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SOLUTION OF THE FIRST PROBLEM OF PLANE ELASTICITY FOR MULTIPLY CONNECTED REGIONS BY THE METHOD OF LEAST SQUARES ON THE BOUNDARY (Part II)

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In Part I of this paper (Apl. mat. 22 (1977), 349–394), the formulation of the problem was given and fundamental theorems on the existence of solution and on its properties were proved (Chaps. 1 and 2). An approximate method – the so-called method of least squares on the boundary – was developed and some numerical examples were shown (Chap. 3).

The present Part II (Chaps. 4 and 5) brings the proof of the main convergence theorem 3.2 from p. 379.

Chapter 4. CONVERGENCE OF THE METHOD

Before giving the proof of the convergence theorem for the method of least squares on the boundary in the case of multiply connected regions, let us summarize shortly some basic results from Chapters 2 and 3.

Let the loading on Γ satisfy the equilibrium conditions both in forces and moments on every of the boundary curves Γ_i $(i = 0, \ldots, k)$ and let the functions g_{i0}, g_{i1} satisfy the relations

$$g_{i0} \in W_2^{(1)}(\Gamma_i)\,, \quad g_{i1} \in L_2(\Gamma_i) \tag{4.1}$$

(briefly $(g_{i0}, g_{i1}) \in W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$). Let $u(x, y)$ be the (unique) very weak solution of the biharmonic problem

$$\Delta^2 u = 0 \quad \text{in} \quad G\,, \tag{4.2}$$

$$u = g_{i0}\,, \frac{\partial u}{\partial \nu} = g_{i1} \quad \text{on} \quad \Gamma_i\,. \tag{4.3}$$

According to Theorem 2.1, p. 373, there exists exactly one very weak Airy function corresponding to the given loading (given by the functions g_{i0}, g_{i1}). This function can be written in the form

$$U(x, y) = u(x, y) - v(x, y)\,, \tag{4.4}$$

where

$$v(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij} r_{ij}(x, y)\,; \tag{4.5}$$

$r_{ij}(x, y)$ are the basic singular biharmonic functions (p. 367), α_{ij} $(i = 1, \ldots, k, j = 1, 2, 3)$ are solutions of the system (2.51).

In Chap. 3, the method of least squares on the boundary was developed to find an approximate solution of the problem (4.2), (4.3). This approximate solution is assumed in the form

$$u_{st}(x, y) = U_{st}(x, y) + \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{stij}\, r_{ij}(x, y)\,, \tag{4.6}$$

where s, t are chosen positive integers $(s \geqq 2)$,

$$U_{st}(x, y) = V_{st}(x, y) + W_{st}(x, y) + \sum_{i=1}^{k} c_{sti} \ln\left[(x - x_i)^2 + (y - y_i)^2\right] \tag{4.7}$$

and

$$V_{st}(x, y) = \sum_{p=1}^{4s-2} a_{stp}\, z_p(x, y)\,, \tag{4.8}$$

$$W_{st}(x, y) = \sum_{i=1}^{k} \sum_{q=1}^{4t} b_{stiq}\, v_{iq}(x, y)\,. \tag{4.9}$$

Here $z_p(x, y)$ are the basic biharmonic polynomials of degrees $\leqq s$, $v_{iq}(x, y)$ in (4.9) are the basic rarional biharmonic functions defined by

$$\begin{aligned} v_{i,4l+1}(x, y) &= \operatorname{Re}\left(\frac{\bar{z}}{(z - z_i)^{l+1}}\right), \quad v_{i,4l+2}(x, y) = \operatorname{Im}\left(\frac{\bar{z}}{(z - z_i)^{l+1}}\right), \\ v_{i,4l+3}(x, y) &= \operatorname{Re}\left(\frac{1}{(z - z_i)^{l+1}}\right), \quad v_{i,4l+4}(x, y) = \operatorname{Im}\left(\frac{1}{(z - z_i)^{l+1}}\right) \end{aligned} \tag{4.10}$$

$(l = 0, 1, \ldots, t - 1)$, $P_i(x_i, y_i)$ are arbitrary but fixed points lying in the interior of Γ_i $(i = 1, \ldots, k)$ (thus in the exterior of $\bar{G}$), $z_i = x_i + iy_i$. The (real) coefficients a_{stp}, b_{stiq}, c_{sti} and α_{stij} are uniquely determined (Theorem 3.1) by the condition (3.13) (the condition of the best approximation of boundary conditions in the sense of least squares).

Note that the function

$$v_{st}(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{stij}\, r_{ij}(x, y)\,, \tag{4.11}$$

and the function $U_{st}(x, y)$ represent the singular part of the function $u_{st}(x, y)$ and the Airy part of this function, respectively. In fact, (4.11) is a singular biharmonic function (thus "producing" a multi-valued displacement), provided at least one of the coefficients α_{stij} is different from zero (Lemma 2.4). On the other hand, each of the functions (4.7) is an Airy function: The function $V_{st}(x, y)$ is a biharmonic polynomial and consequently, it is defined not only in G but in every simply connected region $\hat{G} \subset E_2$ containing G. Thus the corresponding complex stress-functions $\varphi(z)$, $\psi(z)$ are holomorphic in G and the formula (2.9),

$$d_1 + \mathrm{i}d_2 = \frac{1}{2\mu}(\varkappa\varphi - z\bar{\varphi}' - \bar{\psi}) \tag{4.12}$$

gives an evidently single-valued displacement. The functions $v_{iq}(x, y)$ corresponding to the functions $\varphi(z) = 1/(z - z_i)^{l+1}$ and $\chi(z) = 1/(z - z_i)^{l+1}$ $(l = 0, 1, \ldots, t - 1)$ according to the formulae (4.10) are also Airy functions in virtue of the same formula (4.12) because the functions $1/(z - z_i)^{l+1}$ and $-(l + 1)/(z - z_i)^{l+1}$ are single-valued functions. From the same formula the same result follows for the functions $\ln [(x - x_i)^2 + (y - y_i)^2]$, because each of these functions is of the form

$$\ln [(x - x_i)^2 + (y - y_i)^2] = \operatorname{Re}(\chi(z))$$

with

$$\chi(z) = 2 \ln (z - z_i)\,,$$

and the function

$$\psi(z) = \chi'(z) = \frac{2}{z - z_i}$$

is a single-valued function.

The proof of convergence of our method consists in proving that

$$U_{st}(x, y) \to U(x, y)\,, \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{stij}\, r_{ij}(x, y) \to v(x, y) \text{ in } L_2(G) \text{ for } s \to \infty\,, t \to \infty\,,$$

where $U(x, y)$ and $v(x, y)$ are functions given by (4.4), (4.5). (Of course the convergence of $U_{st}(x, y)$ to $U(x, y)$ is of particular importance, because $U(x, y)$ is the Airy function corresponding to the given loading and therefore the required solution.)

We start with some auxiliary lemmas: In these lemmas, G is the considered bounded $(k + 1)$-tuply connected region with the Lipschitzian boundary, z_i $(i = 1, \ldots, k)$ are fixed points lying inside the inner boundary curves Γ_i (cf. Lemma 2.2, p. 361), $\tilde{G}$ is a bounded $(k + 1)$-tuply connected region with a smooth boundary $\tilde{\Gamma}$ such that $\bar{G} \subset \tilde{G}$ and that each of the points z_i $(i = 1, \ldots, k)$ lies again inside the inner boundary curve $\tilde{\Gamma}_i$ (Fig. 9). We shall often speak briefly of a *weak* or *very weak biharmonic function*, respectively, instead of a *weak* or *very weak solution of a biharmonic problem*. In a similar sense we shall speak of a *weak* or *very weak Airy* function (cf. p. 355).

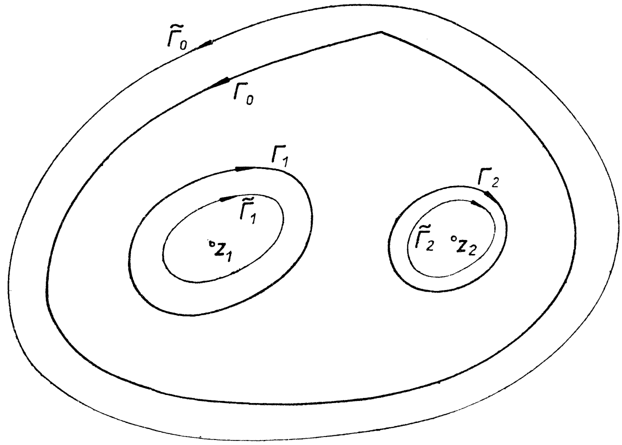

Fig. 9.

Lemma 4.1. *Let $(U(x, y)$ be a weak Airy function in G. Then to every $\delta_1 > 0$ there exists such a region $\tilde{G} \supset \bar{G}$ and such a weak Airy function $\tilde{U}(x, y)$ in $\tilde{G}$ that its restriction on G[1]) satisfies*

$$\|U - \tilde{U}\|_{W_2^{(1)}(\Gamma)} < \delta_1, \left\|\frac{\partial U}{\partial \nu} - \frac{\partial \tilde{U}}{\partial \nu}\right\|_{L_2(\Gamma)} < \delta_1 . \tag{4.13}$$

Roughly speaking: An Airy function in G can be approximated by such an Airy function defined in a "slightly" larger region $\tilde{G}$ that the traces of both these Airy functions on Γ are sufficiently close.

For the proof see Chap. 5, p. 444–448. (The text preceding the relations (5.58).)

Lemma 4.2. *An Airy function $\tilde{U}(x, y)$ in in $\tilde{G}$ can be written in the form*

$$\tilde{U}(x, y) = \operatorname{Re}(\bar{z}\varphi + \chi), \tag{4.14}$$

where $\varphi(z)$ is holomorphic in $\bar{G}$ and $\chi(z)$ is of the form

$$\chi(z) = \chi_0(z) + \sum_{i=1}^{k} c_i \ln(z - z_i),$$

where $\chi_0(z)$ is holomorphic in $\bar{G}$ and c_i are real constants.

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[1]) I.e. the "part" of the function $\tilde{U}(x, y)$, considered only on G.

(Thus if $\tilde{U}(x, y)$ is an Airy function, the form of the corresponding stress-functions is very simple.)

For the proof see Chap. 5, p. 441.

