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# ON A MODIFICATION OF RÉNYI'S TRAFFIC MODEL

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0. The problem of the bunching of traffic on roads has long been recognized and has been the subject of study from different points of view [1], [3] etc. All models presented have been developed under the assumption that vehicles are moving on a two-lane road at different speeds and with overtaking partially restricted, either by presence of other vehicles in the overtaking lane or by highway factors, such as bends, hills, obstructions etc. or by unnecessary caution in overtaking on the part of some drivers. In this paper we give a bunching model which assumes that overtaking is prohibited, caused by a one-lane road or a traffic sign (for example).

In [2] Rényi considers the following traffic model: Vehicles enter a highway at the same entrance and at time instants $\langle \bar{T}_i,\ i = \ldots, -1, 0, 1, \ldots \rangle$, forming a homogeneous Poisson process of intensity $\lambda$. Thus $\lambda$ is the rate at which vehicles enter the highway per unit of time and it is supposed that there are no junctions or exits. It is supposed in this model that the vehicle entering at time $\bar{T}_i$ will choose the velocity $V_i$ and will travel all the time at this constant velocity. The random variables $\langle V_i,\ i = \ldots, -1, 0, 1, \ldots \rangle$ are independently and identically distributed (i.i.d.) and also independent of the process $\langle \bar{T}_i \rangle$ with cumulative distribution function (c.d.f.) $F(v) = P(V_i < v)$ satisfying the conditions $F(0) = 0$ and

$$\int_0^\infty \frac{\mathrm{d}F(v)}{v} < +\infty\,,$$

i.e. the mean value of $V_i^{-1}$ is finite. In reality the positive lower limit to the velocity (a) is such that $F(a) = 0$ ensures that those conditions hold. From the assumptions of Rényi's model it is clear that each vehicle can overtake anytime and anywhere without changing its velocity.

The given model is modified in this paper in the following manner: we suppose that if a vehicle $A$ approaches a slower vehicle $B$ ahead of it, vehicle $A$ has to slow down to the velocity of vehicle $B$ and can never overtake it. The cars leave the section of highway on which the above assumptions hold true in bunches, provided that

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a single vehicle without another one close behind or in front is considered a bunch of size one.

In order to handle our results more easily, it is supposed that the highway section under consideration is of unit length. Then the random variables $X_i = V_i^{-1}$ give the passage time over the section without any restriction for the $i$-th vehicle. The assumptions about $\langle V_i \rangle$ imply that $\langle X_i \rangle$ are (i.i.d.) random variables with (c.d.f.)

$$G(x) = 1 - F\left(\frac{1}{x} + 0\right).$$

Let us denote by $\tau_{i+1}$ the time interval between the arrival of two consecutive vehicles, the $i$-th and the $(i + 1)$-st, at the beginning of the section, i.e.

$$\tau_{i+1} = \overline{T}_{i+1} - \overline{T}_i .$$

From Rényi's assumptions it follows that $\langle \tau_i,\ i = \ldots, -1, 0, 1, \ldots \rangle$ are (i.i.d.) with an exponential distribution.

Some probabilistic properties of the traffic model described are given in the next sections of this paper. The probability that a randomly chosen vehicle does not reach any other vehicle, i.e. that this vehicle travels through the section free of the influence of other vehicles and that it is by the end of this section the leading car of a bunch, is given in the first part.

In the second part the probability is expressed that the given car will leave the section in the $(n + 1)$-st place of a bunch, $n = 0, 1, \ldots$. Then the probability distribution of bunch size under modified Rényi conditions is obtained. The fourth part deals with joint probability distribution of bunch size, the time interval between their departures, and their final velocities. The analysed model is described in terms of a semi-Markov process in the last part of this paper.

1. The randomly chosen vehicle indexed by zero leaves the observed highway section as the leading car of a bunch if the following inequalities hold:

$$\begin{aligned}
&X_{-1} - \tau_0 < X_0 , \\
&X_{-2} - (\tau_0 + \tau_{-1}) < X_0 , \\
&\quad\vdots \\
&X_{-i} - (\tau_0 + \tau_{-1} + \ldots + \tau_{-i-1}) < X_0 , \\
&\quad\vdots
\end{aligned} \tag{1.1}$$

