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Parameters in the constraints

QUALITATIVE ANALYSIS OF BASIC NOTIONS
IN PARAMETRIC CONVEX PROGRAMMING, I

(Parameters in the constraints)

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(Received September 15, 1975)

A great deal of work has been done in the field of parametric linear programming from the theoretical as well as from the computational point of view. From the recent work in this direction, let us mention the book [9] "Theorie der linearen parametrischen Optimierung" by F. Nožička, J. Guddat, H. Hollatz and B. Bank which appeared in 1974. In [9]. the notions of the set of feasible parameters, the solvability set, and the local stability set have been defined and analyzed qualitatively. The same notions have been defined and analyzed qualitatively for convex quadratic programs by J. Guddat in [7]. Other works in parametric quadratic programming discuss the effect of infinitesimal changes in the data of the problem on the solution vector, such as the papers of J. W. Daniel [3] and J. C. G. Boot [2]. Recently, some works have related the notions of stability in extremum problems to certain directions in nonlinear duality research. For example, R. T. Rockafellar in [10] has dealt with stability in the convex case using the conjugate function theory, and he has attempted a further development of Fenchel's theory in both finite — and infinite dimensional spaces. G. B. Dantzing, J. Folkman and N. Shapiro in [4] studied stability in terms of the behaviour of the set of minima in response to right-hand-side perturbations. J. P. Evans and F. J. Gould in [6] established necessary and sufficient conditions for constraint set stability requiring neither convex constraint functions nor convex constraint set, with applications to quasiconvex functions.

In this paper, basic notions in parametric convex programming are defined and analyzed qualitatively for the problem

(I) $\min F(x)$,

subject to

$$\mathfrak{M}(v) = \{ x \in \mathsf{R}^n / g_r(x) \leqq v_r, \ r = 1, 2, \ldots, l \} ,$$

where $F(x)$; $g_r(x)$, $r = 1, 2, \ldots, l$ are convex functions, possessing continuous first order partial derivatives on the n-dimensional vector space R^n (the space of all ordered n-tuples of real numbers) and v_r, $r = 1, 2, \ldots, l$ are arbitrary real numbers. These notions are the set of feasible parameters, the solvability set, the stability set of the first kind, and the stability set of the second kind.

I. CHARACTERIZATION OF THE SET OF FEASIBLE PARAMETERS

Definition 1. *The set of feasible parameters for problem* (I) *denoted by* $\mathfrak{A}$, *is defined by*

(1) $$\mathfrak{A} = \left\{ v \in {}'R^l / \mathfrak{M}(v) \neq \emptyset \right\},$$

where ${}'R^l$ *is the* l-*dimensional vector space of parameters.*

Remark 1. The set $\mathfrak{A}$ is nonempty, unbounded [6] and moreover, if $\bar{v} \in \mathfrak{A}$, then all v in the nonnegative orthant of the parametric vector space ${}'R^l$ with the origin at $v = \bar{v}$ belong to the set $\mathfrak{A}$.

Lemma 1. *The set* $\mathfrak{A}$ *is convex.*

Proof. Assume that v^1, v^2 are two points in $\mathfrak{A}$, then there exist points x^1, x^2 in R^n respectively such that $g_r(x^1) \leq v_r^1$, $r = 1, 2, \ldots, l$ and $g_r(x^2) \leq v_r^2$, $r = 1, 2, \ldots, l$. Therefore, $(1 - \omega) g_r(x^1) + \omega g_r(x^2) \leq (1 - \omega) v_r^1 + \omega v_r^2$, $r = 1, 2, \ldots, l$, for all $0 \leq \omega \leq 1$. From the convexity of the functions $g_r(x)$, $r = 1, 2, \ldots, l$, [8], [11], [12] it follows that $g_r[(1 - \omega) x^1 + \omega x^2] \leq (1 - \omega) v_r^1 + \omega v_r^2$, $r = 1, 2, \ldots, l$; $0 \leq \omega \leq 1$. Then $\mathfrak{M}[(1 - \omega) v^1 + \omega v^2] \neq \emptyset$, i.e. $(1 - \omega) v^1 + \omega v^2 \in \mathfrak{A}$ for all $0 \leq \omega \leq 1$. Hence the set $\mathfrak{A}$ is convex.