Lemma 4.3. *Let $\varphi(z)$ be a holomorphic function in $\tilde{G} \supset \bar{G}$. Then this function and its derivatives up to the r-th order can be approximated uniformly on $\bar{G}$ by polynomials and rational functions with poles at the points z_i $(i = 1, \ldots, k)$ and by their corresponding derivatives. More precisely: To the function $\varphi(z)$ holomorphic in $\tilde{G} \supset \bar{G}$, to every positive integer r and to every $\delta_2 > 0$ it is possible to find positive integers s, t and constants A_{sj}, B_{itj} such that the function*

$$\varphi_{st}(z) = \sum_{j=0}^{s-1} A_{sj} z^j + \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{B_{itj}}{(z - z_i)^j} \tag{4.15}$$

satisfies in G

$$\begin{aligned} |\varphi(z) &- \varphi_{st}(z)| < \delta_2 \,, \\ |\varphi'(z) &- \varphi'_{st}(z) < \delta_2 \,, \\ &\cdots\cdots\cdots\cdots\cdots \\ |\varphi^{(r)}(z) &- \varphi_{st}^{(r)}(z)| < \delta_2 \,. \end{aligned} \tag{4.16}$$

This lemma is an immediate consequence of Theorem 10.27 (p. 214) and Theorem 13.6 (p. 256) in [6].

Lemma 4.4. (See Lemma 5.4, p. 436.) *Let a sequence $\{u_n(x, y)\}$ of very weak biharmonic functions converge in $L(G)$ to a very weak biharmonic function $u(x, y)$. Then this convergence is uniform on every subregion $G' \subset \bar{G}' \subset G$. Moreover, the sequence of $D^j u_n(x, y)$, where $D^j u_n(x, y)$ means an arbitrary* (partial) *derivative of $u_n(x, y)$, converges to the corresponding derivative $D^j u(x, y)$ of $u(x, y)$ uniformly on G'.*

In Chap. 2 (Theorem 2.1 and the preceding text) we have seen that every very weak biharmonic function $u(x, y)$ in G can be uniquely expressed in the form

$$u(x, y) = U(x, y) + v(x, y) \,, \tag{4.17}$$

where $U(x, y)$ is an Airy function and

$$v(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij} \, r_{ij}(x, y) \,, \tag{4.18}$$

where $r_{ij}(x, y)$ are basic singular biharmonic functions defined as solutions of the problems (2.27)–(2.35), p. 367. Thus (4.18) is the "singular part" of the function $u(x, y)$. This singular part depends continuously on the function $u(x, y)$:

Lemma 4.5. (See Lemma 5.8, p. 443.) *Let a sequence $\{u_n(x, y)\}$ of very weak biharmonic functions converge in $L_2(G)$ to a very weak biharmonic function $u(x, y)$. Then the sequence of the corresponding "singular parts"*

$$v_n(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ijn} r_{ij}(x, y) \tag{4.19}$$

converges in $L_2(G)$ to the corresponding "singular part"

$$v(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij} r_{ij}(x, y) \tag{4.20}$$

of the function $u(x, y)$.

Using Lemmas 4.1–4.3 we are able to prove the fundamental lemma of this chapter:

Lemma 4.6 (*on density*). *Let the functions $(g_{i0}, g_{i1}) \in W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$, $i = 0, 1, \ldots, k$ be given on the boundary Γ of G.*[2]*) Then to every $\varepsilon > 0$ there exists such a weak biharmonic function $\tilde{u}_{st}$ of the form* (4.6) (i.e. there exist such positive integers s, t and such real constants $\tilde{a}_{stp}$, $\tilde{b}_{stiq}$, $\tilde{c}_{sti}$ and $\tilde{\alpha}_{stij}$) *that*

$$\|\tilde{u}_{st} - g_{i0}\|_{W_2^{(1)}(\Gamma_i)} < \varepsilon\,, \quad \left\|\frac{\partial \tilde{u}_{st}}{\partial v} - g_{i1}\right\|_{L_2(\Gamma_i)} < \varepsilon \tag{4.21}$$

for all $i = 0, 1, \ldots, k$.

Proof: Denote

$$\eta = \frac{\varepsilon}{3} > 0\,. \tag{4.22}$$

The traces $(u, \partial u/\partial v)$ of functions $u(x, y)$ from the space $W_2^{(2)}(G)$ are dense in $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$ (see [2], Lemma 5.4.4). Consequently, to the given functions g_{i0}, g_{i1} and to this η it is possible to find such a function $z \in W_2^{(2)}(G)$ that

$$\|z - g_{i0}\|_{W_2^{(1)}(\Gamma_i)} < \eta\,, \quad \left\|\frac{\partial z}{\partial v} - g_{i1}\right\|_{L_2(\Gamma_i)} < \eta \quad \text{for all } i = 0, 1, \ldots, k\,. \tag{4.23}$$

Let $u_0 \in W_2^{(2)}(G)$ be the (unique) weak biharmonic function satisfying the conditions

$$u_0 = z\,, \quad \frac{\partial u_0}{\partial v} = \frac{\partial z}{\partial v} \quad \text{on} \quad \Gamma_i\,, \quad i = 0, 1, \ldots, k\,. \tag{4.24}$$

By theorem 2.1, p. 373, the function $u_0(x, y)$ can be uniquely written in the form

$$u_0(x, y) = U_0(x, y) + v_0(x, y)\,, \tag{4.25}$$

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[2]) Boundedness of G and a Lipschitzian boundary (see Convention 1.1) are always assumed.

where $U_0(x, y)$ is an Airy function and

$$(4.26) \qquad v_0(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{0ij}\, r_{ij}(x, y)$$

is the corresponding "singular part" of $u_0(x, y)$.

$U_0(x, y)$ being a weak Airy function, it follows from Lemma 4.1 that there exists such an Airy function $\tilde{U}(x, y)$ defined in a region $\tilde{G} \supset \bar{G}$ that we have

$$(4.27) \quad \|U_0 - \tilde{U}\|_{W_2^{(1)}(\Gamma_i)} < \eta\,, \quad \left\|\frac{\partial U_0}{\partial v} - \frac{\partial \tilde{U}}{\partial v}\right\|_{L_2(\Gamma_i)} < \eta \quad \text{for all} \quad i = 0, 1, \ldots, k\,.$$

According to Lemma 4.2, this function can be expressed in $\tilde{G}$ in the form

$$(4.28) \qquad \tilde{U}(x, y) = \operatorname{Re}(\bar{z}\varphi + \chi)\,,$$

where $\varphi(z)$ is a holomorphic function in $\tilde{G}$ and $\chi(z)$ is of the form

$$(4.29) \qquad \chi(z) = \chi_0(z) + \sum_{i=1}^{k} c_i \ln(z - z_i)$$

with $\chi_0(z)$ holomorphic in $\tilde{G}$ and c_i real. (Concerning z_i see the text preceding Lemma 4.1.) According to Lemma 4.3, the functions $\varphi(z)$ and $\chi_0(z)$ and their derivatives can be approximated with an arbitrary accuracy (in the sense of (4.16)) by polynomials and simple rational functions with poles at the points z_i, and by their corresponding derivatives. More precisely, let $\delta > 0$ be chosen. Then it is possible to find such positive integers s, t and such constants A_{sj}, C_{sj}, B_{itj}, D_{itj} that the functions

$$(4.30) \quad \varphi_{st} = \sum_{j=0}^{s-1} A_{sj} z^j + \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{B_{itj}}{(z - z_i)^j}\,, \quad \chi_{0st} = \sum_{j=0}^{s} C_{sj} z^j + \sum_{i=1}^{k} \sum_{j=1}^{t} \frac{D_{itj}}{(z - z_i)}$$

satisfy in $\bar{G}$

$$(4.31) \qquad |\varphi(z) - \varphi_{st}(z)| < \delta\,, \quad |\chi_0(z) - \chi_{0st}(z)| < \delta$$

and, simultaneously,

$$(4.32) \qquad |\varphi'(z) - \varphi'_{st}(z)| < \delta, \quad |\chi'_0(z) - \chi'_{0st}(z)| < \delta\,,$$

$$(4.33) \qquad |\varphi''(z) - \varphi''_{st}(z)| < \delta, \quad |\chi''_0(z) - \chi''_{0st}(z)| < \delta\,.$$

If we substitute $\varphi_{st}(z)$ and $\chi_{0st}(z) + \sum_{i=1}^{k} c_i \ln(z - z_i)$ into (4.28) for $\varphi(z)$ and $\chi(z)$, we get an approximation $\tilde{U}_{st}(x, y)$ of $\tilde{U}(x, y)$ in the form

$$(4.34) \qquad \tilde{U}_{st}(x, y) = P_s(x, y) + Q_t(x, y) + \sum_{i=1}^{k} d_i \ln[(x - x_i)^2 + (y - y_i)^2]\,,$$

where $d_i = c_i/2$,[3]) $Q_t(x, y)$ is a linear combination of biharmonic*) functions of the form (4.10), p. 426, and $P_s(x, y)$ is a biharmonic polynomial of order s (or possibly $< s$) which thus can be expressed as a linear combination of the basic biharmonic polynomials $z_p(x, y)$ of the order $\leqq s$. Consequently, $\tilde{U}_{st}(x, y)$ is of the form

$$(4.35) \quad \tilde{U}_{st}(x, y) = \tilde{V}_{st}(x, y) + \tilde{W}_{st}(x, y) + \sum_{i=1}^{k} \tilde{c}_i \ln\left[(x - x_i)^2 + (y - y_i)^2\right]$$

where

$$\tilde{V}_{st}(x, y) = \sum_{p=1}^{4s-2} \tilde{a}_{stp}\, z_p(x, y),$$

$$\tilde{W}_{st}(x, y) = \sum_{i=1}^{k} \sum_{q=1}^{4t} \tilde{b}_{stiq}\, v_{iq}(x, y).$$[4])

Obviously $\tilde{U}_{st} \in W_2^{(2)}(G)$ and so is the function $\tilde{U}$, being a weak biharmonic function in $\tilde{G}$.[5]) At the same time, the derivatives of the functions $\tilde{U}(x, y)$, $\tilde{U}_{st}(x, y)$ up to the second order (which are required when computing the norm in $W_2^{(2)}(G)$) are constructed from the derivatives of the logarithmic functions appearing in (4.35) and from the derivatives, up to the second order, of the functions $\varphi(z)$, $\chi_0(z)$ and $\varphi_{st}(z)$, $\chi_{0st}(z)$, respectively, as is seen from (4.28).