Let us denote the probability of this system of inequalities under the condition $X_0 = x$ by the symbol $P(x)$. If $\tau_0 = t$ then (1.1) becomes

$$\begin{aligned}
&X_{-1} < X_0 + t , \\
&X_{-2} - \tau_{-1} < X_0 + t , \\
&\quad\vdots \\
&X_{-i} - (\tau_{-1} + \ldots + \tau_{-i-1}) < X_0 + t , \\
&\quad\vdots
\end{aligned} \tag{1.2}$$

The probability of (1.2) under the condition $X_0 = x$ equals

$$G(x + t)\, P(x + t)$$

due to the independence of the random variables under consideration. From the assumption of a Poisson input we obtain that the probability density function of random variables $\tau_i$ is for each $i$

$$f(t) = \lambda \exp(-\lambda t) \quad \text{for} \quad t \geqq 0\,,$$
$$f(t) = 0 \quad \text{for} \quad t < 0\,.$$

So

(1.3) $$P(x) = \lambda \int_0^\infty \exp(-\lambda t)\, G(x + t)\, P(x + t)\, \mathrm{d}t\,.$$

Let us denote

(1.4) $$R(x) = \exp(-\lambda x)\, P(x)\,,$$

then (1.3) yields

$$R(x) = \lambda \int_x^\infty R(y)\, G(y)\, \mathrm{d}y$$

and thus

$$\frac{\partial R(x)}{\partial x} = -\lambda\, R(x)\, G(x)\,.$$

We obtain from this differential equation

(1.5) $$R(x) = C \exp(-\lambda\, \Gamma(x))\,,$$

where

(1.6) $$\Gamma(x) = \int_0^x G(y)\, \mathrm{d}y\,.$$

Putting (1.5) into (1.4) we obtain

(1.7) $$P(x) = C \exp(-\lambda(\Gamma(x) - x))\,.$$

But, as can be seen from (1.1), $P(+\infty) = 1$ since the vehicle with zero velocity has the unit probability of catching up with any car on his journey. So

(1.8) $$C = \lim_{x \to +\infty} \exp(+\lambda(\Gamma(x) - x)) = \exp(-\mu)\,,$$

where $\mu$ is the mean value of the passage time,

(1.9) $$\mu = E(X_i)\,.$$

Putting (1.8) into (1.7) we get

(1.10) $$P(x) = \exp(-\lambda(\mu + \Gamma(x) - x))$$

and the desired probability that the randomly chosen vehicle does not catch up with another, is

$$(1.11)\qquad P = \int_0^\infty \exp\left(-\lambda(\mu + \Gamma(x) - x)\right) \mathrm{d}G(x)$$

using (c.d.f.) $G(x)$ of the passage time.

The same result follows immediately from Rényi's theorem [2, p. 312], which states that the instants when a given car overtakes cars travelling at lower speeds form a homogeneous Poisson process with the intensity

$$\lambda^+(v_0) = \int_0^{v_0} \frac{v_0 - v}{v}\, \mathrm{d}F(v)\,,$$

where $v_0$ is the velocity of the given car. The properties of Poisson process imply

$$(1.12)\qquad P(x) = \exp\left(-\lambda_R(x)\right),$$

where

$$(1.13)\qquad \lambda_R(x) = \lambda^+(x^{-1})\, x^{-1} = \lambda \int_x^\infty (y - x)\, \mathrm{d}G(y) = \lambda(\mu + \Gamma(x) - x)$$

and this is the same result as (1.10).

2. Now we know the probability that the randomly chosen vehicle will be the leading car in a bunch. We shall compute the probability that a given car leaves the highway section in the $(n + 1)$-st place of a bunch where $n = 0, 1, \ldots$. It can be seen that the car leaves a one-way road as $(n + 1)$-st if both of the following conditions are satisfied:

A. The $n$-th vehicle ahead of the given one, indexed by zero, does not catch up with any other again.

B. The next vehicles, i.e. the first, the second, ..., including the given one, would overtake the zero car if overtaking were possible.

For $n = 0$ only condition $A$ remains because the given car becomes the zero car. Condition $B$ may be expressed by the following system of inequalities

$$(2.1)\qquad \begin{aligned} \tau_1 + X_1 &\leqq X_0\,, \\ \tau_1 + \tau_2 + X_2 &\leqq X_0\,, \\ &\vdots \\ \tau_1 + \tau_2 + \ldots + \tau_n + X_n &\leqq X_0 \end{aligned}$$

for $n = 1, 2, \ldots$.