Lemma 2. *If there is* $v \in \mathfrak{A}$ *such that* $\mathfrak{M}(v)$ *is bounded, then* $\mathfrak{A}$ *is closed.*

Proof. Suppose that $\bar{v} \in {}'R^l$ is a frontier point of the set $\mathfrak{A}$, then by Remark 1 and since any neighbourhood of $\bar{v}$ has nonempty intersection with $\mathfrak{A}$, it follows that $\bar{v} + \varepsilon \in \mathfrak{A}$ for any $\varepsilon > 0$, $\varepsilon \in {}'R^l$. (By $\varepsilon > 0$, $\varepsilon \in {}'R^l$ we mean $\varepsilon_r > 0$, $r = 1, 2, \ldots, l$.) Consider the sequence $\mathfrak{M}(\bar{v} + \varepsilon^n)$, $\varepsilon^n \in {}'R^l$; $\varepsilon^n > 0$; $\varepsilon^{n+1} < \varepsilon^n$; $\varepsilon^n \to 0$ $(n = 1, 2, \ldots)$. The set $\mathfrak{M}(\bar{v} + \varepsilon^1)$ is compact, since it is closed [12] and bounded [8], [12] (in [12], there is a lemma stating that "the nonvoid level sets $S(\alpha) := \left\{ x \in R^n / f(x) \leq \alpha \right\}$ of a closed convex function are either all bounded or all unbounded", where closedness of a function is equivalent to its lower semicontinuity on R^n [8], [12]). All the sets $\mathfrak{M}(\bar{v} + \varepsilon^n)$, $n = 1, 2, \ldots$ are closed in R^n [12] and therefore closed with respect to the compact set $\mathfrak{M}(\bar{v} + \varepsilon^1)$ [5], [8] and it is clear that $\mathfrak{M}(\bar{v} + \varepsilon^n) \subseteq \mathfrak{M}(\bar{v} + \varepsilon^1)$, $n = 1, 2, \ldots$.

Since $\bigcap\limits_{j=1}^{m} \mathfrak{M}(\bar{v} + \varepsilon^{i_j}) = \mathfrak{M}(\bar{v} + \varepsilon^{i_k}) \neq \emptyset$, $\varepsilon^{i_j} \in {}'R^l$; $j = 1, 2, \ldots, m$, where $i_k = \min\limits_{j \in \{1, 2, \ldots, m\}} i_j$, this holds for the intersection of any finite number of elements of the sequence $\mathfrak{M}(\bar{v} + \varepsilon^n)$. Therefore, it follows from the finite intersection property of compact sets $[8]$ that $\bigcap\limits_{n=1}^{\infty} \mathfrak{M}(\bar{v} + \varepsilon^n) = \mathfrak{M}(\bar{v}) \neq \emptyset$, i.e. $\bar{v} \in \mathfrak{A}$ and hence the result.

2. CHARACTERIZATION OF THE SOLVABILITY SET

Definition 2. *The solvability set for problem* (I) *denoted by* $\mathfrak{B}$, *is defined by*

(2) $$\mathfrak{B} = \{v \in {}'R^l / m_{\mathrm{opt}}(v) \neq \emptyset\},$$

where $m_{\mathrm{opt}}(v)$ *is the set of all optimal points of problem* (I), *i.e.*

(3) $$m_{\mathrm{opt}}(v) = \{x^* \in R^n / F(x^*) = \min_{x \in \mathfrak{M}(v)} F(x)\}.$$

Theorem 1. *If for one* $v \in \mathfrak{B}$ *it holds that the set* $m_{\mathrm{opt}}(v)$ *is bounded, then* $\mathfrak{B} = \mathfrak{A}$ *where* $m_{\mathrm{opt}}(v)$ *is given by* (3).

Pr oof. Suppose that $\tilde{k} = \min\limits_{x \in \mathfrak{M}(\tilde{v})} F(x)$, then it follows from the assumptions that the set $M(\tilde{v}, \tilde{k})$ defined by

$$M(\tilde{v}, \tilde{k}) = \{x \in R^n / g_r(x) \leqq \tilde{v}_r, \ r = 1, 2, \ldots, l, \ F(x) \leqq \tilde{k}\}$$

is bounded. Hence, the set $M(v, k)$ given by

$$M(v, k) = \{x \in R^n / g_r(x) \leqq v_r, \ r = 1, 2, \ldots, l, \ F(x) \leqq k, k \in {}'R\}$$

is bounded for all $(v, k) \in {}'R^{l+1}$ for which $M(v, k) \neq \emptyset$ (see $[12]$ and the proof of Lemma 2).

Let us define the set A as follows:

$$A = \{(v, k) \in {}'R^{l+1} / M(v, k) \neq \emptyset\}.$$

The set A is unbounded and from Lemma 1, Lemma 2 it follows that it is convex and closed. Suppose that $v^* \in \mathfrak{A}$, then $\mathfrak{M}(v^*) \neq \emptyset$ and the set

$$A^* = \{k \in {}'R / M(v^*, k) \neq \emptyset\}$$

is convex, unbounded, and assumes the form $[k^*, \infty)$. Therefore $\min\limits_{x \in \mathfrak{M}(v)} F(x) = k^*$, which implies that $v^* \in \mathfrak{B}$. Then $\mathfrak{A} \subset \mathfrak{B}$, and hence $\mathfrak{B} = \mathfrak{A}$.