The estimates (4.31)–(4.33) imply that the norm

$$(4.36) \qquad \|\tilde{U} - \tilde{U}_{st}\|_{W_2^{(2)}(G)}$$

can be made arbitrarily small if δ has been chosen sufficiently small. Moreover, the operator of traces from $W_2^{(2)}(G)$ into $W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$ is continuous. Consequently, if (4.36) is "small", then

$$\|\tilde{U} - \tilde{U}_{st}\|_{W_2^{(1)}(\Gamma_i)}, \quad \left\|\frac{\partial \tilde{U}}{\partial \nu} - \frac{\partial \tilde{U}_{st}}{\partial \nu}\right\|_{L_2(\Gamma_i)}, \quad i = 0, 1, \ldots, k$$

is "small" as well.

Summarizing, we have: To the function $\tilde{U}(x, y)$ and to the given $\eta > 0$ it is possible to find such a function (4.35) (with s and t sufficiently large) that

$$(4.37) \quad \|\tilde{U} - \tilde{U}_{st}\|_{W_2^{(1)}(\Gamma_i)} < \eta, \quad \left\|\frac{\partial \tilde{U}}{\partial \nu} - \frac{\partial \tilde{U}_{st}}{\partial \nu}\right\|_{L_2(\Gamma_i)} < \eta \quad \text{for all} \quad i = 0, 1, \ldots, k.$$

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[3]) Because $\ln[(x - x_i)^2 + (y - y_i)^2] = \ln r_i^2 = 2 \ln r_i$, where $r_i = |z - z_i|$.

*) The functions $\varphi_{st}(z)$, $\chi_{0st}(z)$ being holomorphic, the function $\mathrm{Re}\,(\bar{z}\varphi_{st} + \chi_{0st})$ is biharmonic.

[4]) Note that $\tilde{U}_{st}(x, y)$ is of the form (4.7); however, the coefficients $\tilde{a}_{stp}$, $\tilde{b}_{stiq}$, $\tilde{c}_i$ are generally different from the coefficients a_{stp}, b_{stiq}, c_{sti} determined by the condition (3.13) of the method of least squares on the boundary.

[5]) Both these functions belong even to $W_2^{(2)}(\tilde{G})$, but this fact is of no use for us here.

Putting $\tilde{u}_{st}(x, y) = \tilde{U}_{st}(x, y) + v_0(x, y)$ (see (4.26)), the relations (4.23), (4.24), (4.25), (4.27) and (4.37) yield (4.21).

Now, it easy to prove the main convergence theorem:

Theorem 4.1 *Let* $(g_{i0}, g_{i1}) \in W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$, $i = 0, 1, \ldots, k$. *Let* $u(x, y)$ *be the very weak solution of the problem*

(4.38) $$\Delta^2 u = 0 \quad in \quad G,$$

(4.39) $$u = g_{i0}, \frac{\partial u}{\partial v} = g_{i1} \quad \text{on} \quad \Gamma_i.$$

Then the functions (4.6) *constructed by the method of least squares on the boundary satisfy*

(4.40) $$\lim_{\substack{s \to \infty \\ t \to \infty}} u_{st}(x, y) = u(x, y) \quad \text{in} \quad L_2(G).$$

Proof. We have to prove that to every $\varepsilon > 0$ there exist such s_0 and t_0 that for every positive integers $s > s_0$, $t > t_0$ it holds

(4.41) $$\|u - u_{st}\|_{L_2(G)} < \varepsilon.$$

Let ε_n be a decreasing sequence of positive numbers, $\lim \varepsilon_n = 0$ for $n \to \infty$. According to Lemma 4.6, to each of these ε_n there exist such positive integers s_n, t_n and (real) constants $\tilde{a}_{s_n t_n p}$, $\tilde{b}_{s_n t_n iq}$, $\tilde{c}_{s_n t_n i}$, $\tilde{\alpha}_{s_n t_n ij}$ that

(4.42) $$\|\tilde{u}_{s_n t_n -} - g_{i0}\|_{W_2^{(1)}(\Gamma_i)} < \varepsilon_n, \left\|\frac{\partial \tilde{u}_{s_n t_n}}{\partial v} - g_{i1}\right\|_{L_2(\Gamma_i)} < \varepsilon_n \text{ for all } i = 0, 1, \ldots, k,$$

where $\tilde{u}_{s_n t_n}$ is the function (4.6) with a_{stp} replaced by $\hat{a}_{s_n t_n p}$, etc. From (4.42) it follows

(4.43) $$\sum_{i=0}^{k} \|\tilde{u}_{s_n t_n} - g_{i0}\|^2_{W_2^{(1)}(\Gamma_i)} + \sum_{i=0}^{k} \left\|\frac{\partial \tilde{u}_{s_n t_n}}{\partial v} - g_{i1}\right\|^2_{L_2(\Gamma_i)} < 2(k + 1)\,\varepsilon_n^2.$$

But s_n, t_n being found, the inequality (4.43) will the more hold for the function $u_{s_n t_n}(x, y)$ with the coefficients $a_{s_n t_n p}, \ldots, \alpha_{s_n t_n ij}$ determined by the condition (3.13), p. 377.[6]) Thus a subsequence $\{u_{s_n t_n}(x, y)\}$ from the (double) sequence of functions (4.6) can be found, converging in $W_2^{(1)}(\Gamma)_i \times L_2(\Gamma_i)$ to the given functions g_{i0}, g_{i1}:

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[6])Note that the condition (3.13) can be written in the form

$$\left[\sum_{i=0}^{k} \|u_{st} - g_{i0}\|^2_{W_2^{(1)}(\Gamma_i)} + \sum_{i=0}^{k} \left\|\frac{\partial u_{st}}{\partial v} - g_{i1}\right\|^2_{L_2(\Gamma_i)}\right] = \min.,$$

which implies immediately the assertion just mentioned.

$$\lim_{n\to\infty} u_{s_n t_n} = g_{i0} \quad \text{in} \quad W_2^{(1)}(\Gamma_i),$$

$$\lim_{n\to\infty} \frac{\partial u_{s_n t_n}}{\partial \nu} = g_{i1} \quad \text{in} \quad L_2(\Gamma_i),$$

$i = 0, 1, \ldots, k$. We assert that the whole double sequence $\{u_{st}(x, y)\}$ converges in $W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$ to g_{i0}, g_{i1}, more precisely that to every $\gamma > 0$ there exist numbers S and T such that if $s > S$, $t > T$, then

$$\|u_{st} - g_{i0}\|_{W_2^{(1)}(\Gamma_i)} < \gamma, \quad \left\|\frac{\partial u_{st}}{\partial \nu} - g_{i1}\right\|_{L_2(\Gamma_i)} < \gamma. \tag{4.44}$$

In fact, if (4.43) is fulfilled for s_n, t_n, then the more it is fulfilled for every couple $s \geqq s_n$, $t \geqq t_n$, because the coefficients in $u_{st}(x, y)$ are determined by the method of least squares on the boundary, and consequently, the approximation in the sense of (4.43) by biharmonic polynomials or rational functions of higher orders can be only better. Thus we have

$$\lim_{\substack{s\to\infty\\ t\to\infty}} u_{st} = g_{i0} \quad \text{in} \quad W_2^{(1)}(\Gamma_i)$$

and

$$\lim_{\substack{s\to\infty\\ t\to\infty}} \frac{\partial u_{st}}{\partial \nu} = g_{i1} \quad \text{in} \quad L_2(\Gamma_i)$$

(in the sense of (4.44)). Finally, $u(x, y)$ being the very weak solution of (4.38), (4.39), it follows immediately that

$$\lim_{\substack{s\to\infty\\ t\to\infty}} u_{st}(x, y) = u(x, y) \quad \text{in} \quad L_2(G) \tag{4.45}$$

(in the sense of (4.41)) which completes the proof.

Remark 4.1. Lemma 4.5 implies that the "singular parts" of the functions (4.6), i.e. the functions

$$v_{st}(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{stij}\, r_{ij}(x, y)$$

converge in $L_2(G)$ for $s \to \infty$, $t \to \infty$ to the "singular part"

$$v(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\, r_{ij}(x, y)$$

of the function $u(x, y)$. Consequently, the "Airy parts" $U_{st}(x, y)$ of the functions $u_{st}(x, y)$ converge in $L_2(G)$ to the "Airy part" $U(x, y)$ of the function $u(x, y)$. According to Lemma 4.4, this convergence (and the convergence of the derivatives) is uniform on every closed region $\bar{G}' \subset G$. Hence we have

Theorem 4.2. *For $s \to \infty$, $t \to \infty$ the "Airy parts" $U_{st}(x, y)$ of the functions* (4.6) *constructed by the method of least squares on the boundary converge* in $L_2(G)$ (in the sense of (4.41)) *to the "Airy part" $U(x, y)$ of the very weak solution of the problem* (4.38), (4.39), *thus to the very weak Airy function corresponding to the given loading. Moreover, this convergence is uniform on every closed region $\bar{G}' \subset G$. The same assertion holds for the convergence of the sequence of $D^j U_{st}(x, y)$ on $\bar{G}'$, where $D^j U_{st}(x, y)$ means an arbitrary* (partial) *derivative of $U_{st}(x, y)$, to the corresponding derivative $D^j U(x, y)$ of $U(x, y)$.*

Remark 4.2. Theorems 4.1 and 4.2 imply Theorem 3.2, p. 379.

Remark 4.3. Theorem 4.2 implies that although the basic singular biharmonic functions $r_{ij}(x, y)$ play a fundamental role in our theoretical considerations, they actually need not be constructed since they play only an auxiliary role in our method, as mentioned in Chap. 2.

Chapter 5. SOME AUXILIARY RESULTS. PROOFS OF SOME THEOREMS AND LEMMAS USED IN THE PRECEDING CHAPTERS

1. *Smoothness of very weak solutions in G*

We start with two lemmas, the first of which follows immediately from Theorem 5.4.2 in [2], p. 274, the second being a consequence of Theorem 4.1.3 (with $A = \Delta^2$ and $\varkappa = 1$) in [2], p. 200, and of the well-known Sobolev immersion theorems. (Convention 1.1 on boundedness of G and on the Lipschitzian boundary is always preserved.)

