.$$

The inclusions $S_i \subset \sigma$, $X \subset (0, \infty)$ together with (36)–(38), (48) and (21) imply

(49)
$$\Phi_{m-i+1}(x_i) \leqq f_{m-i+1}(x_i) \quad (x_i \in X)$$

for $i = 1, 2, \ldots, m-1$. For $x_1 > 0$ the function $\Phi_m(x_1)$ is strictly convex and has for

(50)
$$x_1 = \gamma_0^{(1)}$$

the unique minimum equal to

(51)
$$\min_{x_1 > 0} \Phi_m(x_1) = \sum_{n=0}^{m-2} A_n\,.$$

From (23), (25), (49) and (51) we conclude $E_1 \leqq E$.

The following theorem yields another representation of the lower bound based on dynamic programming, which provides a clearer view of the structure of the lower bound expression.

Theorem 4. *The function $\Phi_m(x_1)$ which determines the lower bound E_1 of the optimal cost E through* (37), (38), (51) *and* (25) *is a solution of the dynamic programming problem*

$$\Phi_{m-i+1}(x_i) = \min_{k_i \in \sigma} \left[G_i(x_i, k_i) + \Phi_{m-i}(k_i x_i)\right] \tag{52}$$

$$(x_i > 0\,,\ i = 1, 2, \ldots, m-2)\,,$$

$$\Phi_2(x_{m-1}) = f_2(x_{m-1})\,,$$

where

$$G_i(x_i, k_i) = \begin{cases} g_i(x_i, k_i) & (\lambda_i^{(1)} = \lambda_i^{(2)}) \\ \dfrac{C_{1i}}{x_i} + x_i\, C_{2i}[B_i + k_i(D_i\, l(D_i) - B_{i+1})] & (\lambda_i^{(1)} \neq \lambda_i^{(2)})\,, \end{cases} \tag{53}$$

whose unique optimal policy is

$$k_i = \begin{cases} \gamma_i^{(1)}/x_i & (x_i \leqq \gamma_i^{(1)} \wedge \lambda_i^{(1)} = \lambda_i^{(2)}) \\ 1 & (\gamma_i^{(1)} < x_i \leqq \gamma_i^{(2)} \wedge \lambda_i^{(1)} = \lambda_i^{(2)}) \\ \gamma_i^{(2)}/x_i & (\text{otherwise})\,. \end{cases}$$

Proof can be easily obtained by combining the relevant elements of the proof of Theorem 3.

The following corollary demonstrating one interesting special situation follows from Theorem 4.

Corollary. *If*

$$\operatorname{sgn}\left(1 - \frac{\gamma_{i+1}^{(1)}}{\gamma_i^{(2)}}\right) = \operatorname{sgn}\left(1 - \frac{\gamma_{i+1}^{(1)}}{\gamma_i^{(1)}}\right), \quad \operatorname{sgn}\left(1 - \frac{\gamma_{i+1}^{(2)}}{\gamma_i^{(2)}}\right) = \operatorname{sgn}\left(1 - \frac{\gamma_{i+1}^{(2)}}{\gamma_i^{(1)}}\right) \tag{55}$$

holds for $i = 1, 2, \ldots, m-3$ then $\Phi_m(x_1) = \varphi_m(x_1)$ where $\varphi_m(x_1)$ is a solution of the dynamic programming problem

$$\varphi_{m-i+1}(x_i) = \min_{k_i \in \sigma} \left[g_i(x_i, k_i) + \varphi_{m-i}(k_i x_i)\right] \quad (x_i > 0,\ i = 1, 2, \ldots, m-2)\,,$$

$$\varphi_2(x_{m-1}) = f_2(x_{m-1})\,,$$

the unique optimal policy of which is given by (41).

Proof is elementary as soon as we find, by analyzing relations (28) and (32), that (55) is equivalent to $\lambda_i^{(1)} = \lambda_i^{(2)}$.

The following theorem provides a simple expression for a lower bound of E by modifying the procedure given in the proof of Theorem 3.