Let us denote the probability of (2.1) under the condition $X_0 = x$ by $Q_n(x)$ and let us define, in agreement with our previous remark,

$$Q_0(x) = 1 \,. \tag{2.2}$$

The system (2.1) under the condition $\tau_1 = t$ can be rewritten as

$$\begin{aligned} X_1 &\leqq X_0 - t \,, \\ \tau_2 + X_2 &\leqq X_0 - t \,, \\ &\vdots \\ \tau_2 + \tau_3 + \ldots + \tau_n + X_n &\leqq X_0 - t \,. \end{aligned} \tag{2.3}$$

From now on, let us assume that $G(x)$ is continuous. If the probability of (2.1) under the conditions $\tau_1 = t$ and $X_0 = x$ is denoted by $Q_n(x, t)$, then (2.3) implies

$$Q_n(x, t) = G(x - t)\, Q_{n-1}(x - t)$$

for $n = 1, 2, \ldots$.

For the same reason as (1.3),

$$Q_n(x) = \lambda \int_0^x \exp(-\lambda t)\, G(x - t)\, Q_{n-1}(x - t)\, \mathrm{d}t \tag{2.4}$$

for $n = 1, 2, \ldots$.

Defining

$$R_n(x) = \exp(\lambda x)\, Q_n(x) \tag{2.5}$$

for $n = 0, 1, 2, \ldots$, (2.4) and (2.2) imply

$$R_n(x) = \lambda \int_0^x G(y)\, R_{n-1}(y)\, \mathrm{d}y \,, \quad \text{for} \quad n = 1, 2, \ldots$$

and

$$R_0(x) = \exp(\lambda x) \,.$$

Let us denote

$$R(x, z) = \sum_{n=0}^{\infty} z^n R_n(x) = \exp(\lambda x) + z\lambda \int_0^x G(y)\, R(y, z)\, \mathrm{d}y \,. \tag{2.6}$$

Then in a way similar to the method used in the previous section we get

$$R(x, z) = \exp(\lambda z\, \Gamma(x)) \cdot \left(1 + \lambda \int_0^x \exp(\lambda y - \lambda z\, \Gamma(y))\, \mathrm{d}y\right), \tag{2.7}$$

where $\Gamma(x)$ is given by the relation (1.6).

Putting (2.7) back into (2.5) and (2.6) we obtain

$$(2.8) \qquad Q_n(x) = \frac{\lambda^n}{n!} \exp(-\lambda x) \left[ \Gamma^n(x) + \lambda \int_0^x \exp(\lambda y) [\Gamma(x) - \Gamma(y)]^n \, dy \right]$$

for $n = 0, 1, 2, \ldots$.

Let $P_{n+1}(x)$ ($n = 0, 1, 2, \ldots$) denote the probability that the vehicle comes in the $(n + 1)$-st place in a bunch under the condition that the speed of the leading vehicle of this bunch equals $x^{-1}$. Following conditions $A$ and $B$ and using (1.12) and (2.8), we conclude

$$(2.9) \quad P_{n+1}(x) = \exp(-\lambda(\mu + \Gamma(x))) \frac{\lambda^n}{n!} \left[ \Gamma^n(x) + \lambda \int_0^x \exp(\lambda y) [\Gamma(x) - \Gamma(y)]^n \, dy \right]$$

for $n = 0, 1, 2, \ldots$.

Similarly to (1.11),

$$(2.10) \qquad P_{n+1} = \exp(-\lambda\mu) \frac{\lambda^n}{n!} \cdot$$

$$\cdot \int_0^\infty \exp(-\lambda\, \Gamma(x)) \left[ \Gamma^n(x) + \lambda \int_0^x \exp(\lambda y) [\Gamma(x) - \Gamma(y)]^n \, dy \right] dG(x)$$

for $n = 0, 1, \ldots$ is the probability that a randomly chosen vehicle comes in the $(n + 1)$-st place in a bunch. Obviously (1.10) or (1.11), and (2.9) or (2.10) for $n$ equal to zero are identical.