Lemma 5.1. *Let $u(x, y)$ be the very weak solution of the biharmonic problem*

$$\Delta^2 u = 0 \quad \text{in} \quad G, \tag{5.1}$$

$$u = g_{i0}, \quad \frac{\partial u}{\partial v} = g_{i1} \quad \text{on} \quad \Gamma_i, \quad i = 0, 1, \ldots, k, \tag{5.2}$$

with $(g_{i0}, g_{i1}) \in W_2^{(1)}(\Gamma_i) \times L_2(\Gamma_i)$. Then there exists such a constant $c_1 > 0$ depending only on G (and independent of g_{i0}, g_{i1}) *that*

$$\|u\|_{L_2(G)} \leqq c_1\Big(\sum_{i=0}^{k} \|g_{i0}\|_{W_2^{(1)}(\Gamma_i)} + \sum_{i=0}^{k} \|g_{i1}\|_{L_2(\Gamma_i)}\Big). \tag{5.3}$$

Lemma 5.2. *Let $u(x, y)$ be the very weak solution of* (5.1), (5.2). *Then $u(x, y)$ has in G derivatives of all orders. To every subregion $G' \subset \bar{G}' \subset G$ there exists a constant $c_2(G') > 0$ such that we have*

$$\|u\|_{C(G')} \leqq c_2 \|u\|_{L_2(G)}, \tag{5.4}$$

where $\|u\|_{C(G')}$ means the norm in the space $C(G')$ of continuous functions in $\bar{G}'$.

More generally, there exists a constant $c_3(G', j) > 0$ such that every partial derivative $D^j u$ satisfies

$$\|D^j u\|_{C(G')} \leqq c_3 \|u\|_{L_2(G)}. \tag{5.5}$$

From Lemmas 5.1 and 5.2 we conclude

Lemma 5.3. (G' is the subregion of G from Lemma 5.2.) *To every D^j there exists such a constant $c_4(G', j) > 0$ that*

$$\|D^j u\|_{C(G')} \leqq c_4\Big(\sum_{i=0}^{k} \|g_{i0}\|_{W_2^{(1)}(\Gamma_i)} + \sum_{i=0}^{k} \|g_{i1}\|_{L_2(\Gamma_i)}\Big). \tag{5.6}$$

Remark 5.1. Consequently, in a fixed subregion G' the derivative $D^j u(x, y)$ of the very weak solution $u(x, y)$ is "small" provided the boundary functions are "small" (in the sense of (5.6)). Or, because of the linearity of the problem: The derivative $D^j v(x, y)$ of the difference $v(x, y)$ of two very weak solutions of (5.1), (5.2) is "small" provided the differences between the corresponding boundary functions are "small".

Remark 5.2. Similarly as in Chap. 4 we shall often speak, in what follows, of a *weak* or *very weak biharmonic function* instead of a *weak* or *very weak solution of a biharmonic problem* (5.1), (5.2). In a similar sense we shall speak of a *weak* or *very weak Airy function*.

Lemma 5.4. *Let a sequence of very weak biharmonic functions $u_n(x, y)$ converge in $L_2(G)$ to a very weak biharmonic function $u(x, y)$. Then this convergence is uniform on every subregion $G' \subset \bar{G}' \subset G$. Moreover, the sequence of $D^j u_n(x, y)$, where $D^j u_n(x, y)$ means an arbitrary* (partial) *derivative of $u_n(x, y)$ converges to the corresponding derivative $D^j u(x, y)$ of $u(x, y)$ uniformly in G'.*

The proof is very easy: The first assertion follows from (5.4) because

$$\|u - u_n\|_{C(G')} \leqq c_2 \|u - u_n\|_{L_2(G)}. \tag{5.7}$$

The second follows analogously from (5.5).

2. *The complex stress-functions*

In Chap. 1 (Lemma 1.1, p. 352) we introduced the concept of an Airy function: To sufficiently smooth functions

$$\sigma_x(x, y)\,, \quad \sigma_y(x, y)\,, \quad \tau_{xy}(x, y) \tag{5.8}$$

satisfying the relations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0\,, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0\,, \tag{5.9}$$

$$\Delta(\sigma_x + \sigma_y) = 0 \tag{5.10}$$

(the so-called equations of equilibrium and compatibility) in a simply connected region G, there exists a biharmonic function $u(x, y)$ in G such that

$$\sigma_x = \frac{\partial^2 u}{\partial y^2}\,, \quad \sigma_y = \frac{\partial^2 u}{\partial x^2}\,, \quad \tau_{xy} = -\frac{\partial^2 u}{\partial x\, \partial y}\,. \tag{5.11}$$

On the other hand, if $u(x, y)$ is a biharmonic function in G, then the function (5.11) satisfy (5.9), (5.10).

In Chap. 2 (Lemma 2.1) we mentioned that in a simply connected region G every biharmonic function $u(x, y)$ can be expressed in the form

$$u(x, y) = \operatorname{Re}(\bar{z}\, \varphi(z) + \chi(z))\,, \tag{5.12}$$

where

$$\varphi(z)\,, \quad \chi(z) \tag{5.13}$$

are holomorphic functions in G, the so-called stress-functions. It follows that if the functions (5.8) satisfying (5.9), (5.10) are given, then by means of the Airy function and (5.12) a pair of stress-functions (5.13) can be found. From Chap. 2 we know that these functions are uniquely determined by the functions (5.8) up to some linear expressions in z, and that between the functions (5.8) and (5.12) the following relations hold:

$$\sigma_x + \sigma_y = 4 \operatorname{Re}(\varphi')\,, \tag{5.14}$$

$$\sigma_y - \sigma_x + 2\mathrm{i}\tau_{xy} = 2(\bar{z}\varphi'' + \chi'')\,. \tag{5.15}$$

Moreover, if the functions (5.8) are interpreted as components of a stress-tensor in G, then the components $d_1(x, y)$, $d_2(x, y)$ of the corresponding vector of diplacement satisfy

$$d_1 + \mathrm{i}d_2 = \tfrac{1}{2}\mu(\varkappa\varphi - z\bar{\varphi}' - \bar{\chi}')\,, \tag{5.16}$$

where μ and $\varkappa$ are positive constants (given by the material considered). The functions

(5.8) being given, the vector of displacement is uniquely determined up to a linear expression in z which can be interpreted as a "small" displacement and rotation of G as of a rigid body.

If G is multiply connected, then to the given (sufficiently smooth) functions (5.8) satisfying (5.9), (5.10) it is also possible to construct the corresponding Airy function and the stress-functions (5.13) such that the relations (5.11), (5.12), (5.14), (5.15) and (5.16) hold. In contrast to the former case, neither the Airy function nor the stress-functions need be single-valued functions in G. As stated in Chap. 2, we do not introduce the concept of a multi-valued real function and of its derivatives here, so that we shall speak of an Airy function corresponding to the functions (5.8) only if it is a single-valued function.[1] According to Lemma 2.2., p. 361, in the case of a $(k+1)-$tuply connected region considered in Chap. 2 the stress-functions $\varphi(z)$, $\psi(z) = \chi'(z)$ can be written in the form

$$\varphi(z) = z \sum_{i=1}^{k} A_i \ln(z - z_i) + \sum_{i=1}^{k} B_i \ln(z - z_i) + \varphi_0(z)\,, \tag{5.17}$$

$$\psi(z) = \sum_{i=1}^{k} C_i \ln(z - z_i) + \psi_0(z)\,, \tag{5.18}$$

where $z_i\,(i = 1, \ldots, k)$ are fixed points chosen inside the inner boundary curves Γ_i, A_i are real constants uniquely determined by the functions $\varphi(z)$, $\psi(z)$, B_i, C_i are complex constants depending generally also on the choice of the points z_i, and $\varphi_0(z)$, $\psi_0(z)$ are holomorphic functions in G.

Remark 5.3. For the proof of Lemma 2.2, the reader has been referred to the book [4]. However, this proof enables us to draw some useful consequences. Therefore we sketch it briefly here. Following [4], Sec. 2.10, we shall assume that the boundary

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_k$$

is sufficiently smooth (there is no need to make this concept more precise here) and that the functions (5.8) have continuous partial derivatives up to the second order in the closed region $\bar{G}$. In Remark 5.4 we show how to remove these assumptions.

Let Γ be sufficiently smooth and let the functions (5.8) with continuous partial derivatives up to the second order in $\bar{G}$ satisfy (5.9), (5.10). Denote

$$\tfrac{1}{4}(\sigma_x + \sigma_y) = h(x, y).\text{[2]} \tag{5.19}$$

[1]) At the same time, the functions (5.13) need not be single-valued, as shown in Remark 2.4, p. 362.

[2]) Thus $h(x, y) = \frac{1}{4}\,\Delta u$ if the functions (5.8) are derived from the function $u(x, y)$ according to (5.11).

Then according to (5.10) we have

$$\Delta h = 0 \tag{5.20}$$

and the expression

$$-\frac{\partial h}{\partial y}\,\mathrm{d}x + \frac{\partial h}{\partial x}\,\mathrm{d}y \tag{5.21}$$

is locally a total differential in G. Denote

$$A_i = \tfrac{1}{2}\pi \int_{\Gamma_i} \left(\frac{\partial h}{\partial y}\,\mathrm{d}x - \frac{\partial h}{\partial x}\,\mathrm{d}y\right). \tag{5.22}$$

An easy computation (for details see again [4]) shows that then the function $h^*(x, y)$ defined by

$$h^*(x, y) = \int_{z_0}^{z} \left[\left(-\frac{\partial h}{\partial y}\,\mathrm{d}x + \frac{\partial h}{\partial x}\,\mathrm{d}y\right) + \right. \tag{5.23}$$

$$\left. + \sum_{i=1}^{k} A_i \left(\frac{\partial \operatorname{Re}(\ln(z - z_i))}{\partial y}\,\mathrm{d}x - \frac{\partial \operatorname{Re}(\ln(z - z_i))}{\partial x}\,\mathrm{d}y\right)\right],$$

where $z_0 = x_0 + \mathrm{i}y_0$ is an arbitrary (but fixed) point in G, is a single-valued function in G and that the function

$$\Phi_0(z) = h(x, y) - \sum_{i=1}^{k} A_i - \operatorname{Re}\left(\sum_{i=1}^{k} A_i \ln(z - z_i)\right) + \mathrm{i}h^*(x, y)$$

is holomorphic in G and continuous in $\bar{G}$. Put

$$\varphi'(z) = \Phi_0(z) + \sum_{i=1}^{k} A_i \ln(z - z_i) + \sum_{i=1}^{k} A_i\,. \tag{5.24}$$

Construct the function

$$\Phi^*(z) = \Phi_0(z) + \sum_{i=1}^{k} \frac{k_i}{z - z_i}$$

choosing the complex constants k_i in such a way that

$$\int_{\Gamma_i} \Phi^*(z)\,\mathrm{d}z = 0\,, \quad i = 1, \ldots, k\,. \tag{5.25}$$

This is possible, even uniquely, because

$$\int_{\Gamma_j} \sum_{i=1}^{k} \frac{k_i}{z - z_i}\,\mathrm{d}z = -2\pi\,\mathrm{i}k_j\,, \quad j = 1, \ldots, k\,. \tag{5.26}$$

If we denote

$$\varphi_0(z) = \int_{z_0}^{z} \Phi^*(t)\,\mathrm{d}t\,,$$

then (5.17) is a primitive function to the function (5.24). Here

$$B_i = -k_i - A_i z_i\,. \tag{5.27}$$

In a similar way one constructs the function (5.18) and checks the validity of the relations (5.14), (5.15).