Theorem 5. *A solution of the dynamic programming problem*

(56) $$\psi_{m-i+1}(x_i) = \min_{k_i \in \sigma} \left\{ \frac{C_{1i}}{x_i} + x_i C_{2i}[B_i + (D_i\, l(D_i) - B_{i+1})\, k_i] + \psi_{m-i}(k_i x_i) \right\}$$

$$(x_i > 0,\ i = 1, 2, \ldots, m-2)$$

$$\psi_2(x_{m-1}) = f_2(x_{m-1})$$

satisfies

(57) $$y_m \min_{x_1 > 0} \psi_m(x_1) = 2y_m[(C_{11}C_{21}B_1)^{1/2} + \sum_{i=1}^{m-2} (C_{1,i+1}\alpha_i^{(2)})^{1/2}] \leqq E_1$$

where $\alpha_i^{(2)}$ *is given by* (31), E_1 *by* (25).

Proof of the left equality in (57) is straightforward (the unique optimal policy of (56) is

$$k_i = (1/x_i)\,(C_{1,i+1}/\alpha_i^{(2)})^{1/2})\,.$$

The proof of the right inequality in (57) is based on the facts: If the definition of the sequences $\{\lambda_n^{(v)}\}$ for $n = 0, 1, 2, \ldots, m-2$, $v = 1, 2$ given in (29), (32) is modified in such a way that the terms of these sequences are arbitrary constants satisfying $\lambda_n^{(1)} + \lambda_n^{(2)}$ $(n = 0, 1, 2, \ldots, m-2)$, then it is evident that the function $\Phi_{m-i+1}(x_i)$ $(i = 1, 2, \ldots, m-2)$, defined by (38), (26)–(34) is replaced by the function

$$\psi_{m-i+1}(x_i) = f_2(x_i) + 2\sum_{n=i}^{m-2} (C_{1,i+1}\alpha_i^{(2)})^{1/2}\,.$$

By a way analogous to (46), (47), (48) we can prove easily

$$\psi_{m-i+1}(x_i) \leqq \Phi_{m-i+1}(x_i)$$

inductively for $i = m-2, m-3, \ldots, 1$.

Then with regard to (25) and (51) we can write

$$y_m \min_{x_1 > 0} \psi_m(x_1) \leqq y_m \min_{x_1 > 0} \Phi_m(x_1) = E_1\,,$$

thus completing the proof.

7. NUMERICAL EXAMPLES

Some simple examples will be given here which show the comparison of results obtained by means of the method presented in Section 5, both with the lower bounds of the optimal cost which have been derived in Section 6 and with some results obtained for $T_i \subset [A/x_i, B/x_i]$. Tables II and IV show some examples of results.

obtained by calculation of cost $E^{(\varepsilon)}$ and corresponding results based on the calculation of optimal lot sizes $x_i^{(\varepsilon)}$ for the i-th production stage ($i = 1, 2, \ldots, m - 1$), in two production processes characterized by the data given in Tables I and III, respectively. Here $E^{(1)}$ is the lower bound of the optimal cost expressed by the left-hand side of Ineq. (57), $E^{(2)} = E_1$ is the lower bound of the optimal cost calculated from (25), $E^{(3)} = E$ is the optimal cost (15) calculated by means of the algorithm (16)–(23) based on the dynamic programming, $x_1^{(3)} = x_1$ where x_1 minimizes the function $f_m(x_1)$ in (23), $x_{i+1}^{(3)} = K_i x_i^{(3)}$ for $i = 1, 2, \ldots, m - 2$, where K_i is the optimal policy of the solution of the problem (16)–(22). For the given numerical examples the problem (16)–(23) has a unique solution. $E^{(4)}$ equals the optimal cost $F(\hat{x}_m^{(H)}, H(\hat{x}_m^{(H)}))$ calculated for the case when $T_i = \{1\}$ for all $i \neq 2$, $T_2 = (0, 1]$ (cf. Section 2). $E^{(5)}$ equals the optimal cost when $T_i = \{1\}$ for $i = 1, 2, \ldots, m - 2$, so that the optimal value of the lot size satisfies the relation $x_1^{(5)} = x_2^{(5)} = \ldots = x_{m-1}^{(5)}$.