Corollary 1. *Under the same assumptions as in Theorem 1, the set* $\mathfrak{B}$ *is unbounded, convex* (*see Lemma 1*).

Corollary 2. *If the set* $\mathfrak{B} \neq \emptyset$, *and* $\mathsf{F}(x)$ *is strictly convex on* R^n, *then* $\mathfrak{B} = \mathfrak{A}$ *(follows directly from Theorem 1).*

Corollary 3. *If the set* $\mathfrak{B} \neq \emptyset$, *and the set* $\mathfrak{M}(v)$ *is bounded for one* $v \in \mathfrak{A}$, *then* $\mathfrak{B} = \mathfrak{A}$, *and the set* $\mathfrak{B}$ *is unbounded, convex (see Lemma 1) and closed (see Lemma 2).*

Example 1. Consider the problem

$$\min y \, ,$$

subject to the set $\mathfrak{M}(v_1, v_2)$ given by

$$\mathfrak{M}(v_1, v_2) = \left\{ (x_1, x_2) \in \mathsf{R}^2 / -x_2 + e^{x_1} \leqq v_1, \ -x_2 \leqq v_2 \right\} .$$

The problem is solvable for $v_1 = 0$, $v_2 = -2$ with optimal points $x_1 \leqq \log 2$, $x_2 = 2$ which means that $m_{\mathrm{opt}}(0, -2)$ is unbounded. The problem is feasible at $v_1 = 0$, $v_2 = 0$ but it is not solvable there.

Let us consider the dual problem to (I), denoted by $(\mathrm{I})_d$ which assumes the form [8]:

Find $\max\limits_{(x,u) \in Z} \Psi(x, u, v)$ if it exists, where

$(\mathrm{I})_d$
$$\Psi(x, u, v) = \mathsf{F}(x) + \sum_{r=1}^{l} u_r(\mathsf{g}_r(x) - v_r) \, ,$$

and

$$Z =$$
$$= \left\{ (x, u) \in \mathsf{R}^{n+l} \, \bigg/ \, \frac{\partial \mathsf{F}}{\partial x_\alpha} + \sum_{r=1}^{l} u_r \frac{\partial \mathsf{g}_r}{\partial x_\alpha} = 0 \, , \ \alpha = 1, 2, \ldots, n; \ u_r \geqq 0, \ r = 1, 2, \ldots, l \right\} .$$

Definition 3. *The solvability set for problem* $(\mathrm{I})_d$ *denoted by* $\mathfrak{B}_d$, *is defined by*

(4)
$$\mathfrak{B}_d = \left\{ v \in {}'\mathsf{R}^l / \text{problem } (\mathrm{I})_d \text{ is solvable} \right\} .$$

Lemma 3. *If the functions* $\mathsf{g}_r(x) - \bar{v}_r$, $r = 1, 2, \ldots, l$ *satisfy any one of the constraint qualifications* [8] *(for example Slater) for all* $\bar{v} \in \mathfrak{B}$, *and* $\Psi(x, {}'u, v)$ *is strictly convex at* $'x$ *for all* $v \in \mathfrak{B}_d$, *then* $\mathfrak{B}_d = \mathfrak{B}$, *where*

$$\Psi('x, {}'u, v) = \max_{(x,u) \in Z} \Psi(x, u, v) \, .$$

Proof. Let $v \in \mathfrak{B}$, then by the assumptions and from Wolfe's duality theorem [8], [12] it follows that $v \in \mathfrak{B}_d$ and therefore $\mathfrak{B} \subset \mathfrak{B}_d$. Let $v \in \mathfrak{B}_d$, then by the assumptions and from the strict converse duality theorem [8] it follows that $v \in \mathfrak{B}$ and therefore $\mathfrak{B}_d \subset \mathfrak{B}$. Hence $\mathfrak{B}_d = \mathfrak{B}$.

It is clear that $\Psi(x, u, v)$ is strictly convex in x, if either $F(x)$ is strictly convex, or $g_i(x)$ is strictly convex for at least one index $i \in \{1, 2, \ldots, l\}$ for which u_i is positive [8].

Remark 2. *If $F(x)$ is a convex quadratic function, and $g_r(x)$, $r = 1, 2, \ldots, l$ are linear functions on R^n (i.e. for quadratic convex programs), then the result that $\mathfrak{B}_d = \mathfrak{B}$ can be obtained directly from Dorn's duality and converse duality theorems* [8], [12].