Corollary 5.1 to Lemma 2.2. If, moreover, the components d_1, d_2 of the vector of displacement (5.16) corresponding to the functions (5.8) are assumed to be single-valued functions in G, then

$$A_i = 0 \tag{5.28}$$

and

$$B_i = -\frac{X_i + \mathrm{i}Y_i}{2\pi(1 + \varkappa)}, \quad C_i = \frac{\varkappa(X_i - \mathrm{i}Y_i)}{2\pi(1 + \varkappa)}, \tag{5.29}$$

$i = 1, \ldots, k$, where

$$X_i = \int_{\Gamma_i} X(s)\,\mathrm{d}s\,, \quad Y_i = \int_{\Gamma_i} Y(s)\,\mathrm{d}s \tag{5.30}$$ [3])

and $\varkappa$ is the constant from (5.16).

For the proof of this corollary see [4], Sec. 2.10. It consists in a more detailed analysis of the expression (2.43), p. 370.

Remark 5.4. The assumption concerning the smoothness of the boundary Γ when deriving the form of the functions (5.17), (5.18) as well as the assumption of smoothness of the functions (5.8) up to the boundary can be easily removed. Let G' be a $(k + 1)$–tuply connected region such that $G' \subset \overline{G}' \subset G$ with a smooth boundary

$$\Gamma' = \Gamma'_0 \cup \Gamma'_1 \cup \ldots \cup \Gamma'_k$$

as considered in Chap. 2 (see Fig. 2, p. 370). Each of the points z_i is assumed to lie inside the inner boundary curve Γ'_i, $i = 1, \ldots, k$.

If we assume that the functions (5.8) have continuous derivatives up to the second order in the (open) region G only, then these derivatives are continuous in $\overline{G}'$ and we can carry out all the preceding considerations for the region G'. All the results will be independent of the choice of the region G': In fact, if G'' is another region with the same properties as G', then the integrals appearing in the preceding text will be the same along a curve Γ'_i or Γ''_i, because they are either integrals of expressions which are locally total differentials or of holomorphic functions. In particular, for the constants A_i, B_i, C_i we get always the same values. The intervals over Γ_i in (5.30) are to be replaced by integrals over Γ'_i.

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[3]) Thus X_i, Y_i are the x- and y-components of the total loading which acts on Γ_i (of the so-called main vector on Γ_i).

The simple considerations just performed lead to useful consequences. Note first that if the displacement is a single-valued function and if the main vectors on Γ'_i $(i = 1, \ldots, k)$ are equal to zero, then (5.17), (5.18) and (5.30) (with Γ'_i substituted for Γ_i) imply

$$\varphi(z) = \varphi_0(z)\,, \quad \psi(z) = \chi'(z) = \psi_0(z)\,, \tag{5.31}$$

where φ_0, ψ_0 are holomorphic functions in G.

This case occurs for example if the function $u(x, y)$ connected with the functions (5.8) by (5.11), is an Airy function in G: In fact, according to the definition, an Airy function is a single-valued biharmonic function such that the vector of displacement is a single-valued function. The function $u(x, y)$ being single-valued, so are its derivatives $\partial u/\partial x$ and $\partial u/\partial y$, so that the main vector is equal to zero on every Γ'_i. Thus we have (5.31). Moreover, we assert that in this case the primitive function to $\chi'(z)$ is of the form

$$\chi(z) = \chi_0(z) + \sum_{i=1}^{k} c_i \ln (z - z_i)\,, \tag{5.32}$$

where $\chi_0(z)$ is a holomorphic function in G and c_i are *real* constants. In fact, the form (5.32) of the function $\chi(z)$ with c_i generally complex can be derived in a quite similar way as we have obtained the primitive function

$$\int_{z_0}^{z} \Phi^* \,\mathrm{d}z - \sum_{i=1}^{k} k_i \ln (z - z_i)$$

to the function $\Phi_0(z)$, i.e. by virtue of (5.25), (5.26). But

$$\ln (z - z_i) = \tfrac{1}{2} \ln \left[(x - x_i)^2 + (y - y_i)^2\right] + \mathrm{i}\omega_i\,, \tag{5.33}$$

where ω_i stands for the amplitude of the logarithm. The functions $\varphi_0(z)$, $\chi_0(z)$ being holomorphic, it follows from (5.12), i.e. from

$$u = \operatorname{Re}(\bar{z}\varphi + \chi) \tag{5.34}$$

and from (5.32), (5.33) that the imaginary parts of all the coefficients c_i should be equal to zero in order that $u(x, y)$ be a single-valued function.

Thus *if $u(x, y)$ is an Airy function in G, then the stress functions $\varphi(z)$, $\chi(z)$ are of the form*

$$\varphi(z) = \varphi_0(z)\,, \quad \chi(z) = \chi_0(z) + \sum_{i=1}^{k} c_i \ln (z - z_i)\,, \tag{5.35}$$

where φ_0, χ_0 are holomorphic functions in G and c_i, $i = 1, \ldots, k$ are real constants.

This is the assertion of Lemma 4.2, p. 428.

Now, let $u(x, y)$ be a very weak biharmonic function in G (see Remark 5.2, p. 436). Let (5.17), (5.18) be the corresponding stress-functions so that (5.34) holds. From

Lemma 5.2 and Remarks 5.3, 5.4 it is possible to draw simple conclusions on the dependence of the numbers A_i, B_i, C_i on $\|u\|_{L_2(G)}$[4]). We shall show, roughly speaking, that these numbers are "small" if $\|u\|_{L_2(G)}$ is "small". To this purpose we use construction presented in Remark 5.3, but carried out on a subregion G' according to Remark 5.4, and the fact that $\bar{G}' \subset G$ implies that the derivatives of the function $u(x, y)$ in $\bar{G}'$ are "small" if $\|u\|_{L_2(G)}$ is "small" (Lemma 5.2):

Thus, let G' be a fixed region from Remark 5.4. According to this remark, we can perform all the considerations from Remark 5.3 for this region. Because $G' \subset G$, the function $h(x, y) = \frac{1}{4}\Delta u$ (cf. the footnote 2, p. 438) is small in G' in the sense of (5.5) from Lemma 5.2, if $\|u\|_{L_2(G)}$ is small, and so are its derivatives $\partial h/\partial x$, $\partial h/\partial y$ in G'. According to (5.22) all the numbers A_i $(i = 1, \ldots, k)$ are small because they are integrals from these derivatives along the curves Γ_i' which are fixed, as G' has been chosen fixed. By the same reasoning, the function $h^*(x, y)$ is small and, consequently, so are the numbers k_i in the function $\Phi^*(z)$, because according to (5.25), (5.26) they are proportional to the integrals of the function $\Phi_0(z)$. (5.27) then gives that also the numbers B_i, $i = 1, \ldots, k$ are small. Evidently, all these relations are linear, even homogeneous.[5]) A quite similar reasoning can be carried out for the numbers C_i, $i = 1, \ldots, k$. Thus we can summarize:

Lemma 5.5. *Let $u(x, y)$ be a very weak biharmonic function in G* (cf. Remark 5.2) *and let* (5.17), (5.18) *be the corresponding stress-functions* (so that (5.12) holds) *with fixed points z_i, $i = 1, \ldots, k$. Then there exists such a constant $c > 0$ that the relations*

$$(5.36) \qquad |A_i| \leqq c\|u\|_{L_2(G)}, \quad |B_i| \leqq c\|u\|_{L_2(G)}, \quad |C_i| \leqq c\|u\|_{L_2(G)}$$

hold for all $i = 1, \ldots, k$.

In Chap. 2, p. 370 we have introduced the numbers γ_{ij}, $i = 1, \ldots, k$, $j = 1, 2, 3$ by the relations (2.45),

$$(5.37) \qquad \gamma_{i_1} = (\varkappa + 1)\, A_i\,, \quad \gamma_{i2} = \operatorname{Re}(\varkappa B_i + \bar{C}_i)\,, \quad \gamma_{i3} = \operatorname{Im}(\varkappa B_i + \bar{C}_i)\,.$$

We have immediately

Lemma 5.6. *Let $u(x, y)$ be the very weak biharmonic function from Lemma* 5.5, *γ_{ij}, $i = 1, \ldots, k$, $j = 1, 2, 3$ the numbers* (5.37). *Then there exists such a constant $C < 0$ that*

$$(5.38) \qquad |\gamma_{ij}| \leqq C\|u\|_{L_2(G)}\,.$$

[4]) The points z_i are always assumed fixed.

[5]) So that if we say, for example, that A_i is small if $\|u\|_{L_2(G)}$ is small, it means that a constant $K > 0$ exists, depending on the constant c_3 in (5.5) and on the length of the curve Γ_i', such that

$$|A_i| \leqq K\|u\|_{L_2(G)}$$

holds, etc.

The points z_i in (5.17), (5.18) being chosen fixed, every very weak biharmonic function $u(x, y)$ produces $3k$ numbers γ_{ij}, uniquely determined by this function. Lemma 5.6 implies

Lemma 5.7. *Let the sequence $\{u_n(x, y)\}$ of very weak biharmonic functions converge in $L_2(G)$ to a very weak biharmonic function $u(x, y)$. Let γ_{ijn} ($i = 1, \ldots, k$, $j = 1, 2, 3$, $n = 1, 2, \ldots$), γ_{ij} be the above mentioned numbers corresponding to the functions $u_n(x, y)$, $u(x, y)$, respectively. Then we have*

$$\lim_{n\to\infty} \gamma_{ijn} = \gamma_{ij} \quad \textit{for all} \quad i = 1, \ldots, k, \quad j = 1, 2, 3. \tag{5.39}$$

The proof follows immediately from (5.38), because

$$|\gamma_{ij} - \gamma_{ijn}| \leqq C\|u - u_n\|_{L_2(G)}$$

and $\|u - u_n\|_{L_2(G)} \to 0$ for $n \to \infty$.