3. Because $P_{n+1}$ given by the relation (2.10) is the probability that a randomly chosen vehicle leaves in the $(n + 1)$-st place of a bunch, it can be expected that for a large number $N$ of passing vehicles, $NP_{n+1}$ vehicles are in the $(n + 1)$-st place of bunches. Thus the number of bunches with sizes greater or equal to $(n + 1)$ is the same. Similarly $NP_{n+2}$ is proportional to the number of bunches greater or equal to $n + 2$. So $N(P_{n+1} - P_{n+2})$ is proportional to the number of bunches of size equal exactly to $n + 1$ and $\sum_{n=0}^{\infty} N(P_{n+1} - P_{n+2})$ to the number of all bunches. Then the probability of the appearance of a bunch with exactly $n + 1$ members is

$$(3.1) \qquad p_{n+1} = (P_{n+1} - P_{n+2}) \cdot P_1^{-1} ;$$

using (2.10) we obtain

$$(3.2) \quad p_{n+1} = \int_0^\infty \exp(-\lambda_R(x)) (Q_n(x) - Q_{n+1}(x)) \, dG(x) \left( \int_0^\infty \exp(-\lambda_R(x)) \, dG(x) \right)^{-1} =$$

$$(3.3) \qquad = \frac{\lambda^n}{n!} \left( 1 - \frac{\lambda}{n + 1} \right) \cdot \frac{W}{\int_0^\infty \exp(-\lambda(\Gamma(x) - x)) \, dG(x)},$$

where

$$W = \int_0^\infty e^{-\lambda \Gamma(x)} \left( \Gamma^n(x) + \lambda \int_0^x e^{\lambda y} [\Gamma(x) - \Gamma(y)]^n \, \mathrm{d}y - \Gamma^{n+1}(x) + \right.$$

$$\left. + \lambda \int_0^x e^{\lambda y} [\Gamma(x) - \Gamma(y)]^{n+1} \, \mathrm{d}y \right) \mathrm{d}G(x)$$

for $n = 0, 1, \ldots$ .

The same result can be obtained if we realize the fact that $P_{n+1} - P_{n+2}$ is the probability that the randomly chosen vehicle is at the end of a bunch with exactly $n + 1$ members. Then $\sum_{n=0}^{\infty} (P_{n+1} - P_{n+2})$ is the probability that the randomly chosen vehicle leaves the section as the last vehicle in a bunch and this also leads us to the relation (3.1).

The distribution (3.3) would be similar to some of those presented in [3] but its complicated form does not permit their easy comparison.

Let us now compute the generating function of the distribution (3.3). Using (3.2),

$$\Pi(z) = \sum_{n=0}^{\infty} z^{n+1} p_{n+1} = \sum_{n=0}^{\infty} z^{n+1} \frac{\int_0^\infty e^{-\lambda_R(x)} (Q_n(x) - Q_{n+1}(x)) \, \mathrm{d}G(x)}{\int_0^\infty \exp(-\lambda_R(x)) \, \mathrm{d}G(x)} .$$

From (2.5) and (2.6) we can see that

$$\sum_{n=0}^{\infty} z^{n+1} (Q_n(x) - Q_{n+1}(x)) = \exp(-\lambda x)(z - 1) R(x, z) + 1 ,$$

and putting this into the relation above, we have

$$\Pi(z) = \frac{\int_0^\infty (z - 1) R(x, z) \exp(-\lambda \Gamma(x)) \, \mathrm{d}G(x)}{\int_0^\infty \exp(-\lambda(\Gamma(x) - x)) \, \mathrm{d}G(x)} + 1 , \tag{3.4}$$

where $\lambda_R(x)$ is taken from (1.13). We can obtain the moments of the random variable with this probability distribution and get

$$E(N) = \left( \frac{\partial \Pi(z)}{\partial z} \right)_{z=1} = \frac{1 + \lambda \int_0^\infty \int_0^x \exp(-\lambda(\Gamma(y) - y)) \, \mathrm{d}y \, \mathrm{d}G(x)}{\int_0^\infty \exp(-\lambda(\Gamma(x) - x)) \, \mathrm{d}G(x)} \tag{3.5}$$

and

$$(3.6)\qquad E(N^2) = \left(\frac{\partial^2 \Pi(z)}{\partial z^2}\right)_{z=1} + \left(\frac{\partial \Pi(z)}{\partial z}\right)_{z=1} =$$

$$= 2\lambda + \int_0^\infty (2\lambda\, G(x) + 1)\left(1 + \int_0^x \exp\left(-\lambda(\Gamma(y) - y)\right) \mathrm{d}y\right) \mathrm{d}G(x) \times$$

$$\times \left(\int_0^\infty \exp\left(-\lambda(\Gamma(x) - x)\right) \mathrm{d}G(x)\right)^{-1}.$$

Moments of higher orders can be obtained in the usual way.

4. Not only the size of bunches but also the time intervals between their arrivals and their speed at the end of a one-lane road may be a valuable information about the traffic flow.