In the given examples the following values have been chosen: $A = 500$, $B = 25\,000$, $\Delta = 250$.

Results for 170 examples with $m \leqq 27$ are given in references [4] and [4a].

Table I

Example 1 (m = 6) — Inputs

i	1	2	3	4	5	6
y_i	16 200	45 000	540	45 000	20 000	263
C_{1i}	595	10^{-8}	510	10^{-8}	10^{-8}	—
$10^9 C_{2i}$	569 976	621 414	4 479 264	4 530 702	4 633 578	—

Table II

Example 1 — Results

ε	1	2	3	4	5
$E^{(\varepsilon)}$	36·29	36·65	37·77	55·65	64·09
$x_1^{(\varepsilon)}$	—	—	23 000	25 000	9 000
$x_2^{(\varepsilon)}$	—	—	11 500	25 000	9 000
$x_3^{(\varepsilon)}$	—	—	11 500	6 250	9 000
$x_4^{(\varepsilon)}$	—	—	500	6 250	9 000
$x_5^{(\varepsilon)}$	—	—	500	6 250	9 000

Table III

Example 2 (m = 7) — Inputs

i	1	2	3	4	5	6	7
y_i	16 200	45 000	860	1 760	32 400	32 400	526
C_{1i}	595	10^{-8}	595	170	10^{-8}	10^{-8}	—
$10^9 C_{2i}$	829 476	880 914	3 349 938	4 533 012	4 584 450	4 635 888	—

Table IV

Example 2 — Results

ε	1	2	3	4	5
$E^{(\varepsilon)}$	69·78	71·65	71·79	96·46	103·14
$x_1^{(\varepsilon)}$	—	—	24 000	23 000	13 750
$x_2^{(\varepsilon)}$	—	—	24 000	23 000	13 750
$x_3^{(\varepsilon)}$	—	—	24 000	11 500	13 750
$x_4^{(\varepsilon)}$	—	—	8 000	11 500	13 750
$x_5^{(\varepsilon)}$	—	—	500	11 500	13 750
$x_6^{(\varepsilon)}$	—	—	500	11 500	13 750

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Souhrn

STANOVENÍ OPTIMÁLNÍCH VYROBNÍCH DÁVEK VÍCESTUPŇOVÉHO VYROBNÍHO SYSTÉMU

JINDŘICH L. KLAPKA

Práce se zabývá optimalisací součtu seřizovacích nákladů a ztrát z vázání zásob pro vícestupňový výrobní systém s nekonečným časovým horizontem volbou velikostí výrobních dávek. Je v ní studována jistá třída vícestupňových jednovýrobkových výrobně-skladovacích systémů se seriově uspořádanými výrobními stupni, kde výrobní rychlosti jednotlivých výrobních stupňů jsou konečné, konstantní a vzájemně různé, odběr hotového výrobku je konstantní a každý výrobní stupeň střídá periodicky období, v němž vyrábí, s obdobím, v němž nevyrábí. Některé systémy, které zkoumali Thomas [8], Schussel [5], Taha a Skeith [7], Crowston, Wagner a Williams [2], Streck [6] a Klapka [4], mohou být chápány jako speciální případy systémů této třídy. V práci je proveden výběr systému z této třídy, jehož optimální náklady jsou minimální. Je odvozen algoritmus přesné nákladové optimalisace tohoto systému, založený na dynamickém programování. Tohoto algoritmu je využito k odvození dvou dolních hranic optimálních nákladů pro zkoumanou třídu systémů. Tyto dolní hranice jsou lepší než ta, kterou pro jistý speciální případ výrobního systému odvodili Crowston, Wagner a Williams.

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