Every very weak biharmonic function $u(x, y)$ can be uniquely decomposed (see Theorem 2.1 and its proof, if necessary; this decomposition is independent of the choice of the points z_i) into the "Airy part" $U(x, y)$ and the "singular part"

$$v(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\, r_{ij}(x, y)$$

with α_{ij} uniquely determined. Here, α_{ij} are the solutions of the system (2.51),

$$\sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\beta_{ijpq} = \gamma_{pq}, \quad p = 1, \ldots, k, \quad q = 1, 2, 3. \tag{5.40}$$

For the notation see Chap. 2, p. 371. The determinant D of this system is different from zero (Lemma 2.5). Consequently, the solutions of this system depend continuously on its right-hand side. This fact and Lemma 5.7 immediately imply

Lemma 5.8. *Let the sequence $\{u_n(x, y)\}$ of very weak biharmonic functions converge in $L_2(G)$ to a very weak biharmonic function $u(x, y)$. Then the sequence of the corresponding singular parts*

$$v_n(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ijn}\, r_{ij}(x, y)$$

converges in $L_2(G)$ to the singular part

$$v(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\, r_{ij}(x, y)$$

of the function $u(x, y)$.

The proof is very easy: $D \neq 0$, $\gamma_{ijn} \to \gamma_{ij}$ according to Lemma 5.7, thus $\alpha_{ijn} \to \alpha_{ij}$. The functions $r_{ij}(x, y)$ belong to the space $L_2(G)$, hence we have $v_n(x, y) \to v(x, y)$ in $L_2(G)$.[6)]

3. *Proof of Lemma* 4.1, *p.* 428.

Remark 5.5. In what follows G is a bounded $(k + 1)$-tuply connected region with a Lipschitzian boundary, $z_i(i = 1, \ldots, k)$ are as usual arbitrary points lying inside the inner boundary curves Γ_i; $\tilde{G}$ is a bounded $(k + 1)$-tuply connected region with a smooth boundary, such that $G \subset \bar{G} \subset \tilde{G}$ and that each of the points, z_i, $i = 1, \ldots, k$, lies inside the inner boundary curve Γ'_i. (Cf. Fig. 9, p. 428.)

We start with the following lemma proved in [1], p. 122[7)]

Lemma 5.9. *To every function* $z \in W_2^{(2)}(G)$ *and to every* $\eta > 0$ *it is possible to find such a function* $\hat{z}(x, y)$ *biharmonic in G that*

$$\hat{z} \in C^{(\infty)}(\bar{G}) \tag{5.41}$$

and

$$\|z - \hat{z}\|_{W_2^{(1)}(\Gamma)} < \eta\,, \quad \left\|\frac{\partial z}{\partial v} - \frac{\partial \hat{z}}{\partial v}\right\|_{L_2(\Gamma)} < \eta\,. \tag{5.42}$$

In [1] a simply connected region is treated, so that Lemma 5.9 is formulated there for this case.[8)] But in its proof this assumption is nowhere used so that this lemma holds for multiply connected regions considered in Remark 5.5. Inequalities (5.42) can then be written in the form

$$\|z - \hat{z}\|_{W_2^{(1)}(\Gamma_i)}\| < \eta\,, \quad \left\|\frac{\partial z}{\partial v} - \frac{\partial \hat{z}}{\partial v}\right\|_{L_2(\Gamma_i)} < \eta \quad \text{for all} \quad i = 0, 1, \ldots, k\,. \tag{5.43}$$

Let us give a sketch of the proof of Lemma 5.9 (for details see [1], p. 122–128), because useful conclusions can be drawn from it.

Let $\{\tilde{G}_l\}$ be a sequence of regions of the type shown in Remark 5.5 (thus every point z_i lies always inside the corresponding inner boundary curve $\tilde{\Gamma}_{li}$, $i = 1, \ldots, k$) such that

$$\bar{G} \subset \tilde{G}_l\,, \quad \bar{\tilde{G}}_{l+1} \subset \tilde{G}_l \quad \text{for every} \quad l = 1, 2, \ldots, \tag{5.44}$$

$$\lim_{l \to \infty} m(\tilde{G}_l - \bar{G}) = 0\,, \tag{5.45}$$

6) The functions $r_{ij}(x, y)$ belonging to $W_2^{(2)}(G)$ (as weak solutions of the problems (2.27) to (2.35)), we have even the convergence in $W_2^{(2)}(G)$. But this fact is of no interest here.

7) Lemma 5.9 itself is not used in the sequel. However, we need Lemma 5.10 which follows from the proof of Lemma 5.9.

8) G is bounded, Γ is Lipschitzian.

where $m(\tilde{G}_l - \bar{G})$ is the Lebesgue measure of the region $\tilde{G}_l - \bar{G}$. Thus the region G is "approximated from outside" by a sequence of regions $\tilde{G}_l$ (see Fig. 10). Such a sequence obviously exists. Let $u_0 \in W_2^{(2)}(G)$ be the weak biharmonic function satisfying $u_0 - z \in \mathring{W}_2^{(2)}(G)$ (thus satisfying, in the sense of traces, the boundary conditions

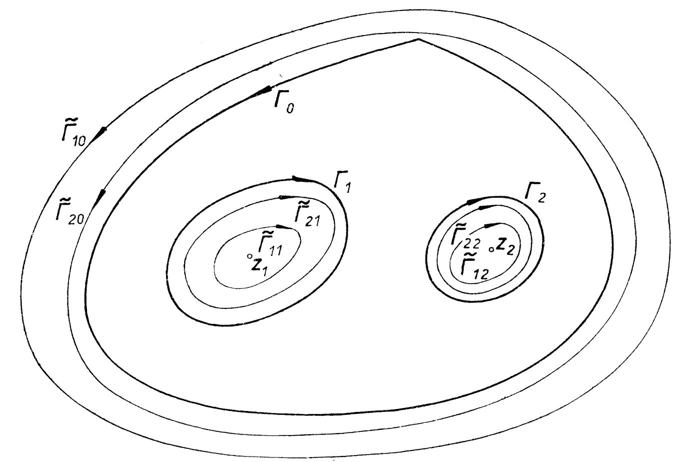

Fig. 10.

$$u_0 = z\,, \frac{\partial u_0}{\partial \nu} = \frac{\partial z}{\partial \nu} \quad \text{on} \quad \Gamma\,,$$

given by the function z). Let us extend the function u_0 in a usual way (cf. e.g. [2], p. 80) onto the whole region $\tilde{G}_1$ so that this extension – denote it by U_0 – belongs to $W_2^{(2)}(\tilde{G}_1)$. Denote by $u_l \in W_2^{(2)}(\tilde{G}_l)$ the weak biharmonic function in $\tilde{G}_l$ satisfying $u_l - U_0 \in \mathring{W}_2^{(2)}(\tilde{G}_l)$, where U_0 is considered as the restriction of U_0 on $\tilde{G}_l$. In [1] it is proved – and, as said above, the assumption that G is a simply connected region is used nowhere in the proof – that a subsequence exists – denote it, for simplicity, by $\{u_l(x, y)\}$ again – such that

$$\lim_{l \to \infty} \|u_0 - \hat{u}_l\|_{W_2^{(2)}(G)} = 0\,, \tag{5.46}$$

where $\hat{u}_l(x, y)$ is the restriction of the function $u_l(x, y)$ on G. The operator of traces from $W_2^{(2)}(G)$ into $W_2^{(1)}(\Gamma) \times L_2(\Gamma)$ being continuous, it is sufficient to take for the desired function $\hat{z}(x, y)$ the restriction $\hat{u}_l(x, y)$ of a function $u_l(x, y)$ with a sufficiently large index l (so that $\|u_0 - \hat{u}_l\|_{W_2^{(2)}(G)}$ is sufficiently small) to obtain (5.42) (or (5.43)). At the same time, $u_l(x, y)$ being biharmonic in $\tilde{G}_l$, $\hat{u}_l(x, y)$ belongs to $C^{(\infty)}(\bar{G})$, which completes the proof.

The way of proof of Lemma 5.9 permits to formulate another lemma which will be more suitable for our purpose:

Lemma 5.10. *Let $u_0(x, y)$ be a weak biharmonic function in G. Then there exists such a sequence of regions $\tilde{G}_l$ with properties* (5.44), (5.45) *and such a sequence of weak biharmonic functions $u_l(x, y)$ defined on $\tilde{G}_l$ that their restrictions $\hat{u}_l(x, y)$ on G satisfy* (5.46).

Thus, if $u_0(x, y)$ is a weak biharmonic function in G, it is possible to find such a weak biharmonic function $u_l(x, y)$ defined in a "larger" region $\tilde{G}_l$ that $\|u_0 - \hat{u}_l\|_{W_2^{(2)}(G)}$ is sufficiently small. In virtue of the continuity of the operator of traces, the numbers

$$\|u_0 - \hat{u}_l\|_{W_2^{(1)}(\Gamma_i)}, \quad \left\|\frac{\partial u_0}{\partial \nu} - \frac{\partial \hat{u}_l}{\partial \nu}\right\|_{L_2(\Gamma_i)}$$

are then alsosufficiently small. This is "almost" Lemma 4.1. However, it is not clear whether it is possible to achieve that $u_l(x, y)$ be an Airy function in $\tilde{G}_l$ provided $u_0(x, y)$ is a (weak) Airy function in G. We shall show that it is possible.