If the following system of conditions on random variables is satisfied, then the zero car will be caught up by exactly $n$ others ($n = 0, 1, \ldots$) and the interval between this bunch and the next one is in the interval $\langle \bar{t}, \bar{t} + \mathrm{d}\bar{t})$:

$$(4.1)\qquad \begin{aligned} \tau_1 + X_1 &\leqq X_0\,,\\ \tau_1 + \tau_2 + X_2 &\leqq X_0\,,\\ &\vdots\\ \tau_1 + \ldots + \tau_n + X_n &\leqq X_0\,, \end{aligned}$$

$$\tau_{n+1} \in \langle X_0 - X_{n+1} + \bar{t} - \tau_1 - \ldots - \tau_n, X_0 - X_{n+1} + \bar{t} + \mathrm{d}\bar{t} - \tau_1 - \ldots - \tau_n)$$

for $n = 0, 1, \ldots$, where $X_{n+1}$ is the random variable representing the time of passage for the leading vehicle of the next bunch. If we add the requirement that the zero car should catch up with no other, then (4.1) are conditions for the arrival of a bunch of size $(n + 1)$, followed by a time interval from $\langle \bar{t}, \bar{t} + \mathrm{d}\bar{t})$ and by the next bunch with the leaving speed $X_{n+1}^{-1}$.

Let us denote the probability of the system (4.1) under the conditions $X_0 = x$, $X_{n+1} = \bar{x}$ by the symbol $Q_n(x, \bar{t}/\bar{x})\,\mathrm{d}\bar{t}$. It is clear that

$$(4.2)\qquad Q_0(x, \bar{t}/\bar{x})\,\mathrm{d}\bar{t} = \lambda \exp\left(-\lambda(x - \bar{x} + \bar{t})\right)\mathrm{d}\bar{t} \quad \text{for} \quad \bar{t} \geqq \bar{x} - x$$

and thus $\qquad \lambda \exp\left(-\lambda(x - \bar{x} + \bar{t})\right) \quad \text{for} \quad \bar{t} \geqq \bar{x} - x$

is the conditional probability density function of the time interval behind the vehicle which has not been caught up by any other.

Under the conditions $\tau_1 = t$, $X_0 = x$ and $X_{n+1} = \bar{x}$, (4.1) becomes

$$\begin{aligned} X_1 &\leqq x - t\,,\\ \tau_2 + X_2 &\leqq x - t\,,\\ &\vdots\\ \tau_2 + \ldots + \tau_n + X_n &\leqq x - t\,, \end{aligned}$$

$$\tau_{n+1} \in \langle x - t - \bar{x} + \bar{t} - \tau_2 - \ldots - \tau_n,\ x - t - \bar{x} + \bar{t} + \mathrm{d}\bar{t} - \tau_2 - \ldots - \tau_n)$$

and the probability of this system is similar to that in the previous section, namely

$$G(x - t)\, Q_{n-1}(x - t, \bar{t}/\bar{x})\, d\bar{t}$$

for $n = 1, 2, \ldots$ and $\bar{t} \geqq \bar{x} - x + t$.

So

$$(4.3) \qquad Q_n(x, \bar{t}/\bar{x})\, d\bar{t} = \lambda \int_0^{x - \bar{x} + \bar{t}} \exp(-\lambda t)\, G(x - t)\, Q_{n-1}(x - t, \bar{t}/\bar{x})\, d\bar{t}\, dt$$

for $n = 1, 2, \ldots$.

For the sake of simplicity let us now assume that the probability density function $g(x)$ of the random variables $X_i$ exists. Let us remove the condition $X_{n+1} = \bar{x}$ from (4.2) and (4.3) and denote the resulting probability distributions by $Q_n(x, \bar{t}, \bar{x})$; then we have recurrent formula

$$(4.4) \qquad \begin{aligned} Q_0(x, \bar{t}, \bar{x}) &= \lambda \exp(-\lambda(x - \bar{x} + \bar{t}))\, g(\bar{x}) \quad \text{for} \quad \bar{x} - \bar{t} \leqq x\,, \\ &= 0 \quad \text{for} \quad \bar{x} - \bar{t} > x \end{aligned}$$

and

$$\begin{aligned} Q_n(x, \bar{t}, \bar{x}) &= \lambda \int_0^{x - \bar{x} + \bar{t}} \exp(-\lambda t)\, G(x - t)\, Q_{n-1}(x - t, \bar{t}, \bar{x})\, dt \\ &\qquad \text{for} \quad \bar{x} - \bar{t} \leqq x\,, \\ &= 0 \qquad \text{for} \quad \bar{x} - \bar{t} > x \end{aligned}$$

for $n = 1, 2, \ldots$.