Theorem 2. *If the functions $g_r(x) - v_r$, $r = 1, 2, \ldots, l$ satisfy any one of the constraint qualifications (for example Slater)* [8], [12] *for all $v \in \mathfrak{B}$ and $F(x)$ is strictly convex on R^n, then the function $\tau(v)$ is convex on $\mathfrak{B}$, where $\tau(v) = \min\limits_{x \in \mathfrak{M}(v)} F(x)$.*

Proof. Suppose that $\tau_d(v) = \max\limits_{(x, u) \in Z} \Psi(x, u, v)$, then it follows by the assumptions and from Wolfe's duality theorem [8], [12], that $\tau_d(v) = \tau(v)$.

Assume that v^1, v^2 are two points in $\mathfrak{B}_d$ with corresponding optimal points (x^1, u^1), (x^2, u^2) respectively. Then

$$\tau_d(v^1) = F(x^1) + \sum_{r=1}^{l} u_r^1 (g_r(x^1) - v_r^1),$$

and

$$\tau_d(v^2) = F(x^2) + \sum_{r=1}^{l} u_r^2 (g_r(x^2) - v_r^2).$$

It follows from Lemma 3, Corollary 2 and Lemma 2 that $v^* = (1 - \omega) v^1 + \omega v^2 \in \mathfrak{B}_d$ for all $0 \leq \omega \leq 1$. Suppose that an optimal point for problem $(I)_d$ corresponding to v^* is (x^*, u^*). Then

$$\tau_d(v^*) = F(x^*) + \sum_{r=1}^{l} u_r^* (g_r(x^*) - v_r^*) =$$

$$= (1 - \omega) \left[F(x^*) + \sum_{r=1}^{l} u_r^* (g_r(x^*) - v_r^1) \right] + \omega \left[F(x^*) + \sum_{r=1}^{l} u_r^* (g_r(x^*) - v_r^2) \right] \leq$$

$$\leq (1 - \omega) \tau_d(v^1) + \omega \tau_d(v^2) \quad \text{for all} \quad 0 \leq \omega \leq 1.$$

Therefore $\tau_d(v)$ is convex on $\mathfrak{B}_d$, and hence it follows from Lemma 3 that $\tau(v)$ is convex on $\mathfrak{B}$.

3. CHARACTERIZATION OF THE STABILITY SET OF THE FIRST KIND

Definition 4. *Suppose that $\bar{v} \in \mathfrak{B}$ with a corresponding optimal point $\bar{x}$, then the stability set of the first kind of problem (I) corresponding to $\bar{x}$ denoted by $\mathfrak{S}(\bar{x})$ is defined by*

$$(5) \qquad\qquad \mathfrak{S}(\bar{x}) = \{ v \in 'R^l / F(\bar{x}) = \min\limits_{x \in \mathfrak{M}(v)} F(x) \}.$$

Lemma 4. *If the functions* $F(x)$; $g_r(x)$, $r = 1, 2, \ldots, l$ *have continuous partial derivatives of the second order on* R^n, *if the matrix* $[\partial^2 F/\partial x_\alpha \, \partial x_\beta]$, $\alpha; \beta = 1, 2, \ldots, n$ *is positive definite and the matrices* $[\partial^2 g_r/\partial x_\alpha \, \partial x_\beta]$, $r = 1, 2, \ldots, l$; $\alpha; \beta = 1, 2, \ldots$ $\ldots, n$ *are positive semi-definite, then the matrix* $[\partial^2 \Psi/\partial x_\alpha \, \partial x_\beta]$, $\alpha; \beta = 1, 2, \ldots, n$ *is positive definite for all* $u_r \geqq 0$, $r = 1, 2, \ldots, l$ *where*

$$(6) \qquad \frac{\partial^2 \Psi}{\partial x_\alpha \, \partial x_\beta} = \frac{\partial^2 F}{\partial x_\alpha \, \partial x_\beta} + \sum_{r=1}^{l} u_r \frac{\partial^2 g_r}{\partial x_\alpha \, \partial x_\beta} ,$$

$$\alpha; \beta = 1, 2, \ldots, \quad n, u_r \geqq 0 , \quad r = 1, 2, \ldots, l .$$

Proof. Let B, C_r, $r = 1, 2, \ldots, l$ and D denote the matrices $[\partial^2 F/\partial x_\alpha \, \partial x_\beta]$, $\alpha; \beta = 1, 2, \ldots n$, $[\partial^2 g_r/\partial x_\alpha \, \partial x_\beta]$, $r = 1, 2, \ldots, l$, $\alpha; \beta = 1, 2, \ldots, n$ and $[\partial^2 \Psi/\partial x_\alpha \, \partial x_\beta]$, $\alpha; \beta = 1, 2, \ldots, n$ respectively, then $(x, Dx) = (x, Bx) + \sum_{r=1}^{l} (x, u_r \, C_r x) =$ $= (x, Bx) + \sum_{r=1}^{l} u_r (x, C_r x)$, where $(.,.)$ denotes the usual scalar product of two vectors. Since $(x, Bx) > 0$ and $(x, C_r x) \geqq 0$, $r = 1, 2, \ldots, l$, it follows that $(x, Dx) < 0$. Hence the matrix D is positive definite.