Every very weak and consequently, every weak biharmonic function $u_0(x, y)$ in G can be uniquely written in the form

$$u_0(x, y) = U(x, y) + \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\, r_{ij}(x, y), \tag{5.47}$$

where $U(x, y)$ is the "Airy part" of $u_0(x, y)$, $r_{ij}(x, y)$ are the basic singular biharmonic functions defined in Chap. 2 and α_{ij}, $i = 1. \ldots, k$, $j = 1, 2, 3$ are uniquely determined constants which are solutions of the system (2.51),

$$\sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\beta_{ijpq} = \gamma_{pq}, \quad p = 1, \ldots, k, \quad q = 1, 2, 3. \tag{5.48}$$

Here γ_{pq} are the numbers (2.47) corresponding to the function $u_0(x, y)$ (cf. also the text preceding Lemma 5.6) and β_{ijpq} are analogous numbers corresponding to the functions $r_{ij}(x, y)$ (keeping the points z_i fixed). Similatly, in $\tilde{G}_l$ $(l = 1, 2, \ldots)$ we can write

$$u_l(x, y) = U_l(x, y) + \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ijl}\, r_{ijl}(x, y), \tag{5.49}$$

where $r_{ijl}(x, y)$ are weak biharmonic functions defined (like the functions $r_{ij}(x, y)$) as solutions of the problems (2.27)–(2.35) considered on the region $\tilde{G}_l$ and α_{ijl} are solutions of the system

$$\sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ijl}\beta_{ijpql} = \gamma_{pql}, \quad p = 1, \ldots, k, \quad q = 1, 2, 3. \tag{5.50}$$

Here γ_{pql} correspond to the function $u_l(x, y)$, β_{ijpql} correspond to the functions $r_{ijl}(x, y)$.

It is not difficult to prove that

(5.51) $$\lim_{l\to\infty} \gamma_{pql} = \gamma_{pq}\,,$$

(5.52) $$\lim_{l\to\infty} \beta_{ijpql} = \beta_{ijpq}\,.$$

(5.51) follows at once from Lemmas 5.10 and 5.6. In fact, the function $\hat{u}_l(x, y)$ is the restriction of the function $u_l(x, y)$ on G so that the numbers γ_{pql} corresponding to this function are the same as those corresponding to the function $u_l(x, y)$. (5.46) and Lemma 5.6 imply (5.51).

To prove (5.52), consider any one of the functions $r_{ij}(x, y)$, say the function $r_{12}(x, y)$ – the solution of the problem (2.30)–(2.32), p. 368 for $i = 1$. Let us extend this function onto the region $\tilde{G}_1$ defining $r_{12}(x, y) = x$ outside G. In this way we get a function which corresponds in the proof of Lemma 5.9 to the function $U_0(x, y)$. However, extending this function in this way, the functions corresponding in the above proof to the functions $u_l(x, y)$ $(l = 1, 2, \ldots)$ will be precisely the functions $r_{12l}(x, y)$. Consequently,

$$\lim_{l\to\infty} \|r_{12} - \hat{r}_{12l}\|_{W_2^{(2)}(G)} = 0\,.$$

In a quite similar way we get, more generally,

(5.53) $$\lim_{l\to\infty} \|r_{ij} - \hat{r}_{ijl}\|_{W_2^{(2)}(G)} = 0 \quad \text{for all} \quad i = 1, \ldots, k\,, \quad j = 1, 2, 3$$

which implies by Lemma 5.6 the required relation (5.52).

The determinant D of the system (5.48) is different from zero (Lemma 2.5, p. 371). So is the determinant D_l of the system (5.50). By (5.52)

$$\lim_{l\to\infty} D_l = D$$

and thus by (5.53)

(5.54) $$\lim_{l\to\infty} \alpha_{ijl} = \alpha_{ij} \quad \text{for all} \quad i = 1, \ldots, k\,, \quad j = 1, 2, 3$$

(independently of the choice of the points z_i). Now (5.53) and (5.54) give

(5.55) $$\lim_{l\to\infty} \Big\| \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ij}\, r_{ij}(x, y) - \sum_{i=1}^{k} \sum_{j=1}^{3} \alpha_{ijl}\, \hat{r}_{ijl}(x, y) \Big\|_{W_2^{(2)}(G)} = 0\,.$$

From (5.47), (5.49) and (5.46), (5.55) we get finally

(5.56) $$\lim_{l\to\infty} \|U - \hat{U}_l\|_{W_2^{(2)}(G)} = 0\,.$$

Let $u_0(x, y)$ be a weak *Airy function* in G. Then in (5.47) we have $u_0(x, y) = U(x, y)$ so that

(5.57) $$\lim_{l\to\infty} \|u_0 - \hat{U}_l\|_{W_2^{(2)}(G)} = 0\,,$$

and in view of the continuity of the operator of traces from $W_2^{(2)}(G)$ into $W_2^{(1)}(\Gamma) \times$ $\times L_2(\Gamma)$

$$(5.58)\ \lim_{l\to\infty} \|u_0 - \hat{U}_l\|_{W_2^{(1)}(\Gamma_i)} = 0\,, \lim_{l\to\infty} \left\|\frac{\partial u_0}{\partial v} - \frac{\partial \hat{U}_l}{\partial v}\right\|_{L_2(\Gamma_i)} = 0\,, \text{ for all } i = 1, \ldots, k\,,$$

which yields immediately the assertion of Lemma 4.1.

4. *Proof of Lemma* 2.4, *p.* 369.

Denote

$$V(x, y) = \sum_{i=1}^{k} \sum_{j=1}^{3} a_{ij}\, r_{ij}(x, y)\,, \tag{5.59}$$

where $r_{ij}(x, y)$ are the basic singular biharmonic functions (defined as solution of (2.27)–(2.35), p. 367) and a_{ij} are (real) constants. Let us assume that $V(x, y)$ is a (weak) Airy function in G. We assert that this assumption implies

$$a_{ij} = 0 \quad \text{for all} \quad i = 1, \ldots, k\,, \quad j = 1, 2, 3\,. \tag{5.60}$$

If we prove (5.60), Lemma 2.4 will be proved completely.

Thus, let $V(x, y)$ be a weak Airy function in G. Denote for brevity

$$\Delta V = S(x, y)\,.$$

According to Remark 5.3 (oı its modification by Remark 5.4), to this function there exists such a single-valued conjugate function $T(x, y)$ that $S + iT$ is a holomorphic function in G and

$$\int_c (S + iT)\, dz = 0 \tag{5.61}$$

over every closed curve c (sufficiently smooth) lying in G. In fact, regardless of the coefficient $1/4$ which has no influence on the validity of the above assertion, these functions have been denoted by $h(x, y)$, $h^*(x, y)$ and $\Phi_0(z)$, respectively, in Remark 5.3. From the assumption that $V(x, y)$ is an Airy function it follows that $A_i = 0$, $B_i = 0$ (see (5.28), (5.29); note that V is a single-valued function so that $X_i = 0$, $Y_i = 0$) and consequently, $k_i = 0$ ($i = 1, \ldots, k$) which implies (see (5.25)) (5.61).

Since the functions r_{ij}, are solutions of problems of the form (2.27)–(2.35), the function V is of the form

$$V = a_{i1} + a_{i2}x + a_{i3}y \quad \text{on} \quad \Gamma_i\,, \quad i = 1, \ldots, k \tag{5.62}$$

and

$$V = 0 \quad \text{on} \quad \Gamma_0\,. \tag{5.63}$$

Let us construct a function $w(x, y)$ sufficiently smooth in G and such that

$$w(x, y) = a_{i1} + a_{i2}x + a_{i3}y \tag{5.64}$$

in a certain neighbourhood of the curve Γ_i $(i = 1, \ldots, k)$ belonging to G and

$$w(x, y) = 0 \tag{5.65}$$

in a certain neighbourhood (belonging to G) of Γ_0. Such a construction is evidently possible.

Further, let us construct a sequence of subregions G_n of G ($\bar{G}_n \subset G$ for every n) tending to G in the sense of Theorem 3.6.7 from [2], p. 173. It is possible to assume that the boundary $\Gamma^{(n)} = \Gamma_0^{(n)} \cup \Gamma_1^{(n)} \cup \ldots \cup \Gamma_k^{(n)}$ of each of these regions is sufficiently smooth. Let us denote by $V_n(x, y)$ the weak solution of the problem

$$\Delta^2 V_n = 0 \quad \text{in} \quad G_n \tag{5.66}$$

$$V_n - w \in \mathring{W}_2^{(2)}(G_n)\,. \tag{5.67}$$

First, extending each of the functions V_n by w on the whole G, we have according to Theorem 3.6.7 in [2]

$$\lim_{n \to \infty} V_n = V \quad \text{in} \quad W_2^{(2)}(G) \tag{5.68}$$

and, in particular,

$$\lim_{n \to \infty} \Delta V_n = \Delta V = S \quad \text{in} \quad L_2(G)\,. \tag{5.69}$$

Further, the function $V(x, y)$ is sufficiently smooth in the interior of G. Thus the Green theorem can be applied,

$$\iint_{G_n} \Delta V_n \, \Delta V \, \mathrm{d}x \, \mathrm{d}y = \int_{\Gamma^{(n)}} \left(\Delta V \frac{\partial V_n}{\partial \nu} - V_n \frac{\partial \Delta V}{\partial \nu} \right) \mathrm{d}s + \iint_{G_n} V_n \, \Delta^2 V \, \mathrm{d}x \, \mathrm{d}y\,.\text{[9]} \tag{5.70}$$

For all n larger than a certain n_0 we have

$$V_n = a_{i1} + a_{i2}x + a_{i3}y\,, \quad \frac{\partial V_n}{\partial \nu} = a_{i2}\nu_x + a_{i3}\nu_y \quad \text{on} \quad \Gamma_i^{(n)} \quad i = 1, \ldots, k\,, \tag{5.71}$$

$$V_n = 0\,, \quad \frac{\partial V_n}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0^{(n)}\,. \tag{5.72}$$

Further, for every $i = 1, \ldots, k$ we get

$$I_i = \int_{\Gamma_i^{(n)}} \left(S \frac{\partial V_n}{\partial \nu} - V_n \frac{\partial S}{\partial \nu} \right) \mathrm{d}s = \int_{\Gamma_i^{(n)}} \left(S \frac{\partial V_n}{\partial \nu} - V_n \frac{\partial T}{\partial s} \right) \mathrm{d}s\,.\text{[10]}$$

---

[9]) In order to be able to write (5.70) we have constructed the sequence of regions G_n, because sufficiently smooth boundaries $\Gamma^{(n)}$ and a sufficiently smooth function w guarantee a sufficient smoothness of the functions V_n in $\bar{G}_n$. If V were sufficiently smooth in G, we could have used (5.70) directly for G with $V_n = V$.