$Q_n(x, \bar{t}, \bar{x})$ can be computed by using once again the generating function

$$(4.6) \qquad R(x, z) = \sum_{n=0}^{\infty} z^n R_n(x, \bar{t}, \bar{x})\,,$$

where $R_n(x, \bar{t}, \bar{x})$ is defined analogously to (2.5). Putting (4.4) and (4.5) in this generating function, we obtain

$$R(x, z) = \lambda \exp(-\lambda(-\bar{x} + \bar{t})) + z\lambda \int_{\bar{x} - \bar{t}}^{x} G(y)\, R(y, z)\, dy\,,$$

where $x \geqq \max\{0, \bar{x} - \bar{t}\}$.

From this equation it follows that

$$R(x, z) = \lambda \exp(\lambda z(\Gamma(x) - \Gamma(\bar{x} - \bar{t})))\exp(-\lambda(-\bar{x} + \bar{t}))\, g(\bar{x})$$

which together with (4.6) implies

$$(4.7) \qquad \sum_{n=0}^{\infty} z^n Q_n(x, \bar{t}, \bar{x}) = \exp(\lambda z(\Gamma(x) - \Gamma(\bar{x} - \bar{t})))\, \lambda \exp(-\lambda(x - \bar{x} + \bar{t}))\, g(\bar{x})\,.$$

Thus

$$(4.8) \qquad Q_n(x, \bar{t}, \bar{x}) = \frac{\lambda^n}{n!} (\Gamma(x) - \Gamma(\bar{x} - \bar{t}))^n \lambda \exp(-\lambda(x - \bar{x} + \bar{t}))\, g(\bar{x})$$

for $n = 0, 1, \ldots$ and $\bar{x} - \bar{t} \leqq x$.

The quantity $Q_n(x, \bar{t}, \bar{x})$ is the conditional joint probability distribution of the number of vehicles which have caught up with a given one, of time interval between this and the next bunch and of the passage time of the leading car in the next bunch under the condition that the time of passage of the given car is $X_0 = x$.

To connect these results with Part 2 of this paper, some relations between $Q_n(x)$ given by (2.8) and $Q_n(x, \bar{t}, \bar{x})$ given by (4.9) should be noted. It can be seen that

$$(4.11) \qquad Q_n(x) - Q_{n+1}(x) = \iint_{\bar{x} - \bar{t} < x} Q_n(x, \bar{t}, \bar{x})\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{t}$$

for each $x \in (0, +\infty)$ and $n = 0, 1, \ldots$.

Then certainly

$$\begin{aligned} P^*_{n+1}(x) &= P_1(x) \iint_{\bar{x} - \bar{t} < x} Q_n(x, \bar{t}, \bar{x})\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{t} = \\ &= P_1(x)\,(Q_n(x) - Q_{n+1}(x)) = \\ &= P_{n+1}(x) - P_{n+2}(x) \end{aligned}$$

gives the probability that the zero car travelling at the speed $x^{-1}$ is the leading car of a bunch with $n = 0, 1, \ldots$ members.

5. Since the quantity $Q_n(x, \bar{t}, \bar{x})$ given by the relation (4.8) can be considered as the conditional joint probability distribution of the size of the $i$-th bunch, of the $(i + 1)$-st lapse of time $T_{i+1}$ (i.e. the time interval between the $i$-th and the $(i + 1)$-st bunches), of the passage time $X_{i+1}$ for the leading car of the $(i + 1)$-st bunch, under the condition that $X_i = x$ (i.e. the passage time for the first car of the $i$-th bunch being known), we can obtain the conditional joint probability distribution of the triplet $(N_{i+1}, T_{i+1}, X_{i+1})$ under the condition $(N_i = m,\ T_i = t,\ X_i = x)$, (i.e. the distribution of the time interval ahead of a bunch, of its size and of the passage time for its first car). We find:

$$(5.1) \qquad \begin{aligned} &P(N_i = n,\ T_{i+1} \in \langle \bar{t}, \bar{t} + \mathrm{d}\bar{t}),\ X_{i+1} \in \langle \bar{x}, \bar{x} + \mathrm{d}\bar{x})/X_i = x) \times \\ &\times \frac{P(N_{i+1} = m/X_{i+1} = \bar{x})}{P(N_i = n/X_i = x)} = \end{aligned}$$

$$= P(N_{i+1} = m,\ T_{i+1} \in \langle \bar{t}, \bar{t} + \mathrm{d}\bar{t}),\ X_{i+1} \in \langle \bar{x}, \bar{x} + \mathrm{d}\bar{x})/N_i = n,\ T_i = t,\ X_i = x)\,.$$