Assume that problem (I) is solvable for $v = \bar{v}$ with a corresponding optimal point $\bar{x}$, and that the functions $g_r(x) - \bar{v}_r$, $r = 1, 2, \ldots, l$ satisfy any one of the constraint qualifications [1], [8] (for example Slater), then there exist $\bar{u} \in R^l$ such that $(\bar{x}, \bar{u})$ solves the Kuhn-Tucker problem [1], [8], i.e.

$$(7) \qquad \frac{\partial F}{\partial x_\alpha}(\bar{x}) + \sum_{r \in I_0} \bar{u}_r \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0 , \quad \alpha = 1, 2, \ldots, n ,$$

$$g_r(\bar{x}) \leqq \bar{v}_r , \quad r = 1, 2, \ldots, l ,$$

$$\bar{u}_r(g_r(\bar{x}) - \bar{v}_r) = 0 , \quad r = 1, 2, \ldots, l ,$$

$$\bar{u}_r \geqq 0 , \quad r \in I_0 \subset \{1, 2, \ldots, l\} ,$$

$$\bar{u}_r = 0 , \quad r \in \{1, 2, \ldots, l\} - I_0 .$$

Assume that the matrix $[\partial^2 \Psi/\partial x_\alpha \, \partial x_\beta]$, $\alpha; \beta = 1, 2, \ldots, n$ which is defined by (6), is positive definite for all $u_r \geqq 0$, $r = 1, 2, \ldots, l$. Then it follows from the implicit function theorem [5], [8] and from (7) that $\bar{x}$ can be expressed uniquely in the form

$$\bar{x} = f(\bar{u}) ,$$

where f is an n-dimensional vector function. Also, it follows that there exists a neighbourhood $V(\bar{x}, \bar{u})$ of $(\bar{x}, \bar{u})$ in R^{n+1} such that

$$(8) \qquad \frac{\partial F}{\partial x_\alpha} + \sum_{r=1}^{l} u_r \frac{\partial g_r}{\partial x_\alpha} = 0 \quad \text{for all points} \quad (x, u) \in V(\bar{x}, \bar{u}), \alpha = 1, 2, \ldots, n .$$

For these points, x can be expressed uniquely in the form

$$(9) \qquad\qquad x = f(u) \,.$$

If $(x^*, u^*) \in V(\bar{x}, \bar{u})$ solves the Kuhn-Tucker problem, i.e.

$$(10) \qquad \frac{\partial F}{\partial x_\alpha}(x^*) + \sum_{r=1}^{l} u_r^* \frac{\partial g_r}{\partial x_\alpha}(x^*) = 0 \,, \quad \alpha = 1, 2, \ldots, n \,,$$

$$g_r(x^*) \leqq v_r \,, \quad r = 1, 2, \ldots, l \,,$$

$$u_r^*(g_r(x^*) - v_r) = 0 \,, \quad r = 1, 2, \ldots, l \,,$$

$$u_r^* \geqq 0 \,, \quad r = 1, 2, \ldots, l \,,$$

then x^* solves problem (I).

In order to obtain the values of the parameters v_r, $r = 1, 2, \ldots, l$ for which problem (I) is solvable with an optimal point $\tilde{x} \in R^n$ such that there exist $\tilde{u}_r \geqq 0$, $r = 1, 2, \ldots, l$ and $(\tilde{x}, \tilde{u}) \in V(\bar{x}, \bar{u})$, we consider the following cases:

i) $\qquad u_r^* \geqq 0 \,, \quad r \in I \subset \{1, 2, \ldots, l\} \,, \quad u_r^* = 0 \,, \quad r \in \{1, 2, \ldots, l\} - I$

$$g_r(f(u^*)) = v_r \,, \quad r \in I \,,$$

$$g_r(f(u^*)) \leqq v_r \,, \quad r \notin I \,,$$

where $f(u^*) = x^*$ (see (8)) and $I \in P$, where P is the set of all proper subsets of $\{1, 2, \ldots, l\}$ with the property $\tilde{u}_r \geqq 0$, $r \in J \,(J \in P)$, $\tilde{u}_r = 0$, $r \notin J$ and $(f(\tilde{u}), \tilde{u}) \in V(\bar{x}, \bar{u})$. We define the sets T_I, W_I and W_1 as follows:

$$(11) \qquad T_I = \{u \in R^l/(f(u), u) \in V(\bar{x}, \bar{u}), \, u_r \geqq 0, \, r \in I, u_r = 0, r \notin I\} \,,$$

$$W_I = \{v \in \,'R^l/v_r = g_r(f(u)), \, r \in I, \, v_r \geqq g_r(f(u)), \, r \notin I, \, u \in T_I\} \,,$$

$$W_1 = \bigcup_{I \in P} W_I \,.$$

ii) $\qquad\qquad\qquad u_r^* = 0 \,, \quad r = 1, 2, \ldots, l \,,$

$$g_r(f(0)) \leqq v_r \,, \quad r = 1, 2, \ldots, l \,.$$

We define the set W_2 by

$$(12) \qquad W_2 = \{v \in \,'R^l/v_r \geqq g_r(f(0)) \,, \quad (f(0), 0) \in V(\bar{x}, \bar{u})\} \,.$$

iii) $\qquad\qquad\qquad u_r^* \geqq 0 \,, \quad r = 1, 2, \ldots, l \,,$

$$g_r(f(u^*)) = v_r \,, \quad r = 1, 2, \ldots, l \,.$$

We define the sets T_3 and W_3 as follows:

$$(13) \qquad T_3 = \{u \in R^l/(f(u), u) \in V(\bar{x}, \bar{u}), \, u_r \geqq 0, \, r = 1, 2, \ldots, l\} \,,$$

$$W_3 = \{v \in \,'R^l/v_r = g_r(f(u)), \, r = 1, 2, \ldots, l, \, u \in T_3\} \,.$$

Let us define the set denoted by W in the form

(14) $\qquad \mathsf{W} = \{ v \in \mathfrak{B} / \text{there exist } (x^*, u^*) \in \mathsf{V}(\bar{x}, \bar{u}) \text{ and } (\bar{x}, u^*) \text{ solves } (10) \}$.

W represents the set of all parameters for which the dual problem (I_d) is solvable in the neighbourhood $\mathsf{V}(\bar{x}, \bar{u})$ of $(\bar{x}, \bar{u})$ in R^{n+l} [8].

It is clear that the first relation in (10) is satisfied by (8), and therefore in the cases i), ii) and iii), the set W defined by (14) can be written in the form

$$\mathsf{W} = \bigcup_{i=1}^{3} \mathsf{W}_i$$

where W_i, $i = 1, 2, 3$ are given by (11), (12) and (13) respectively.

In order to obtain an explicit description for the set defined by (5), let us consider the system

(15) $$\frac{\partial \mathsf{F}}{\partial x_\alpha}(\bar{x}) + \sum_{r=1}^{l} u_r \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0 , \quad \alpha = 1, 2, \ldots, n$$

which represents n linear equations in l unknowns u_r, $r = 1, 2, \ldots, l$. System (15) can be solved explicitly, and depending on its solution the value of v_r, $r = 1, 2, \ldots, l$ are chosen in such a way that $(\tilde{x}, \tilde{u})$ solves the Kuhn-Tucker problem $[1]$, $[8]$, where $\tilde{u}$ solves (15).

Let us consider the following cases.

i) $\qquad \tilde{u}_r > 0$, $\quad r \in \mathsf{J} \subset \{1, 2, \ldots, l\}$, $\quad \tilde{u}_r = 0$, $\quad r \in \{1, 2, \ldots, l\} - \mathsf{J}$.

Let us define the set denoted by $\mathfrak{S}_{\mathsf{J}}(\bar{x})$ in the form

$$\mathfrak{S}_{\mathsf{J}}(\bar{x}) = \{ v \in {}'\mathsf{R}^l / g_r(\bar{x}) = v_r, \ r \in \mathsf{J}, \ g_r(\bar{x}) \leqq v_r, \ r \notin \mathsf{J} \} ,$$

and $\mathsf{J} \in \mathscr{G}$, where $\mathscr{G}$ is the set of all proper subsets of $\{1, 2, \ldots, l\}$ such that $\mathsf{J}^* \in \mathscr{G}$ means that there exists u^* which solves (15), such that $u_r^* > 0$, $r \in \mathsf{J}$, $u_r^* = 0$, $r \in \{1, 2, \ldots, l\} - \mathsf{J}^*$, and let us define the set denoted by $\mathfrak{S}_1(\bar{x})$ in the form

(16) $$\mathfrak{S}_1(\bar{x}) = \bigcup_{\mathsf{J} \in \mathscr{G}} \mathfrak{S}_{\mathsf{J}}(\bar{x}) ;$$

ii) $\qquad \tilde{u}_r = 0$, $\quad r = 1, 2, \ldots, l$.