[10]) The Cauchy-Riemann conditions for the conjugate functions S, T imply

$$\frac{\partial S}{\partial \nu} = \frac{\partial S}{\partial x} \nu_x + \frac{\partial S}{\partial y} \nu_y = \frac{\partial T}{\partial y} \nu_x - \frac{\partial T}{\partial x} \nu_y = \frac{\partial T}{\partial s}\,.$$

Integrating the second term in the right-hand side by parts and noting that V_n and T are single-valued functions, we get

$$I_i = \int_{\Gamma_i^{(n)}} \left(S \frac{\partial V_n}{\partial \nu} + T \frac{\partial V_n}{\partial s} \right) \mathrm{d}s$$

and by (5.71)

$$I_i = a_{i1} \,.\, 0 + a_{i2} \int_{\Gamma_i^{(n)}} (S\nu_x - T\nu_y)\, \mathrm{d}s + a_{i3} \int_{\Gamma_i^{(n)}} (S\nu_y + T\nu_x)\, \mathrm{d}s =$$

$$= a_{i2} \int_{\Gamma_i^{(n)}} (S\, \mathrm{d}y + T \mathrm{d}x) - a_{i3} \int_{\Gamma_i^{(n)}} (S\, \mathrm{d}x - T \mathrm{d}y) = 0$$

in view of (5.61) written in the real form.

Consequently, the first integral on the right-hand side of (5.70) is equal to zero. So is the second, because $\Delta^2 V = 0$ in G. Thus for every $n > n_0$ it holds

$$\iint_{G_n} \Delta V_n\, \Delta V \mathrm{d}x\, \mathrm{d}y = 0$$

and by (5.69)

$$\iint_G (\Delta V)^2\, \mathrm{d}x\, \mathrm{d}y = 0\,. \tag{5.73}$$

(5.73) implies that V is harmonic in G. Using the Green theorem once more, we get[11])

$$-\iint_G V \Delta V \mathrm{d}x\, \mathrm{d}y = -\int_\Gamma V \frac{\partial V}{\partial \nu} \mathrm{d}s + \iint_G \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 \right] \mathrm{d}x\, \mathrm{d}y\,. \tag{5.74}$$

But

$$V = a_{i1} + a_{i2}x + a_{i3}y\,, \quad \frac{\partial V}{\partial \nu} = a_{i2}\, \nu_x + a_{i3}\, \nu_y \quad \text{on} \quad \Gamma_i\,, \quad i = 1, \ldots, k\,.$$

Consequently, for every $i = 1, \ldots, k$ we have

$$\int_{\Gamma_i} V \frac{\partial V}{\partial \nu} \mathrm{d}s = a_{i1} \int_{\Gamma_i} (a_{i2}\, \mathrm{d}y - a_{i3}\, \mathrm{d}x) - \tag{5.75}$$

$$- a_{i2}a_{i3} \int_{\Gamma_i} x\, \mathrm{d}x + a_{i3}a_{i2} \int_{\Gamma_i} y\, \mathrm{d}y + a_{i2}^2 \int_{\Gamma_i} x\, \mathrm{d}y - a_{i3}^2 \int_{\Gamma_i} y\, \mathrm{d}x\,.$$

---

[11]) Here it is no more necessary to consider the regions G_n, because all the symbols in (5.74) have sense.

The first three integrals on the right-hand side of (5.75) obviously vanish. For the remaining two integrals we have in view of the orientation of the curves Γ_i

$$\int_{\Gamma_i} x\,\mathrm{d}y < 0\,, \quad \int_{\Gamma_i} y\,\mathrm{d}x > 0\,.$$

Further,

$$\int_{\Gamma_0} V\frac{\partial V}{\partial \nu}\,\mathrm{d}s = 0$$

by (5.63). Hence (5.74) yields, with regard to $\Delta V = 0$,

$$\iint_G \left[\left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y = 0\,,$$

and by (5.63),

$$V(x, y) \equiv 0 \quad \text{in} \quad G\,.$$

The functions r_{ij} being linearly independent in G, it follows $a_{ij} = 0$ for all $i = 1, \ldots$ $\ldots, k$, $j = 1, 2, 3$ which we were to prove.

5. *Proof of Theorem* 3.1, p. 379.

Let s, t be fixed positive integers. Denote by M the set of all (real) linear combinations of the functions

$$(5.76) \qquad z_p(x, y)\,, \quad v_{iq}(x, y)\,, \quad \ln\left[(x - x_i)^2 (y - y_i)^2\right], \quad r_{ij}(x, y)\,,$$

$$p = 1, \ldots, 4s - 2\,, \quad q = 1, \ldots, 4t\,, \quad i = 1, \ldots, k\,, \quad j = 1, 2, 3\,.$$

(See (3.9), (3.10) and the following text.) By the definitions of the functions (5.76) it follows immediately that they are linearly independent on M[12]).

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[12]) In more detail: (i) The functions r_{ij} are linearly independent in G — this is a trivial consequence of Lemma 2.4. (ii) The functions r_{ij} on the one hand and the remaining functions on the other hand are linearly independent. (It means that no function of one group can be a linear combination in G of functions of the other one.) This follows from the fact that the latter ones are Airy functions, while r_{ij} are not. (iii) The logarithmic functions are linearly independent. Further, the logarithmic functions on the one hand and the functions z_p, v_{iq} on the other hand are linearly independent as well. (iv) The biharmonic polynomials z_p are linearly independent (see [1]); so are the functions v_{iq}, as follows from their construction. (v) The polynomials z_p on the one hand and the rational functions v_{iq} on the other hand are linearly independent (in G): In the opposite case, since they are polynomials and fractional rational functions, they would be linearly dependent not only in G, but in the whole plane E_2 (with the exception of the points (x_i, y_i)). But this is not possible: for $(x_i, y_i) \neq (0, 0)$ and for $(x_i, y_i) = (0, 0)$ and $q > 2$ because of the poles of the functions v_{iq} at the points (x_i, y_i); for $(x_i, y_i) = (0, 0)$ and $q = 1$ or $q = 2$ this fact follows by a direct computation.

For every two functions $u, v \in M$ define

$$(5.77) \qquad (u, v)_\Gamma = \int_\Gamma uv \, \mathrm{d}s + \int_\Gamma \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} \, \mathrm{d}s + \int_\Gamma \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} \, \mathrm{d}s \, .$$

First, (5.77) has sense for every pair of functions u, v from M, because all the functions (5.76) belong to $W_2^{(2)}(G)$. Further, we show that (5.77) is a scalar product on M. It is sufficient to prove

$$(5.78) \qquad (u, u)_\Gamma = 0 \Rightarrow u(x, y) \equiv 0 \quad \text{in} \quad G \, ,$$

since all the remaining axioms of the scalar product are obviously fulfilled. Thus, let $u \in M$ and $(u, u)_\Gamma = 0$. Then (5.77) implies easily

$$(5.79) \qquad u = 0 \quad \text{in} \quad W_2^{(1)}(\Gamma) \, , \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{in} \quad L_2(\Gamma) \, ,$$

because

$$(u, u)_\Gamma = \int_\Gamma u^2 \, \mathrm{d}s + \int_\Gamma \left(\frac{\partial u}{\partial s}\right)^2 \mathrm{d}s + \int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 \mathrm{d}s \, .$$

But the function $u(x, y)$ is a linear combination of weak biharmonic functions (5.76) and by (5.79) it is a weak solution of the problem

$$\Delta^2 u = 0 \quad \text{in} \quad G \, ,$$

$$u = 0 \, , \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \, .$$

By the uniqueness of the weak solution we have

$$u = 0 \quad \text{in} \quad W_2^{(2)}(G) \, .$$

All the functions (5.76) being continuous in G, it follows that $u(x, y) \equiv 0$ in G which proves (5.78).

Consequently, the determinant of the system (3.20) is the Gram determinant constructed of the functions (5.76). But these functions are linearly independent in M. Hence this determinant is different from zero. The system (3.20) is thus uniquely solvable which completes the proof.

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Souhrn

ŘEŠENÍ PRVNÍHO PROBLÉMU ROVINNÉ PRUŽNOSTI PRO VÍCENÁSOBNĚ SOUVISLÉ OBLASTI METODOU NEJMENŠÍCH ČTVERCŮ NA HRANICI

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V případě jednosuše souvislé oblasti lze první problém rovinné pružnosti (str. 352) převést – zhruba řečeno – na biharmonický problém (1.10), (1.11). K jeho přibližnému řešení (v reálném tvaru) lze použít metodu nejmenších čtverců na hranici, rozpracovanou v [1] (viz také str. 374). U této metody není třeba předpokládat, že řešení patří do $W_2^{(2)}(G)$, stačí, aby $g_0 \in W_2^{(1)}(\Gamma)$, $g_1 \in L_2(\Gamma)$. (Podrobně viz v [1] – hranice oblasti se předpokládá lipschitzovská.)

Předložený článek je zobecněním práce [1]. Nejde o zobecnění formální, ale o řešení zcela nových problémů, s kterými se v případě vícenásobně souvislých oblastí setkáme. Jde o obtíže dvojího druhu:

1. Postupujeme-li formálně jako v případě jednoduše souvislé oblasti, dostaneme velmi slabé řešení $u(x, y)$ a k němu prostřednictvím vztahů (1.5), str. 353, funkce σ_x, σ_y, τ_{xy}, které sice splňují rovnice rovnováhy a kompatibility, avšak souřadnice příslušného vektoru posunutí nemusí být jednoznačné funkce. Proto je třeba najít takovou matematickou formulaci problému (v reálném tvaru), která vystihuje jeho fyzikální podstatu (str. 367). Důkaz existence a jednoznačnosti (velmi slabého) řešení takto formulovaného problému je proveden v druhé části kapitoly 2 (existenční věta 2.1, str. 373). Používají se v něm podstatně vlastnosti funkcí r_{ij} zavedených na str. 367.

2. Přibližné řešení metodou nejmenších čtverců na hranici již nelze hledat ve tvaru (0.4), str. 350, ale ve tvaru (3.9), str. 376. Tím se mimo jiné dostanou do výpočtu i funkce r_{ij}. Podstatné pro metodu je to, že při numerickém výpočtu v ní vystupují jen hraniční hodnoty těchto funkcí, a ty jsou velmi jednoduché.

Práce je psána tak, aby první tři kapitoly byly čitelné i pro „konzumenty" matematiky. Po matematické stránce je těžiště článku v kap. 4 (důkaz konvergenční věty 3.2 ze str. 379) a v kap. 5 (pomocná lemmata).

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