Let us denote the right hand side of (5.1) by the symbol $S(m, \bar{t}, \bar{x}/n, x)\,\mathrm{d}\bar{t}\,\mathrm{d}\bar{x}$. With regard to (2.8) and (4.8), we obtain

$$(5.2) \qquad S(m, \bar{t}, \bar{x}/n, x)\,\mathrm{d}\bar{t}\,\mathrm{d}\bar{x} =$$
$$= Q_{n-1}(x, \bar{t}, \bar{x})\,\mathrm{d}\bar{t}\,\mathrm{d}\bar{x}(Q_{n-1}(x) - Q_n(x))^{-1}\,(Q_{m-1}(\bar{x}) - Q_m(\bar{x})) =$$
$$= \frac{\lambda^{m-1}}{(m-1)!}(\Gamma(x) - \Gamma(\bar{x} - \bar{t}))^{n-1}\,\lambda e^{-\lambda(x - \bar{x} + \bar{t})}\,g(\bar{x}) \times$$
$$\times \frac{\Gamma^{m-1}(x)\left(1 - \frac{\lambda}{m}\right) + \lambda\int_0^x e^{\lambda y}[\Gamma(x) - \Gamma(y)]^{m-1}\,(1 - \Gamma(x) + \Gamma(y))\,\mathrm{d}y}{\Gamma^{n-1}(x)\left(1 - \frac{\lambda}{n}\right) + \lambda\int_0^x e^{\lambda y}[\Gamma(x) - \Gamma(y)]^{n-1}\,(1 - \Gamma(x) + \Gamma(y))\,\mathrm{d}y}\,.$$

If we observe a bunch with the leading car passage time $x$ and of the size $n$, then $S(m, \bar{t}, \bar{x}/n, x)\,\mathrm{d}\bar{t}\,\mathrm{d}\bar{x}$ is the probability of the time space $T \in \langle \bar{t}, \bar{t} + \mathrm{d}\bar{t})$ which follows, of the size $N = m$, and of the leading car passage time $X \in \langle \bar{x}, \bar{x} + \mathrm{d}\bar{x})$. When we know this conditional distribution, we can find the stationary distribution.

The process of departures of bunches from the given highway section can be characterized as follows. At the instant of the departure of the $i$-th bunch, let us define the state to be the pair $(X_i = x, N_i = n)$. A random time is spent in this state after which a transition is made to the state $(X_{i+1} = \bar{x}, N_{i+1} = m)$. The probability of the state $(X_{i+1} = \bar{x}, N_{i+1} = m)$ depends on the current state $(X_i = x, N_i = n)$ but is independent of all previous states. Furthermore, $T_i$ depends on the current state as well as on the next one but, given these states, it is independent of the previous $T$'s and states. Thus the given process is a semi-Markov process

$$\{N_i, X_i, T_i, i = \ldots, -1, 0, 1, \ldots\}$$

with the state space $\{1, 2, \ldots\} \times \{x; x > 0\}$.

Let us define

$$(5.3) \qquad A_{n,m}(\bar{t}, \bar{x}/x)\,\mathrm{d}\bar{x} =$$
$$= P(N_{i+1} = m,\ X_{i+1} \in \langle \bar{x}, \bar{x} + \mathrm{d}\bar{x}),\ T_{i+1} \leqq \bar{t}/N_i = n,\ X_i = x)$$

and

$$(5.4) \qquad A_{n,m}(\infty, \bar{x}/x)\,\mathrm{d}\bar{x} =$$
$$= P(N_{i+1} = m,\ X_{i+1} \in \langle \bar{x}, \bar{x} + \mathrm{d}\bar{x}),\ T_{i+1} < \infty/N_i = n,\ X_i = x) =$$
$$= A_{n,m}(\bar{x}/x)\,\mathrm{d}\bar{x}\,.$$