Let us define the set denoted by $\mathfrak{S}_2(\bar{x})$ in the form

(17) $$\mathfrak{S}_2(\bar{x}) = \{ v \in {}'\mathsf{R}^l / g_r(\bar{x}) \leqq v_r, \ r = 1, 2, \ldots, l \} :$$

iii) $\qquad \tilde{u}_r > 0$, $\quad r = 1, 2, \ldots, l$,

let us define the set denoted by $\mathfrak{S}_3(\bar{x})$ in the form

(18) $$\mathfrak{S}_3(\bar{x}) = \{ v \in {}'\mathsf{R}^l / g_r(\bar{x}) = v_r, \ r = 1, 2, \ldots, l \} ,$$

From the Kuhn-Tucker sufficient optimality theorem [1], [8] it follows that the set $\mathfrak{S}(\bar{x})$ defined by (5) assumes the form

$$(19) \qquad \mathfrak{S}(\bar{x}) = \bigcup_{i=1}^{3} \mathfrak{S}_i(\bar{x}),$$

where $\mathfrak{S}_i(\bar{x})$, $i = 1, 2, 3$ are given by (16), (17), (18) respectively. The set $\mathfrak{S}(\bar{x})$ is nonvoid, for by the assumption $\bar{v} \in \mathfrak{S}(\bar{x})$ (see Definition 4).

Lemma 5. *If $\tilde{v}$ is a common point of visibility for all star shaped sets Γ_i, $i = 1, 2, \ldots, k$ [12], then $\bigcup_{i=1}^{k} \Gamma_i$ is a star shaped set with a common point of visibility $\tilde{v}$. (A set L is said to be star shaped if there exists a point $\tilde{\mu} \in \mathsf{L}$ such that for all $\mu \in \mathsf{L}$, the closed line segment $[\tilde{\mu}, \mu] \subseteq \mathsf{L}$, and $\tilde{\mu}$ is said to be a point of common visibility of L.)*

Proof. Suppose that $v \in \bigcup_{i=1}^{k} \Gamma_i$, then $v \in \Gamma_s$ for at least one index $s \in \{1, 2, \ldots, k\}$, then the assumptions imply that $[\tilde{v}, v] \subseteq \Gamma_s$ and hence $[\tilde{v}, v] \subseteq \bigcup_{i=1}^{k} \Gamma_i$. Since $\tilde{v} \in \bigcup_{i=1}^{k} \Gamma_i$, the set $\bigcup_{i=1}^{k} \Gamma_i$ is star shaped with a common point of visibility $\tilde{v}$.

Lemma 6. *If the sets $\mathfrak{S}_i(\bar{x})$, $i = 1, 2, 3$ are defined by (16), (17) and (18) respectively then each of them is star shaped with a common point of visibility v^*, where $v_r^* = g_r(\bar{x})$, $r = 1, 2, \ldots, l$ (see Lemma 5), and closed.*

Proof. It is clear that $v^* \in \mathfrak{S}_i(\bar{x})$, $i = 1, 2, 3$. The first part of the proof will be shown for $\mathfrak{S}_1(\bar{x})$. It can be done similarly for the sets $\mathfrak{S}_2(\bar{x})$ and $\mathfrak{S}_3(\bar{x})$.

Assume that $\tilde{v}$ is any point in $\mathfrak{S}_1(\bar{x})$, then from (16) it follows that there exists an index set $\mathsf{I}_0 \subset \{1, 2, \ldots, l\}$ such that

$$g_r(\bar{x}) = \tilde{v}_r, \quad r \in \mathsf{I}_0 \subset \{1, 2, \ldots, l\}, \quad g_r(\bar{x}) \leqq \tilde{v}_r, \quad r \in \{1, 2, \ldots, l\} - \mathsf{I}_0.$$

The points $\hat{v} = (1 - \omega) v^* + \omega \tilde{v} \in \mathfrak{S}_1(\bar{x})$ for all $0 \leqq \omega \leqq 1$, since

$$g_r(\bar{x}) = v_r, \quad r \in \mathsf{I}_0, \quad g_r(\bar{x}) \leqq \tilde{v}_r, \quad r \in \{1, 2, \ldots, l\} - \mathsf{I}_0.$$

Therefore, $[v^*, \tilde{v}] \subseteq \mathfrak{S}_1(\bar{x})$ for all $\tilde{v} \in \mathfrak{S}_1(\bar{x})$, and hence the set $\mathfrak{S}_1(\bar{x})$ is star shaped with a common point of visibility v^*. The closedness of the sets $\mathfrak{S}_i(\bar{x})$, $i = 1, 2, 3$ follows directly from their definitions (see (16), (17) and (18)).