$A_{n,m}(\bar{x}/x)$ is the one step transition probability function for the underlying Markov chain. The stationarity distribution for $\{N_i, X_i\}$ is characterized as follows:

$$A_m(\bar{x})\,\mathrm{d}\bar{x} = P(N_{i+1} = m,\ X_{i+1} \in \langle \bar{x}, \bar{x} + \mathrm{d}\bar{x}))$$

is the stationary distribution if

(5.5) $$\sum_{m=1}^{\infty} \int_0^{\infty} A_m(\bar{x})\,\mathrm{d}\bar{x} = 1 \quad \text{and} \quad A_m(\bar{x}) \geqq 0$$

and if

(5.6) $$A_m(\bar{x})\,\mathrm{d}\bar{x} = \int_0^{\infty} \sum_{n=1}^{\infty} (A_{n,m}(\bar{x}/x)\,\mathrm{d}\bar{x}\, A_n(x))\,\mathrm{d}x \quad \text{for} \quad m = 1, 2, \ldots .$$

By the same argument as in Part 3 we conclude

(5.7) $$A_n(x)\,\mathrm{d}x = P_1(x)\, g(x)\,\mathrm{d}x(Q_{n-1}(x) - Q_n(x)) \left( \int_0^{\infty} P_1(u)\, g(u)\,\mathrm{d}u \right)^{-1},$$

which is in agreement with (3.2). This implies that

$$P_1(x)\, g(x) \left( \int_0^{\infty} P_1(u)\, g(u)\,\mathrm{d}u \right)^{-1}$$

is the probability density of the random variable representing the passage time of the leading car of a bunch. Thus the distribution of the speed at which a bunch departs can be obtained and compared with $F(v)$.

The stationary distribution $A_n(x)\,\mathrm{d}x$ given by (5.7) clearly fulfils (5.5). Condition (5.6) implies

$$A_m(\bar{x})\,\mathrm{d}\bar{x} = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{P_1(x)\,(Q_{n-1}(x) - Q_n(x))\, g(x)}{\int_0^{\infty} P_1(u)\, g(u)\,\mathrm{d}u} A_{n,m}(\bar{x}/x)\,\mathrm{d}\bar{x}\,\mathrm{d}x .$$

Using (5.2) we obtain

$$A_m(\bar{x})\,\mathrm{d}\bar{x} = \frac{Q_{m-1}(\bar{x}) - Q_m(\bar{x})}{\int_0^{\infty} P_1(u)\, g(u)\,\mathrm{d}u} \int_0^{\infty} \sum_{n=1}^{\infty} \int_{\bar{x}-\bar{t}<x} P(x)\, Q_{n-1}(x, \bar{t}, \bar{x})\,\mathrm{d}\bar{x}\, g(x)\,\mathrm{d}\bar{t}\,\mathrm{d}x .$$

To verify (5.6) we have to show

$$\int_0^{\infty} \sum_{n=1}^{\infty} \int_{\bar{x}-\bar{t}<x} P(x)\, Q_{n-1}(x, \bar{t}, \bar{x})\,\mathrm{d}\bar{x}\, g(x)\,\mathrm{d}\bar{t}\,\mathrm{d}x = P(\bar{x})\, g(\bar{x})\,\mathrm{d}\bar{x} .$$

By (4.10) this is equivalent to

$$\int_0^{\infty} \int_{\bar{x}-\bar{t}}^{\infty} \exp\left(-\lambda(\Gamma(\bar{x} - \bar{t}) - (\bar{x} - \bar{t}))\right) g(x)\,\mathrm{d}x\,\mathrm{d}\bar{t} = \lambda^{-1} \exp\left(-\lambda(\Gamma(\bar{x}) - \bar{x})\right).$$

But this equality is evidently valid, so that (5.6) holds. Thus $A_n(x)$ is the stationary distribution of the bunch size and of the passage time for the first car in this bunch.

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## Souhrn

## O JEDNÉ MODIFIKACI RÉNYIOVA DOPRAVNÍHO MODELU

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Je dán pravděpodobnostní model pohybu vozidel po jednoproudé silnici za těchto předpokladů: rychlosti vozidel při vstupu jsou navzájem nezávislé náhodné veličiny a okamžiky vstupu do daného úseku silnice tvoří Poissonův proces. Při projíždění tohoto úseku každé vozidlo dodržuje svoji rychlost konstantní, pokud není vozidly předcházejícími nuceno ji snížit. Jsou uvedeny pravděpodobnostní rozdělení velikostí a rychlostí shluků, které opouštějí sledovaný úsek a celý proces výstupu je popsán jako semi-markovovský náhodný proces.

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