Theorem 3. *If $\mathfrak{S}(\bar{x})$ is defined by (5), then it is star shaped [12] and closed.*

Proof. The result follows directly from (19), Lemma 5 and Lemma 6.

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Example 2. Consider the problem

$$\min\left(x_1^2 - x_2\right),$$

subject to

$$-x_1 + x_2^2 \leqq v_1,$$

$$-x_1 + x_2 \leqq v_2.$$

For $v_1 = 2$, $v_2 = 0$ an optimal point is found to be $x_1 = \frac{1}{2} = x_2$, and the set $\mathfrak{S}(\frac{1}{2}, \frac{1}{2})$ is given by

$$\mathfrak{S}(\tfrac{1}{2}, \tfrac{1}{2}) = \left\{(v_1, v_2) \in {}'R^2/v_1 \geqq -\tfrac{1}{4}, v_2 = 0\right\} \cup \left\{(v_1, v_2) \in {}'R^2/v_1 = -\tfrac{1}{4}, v_2 \geqq 0\right\}.$$

The set $\mathfrak{S}(\frac{1}{2}, \frac{1}{2})$ is star shaped, closed but not convex.

Lemma 7. *If $\bar{u}_r = 0$, $r = 1, 2, \ldots, l$ solves* (15), *then the set $\mathfrak{S}(\bar{x})$ defined by* (19) *is convex and closed.*

Proof. It is clear that the set $\mathfrak{S}_2(\bar{x})$ is convex (see (17)). By the assumption $\mathfrak{S}_2(\bar{x}) \neq \emptyset$, and therefore it follows from (16), (17) and (18) that $\mathfrak{S}_1(\bar{x}) \subset \mathfrak{S}_2(\bar{x})$ and $\mathfrak{S}_3(\bar{x}) \subset \mathfrak{S}_2(\bar{x})$. Then $\mathfrak{S}(\bar{x}) = \mathfrak{S}_2(\bar{x})$ (see (19)), and hence the set $\mathfrak{S}(\bar{x})$ is convex and closed (Lemma 6).

Remark 3. *The method used in this section to obtain the set given by* (14) *cannot be applied for linear programs, since in that case the matrix $\left[\partial^2 F/\partial x_\alpha \, \partial x_\beta\right]$, $\alpha, \beta = 1, 2, \ldots, n$ is positive semi-defintie (see Lemma 4).*

4. CHARACTERIZATION OF THE STABILITY SET OF THE SECOND KIND

Definition 3. *Suppose that $\bar{v} \in \mathfrak{B}$ (see* (2)*) with a corresponding optimal point $\bar{x}$, and $\bar{x} \in \Sigma(\bar{v}, I)$ where*

(20) $$\Sigma(\bar{v}, I) = \left\{x \in R^n/g_r(x) = \bar{v}_r, r \in I \subset \{1, 2, \ldots, l\}\right\},$$

$$g_r(\bar{x}) < \bar{v}_r, \ r \in \{1, 2, \ldots, l\} - I\}.$$

Then the stability set of the second kind of problem (1) *corresponding to $(\bar{v}, I)$ denoted by $\mathsf{q}(\bar{v}, I)$, is defined by*

(21) $$\mathsf{q}(\bar{v}, I) = \left\{v \in \mathfrak{B}/\boldsymbol{m}_{\text{opt}}(v) \cap \Sigma(v, I) \neq \emptyset\right\},$$

where $\boldsymbol{m}_{\text{opt}}(v)$ is given by (3), *and*

(22) $$\Sigma(v, I) = \left\{x \in R^n/g_r(x) = v_r, r \in I, \ g_r(x) < v_r, r \notin I\right\}.$$

From (20) it is clear that the index set I characteries in general more than one side of $\mathfrak{M}(\bar{v})$ or int $\mathfrak{M}(\bar{v})$.

Lemma 8. *If the function* $F(x)$ *is strictly convex on* $\mathbb{R}^n$ *and* v^1; v^2 *are two distinct points in* $\mathfrak{B}$ *with* $q(v^1, I_1) \neq q(v^2, I_2)$, *then*

$$q(v^1, I_1) \cap q(v^2, I_2) = \emptyset.$$

Proof. From the uniqueness of optimal solutions and by the assumption it follows that $I_1 \neq I_2$. Suppose that $v^* \in q(v^1, I_1) \cap q(v^2, I_2)$, then there exists $x^* \in \Sigma(v^*, I_1) \cap \Sigma(v^*, I_2)$, where

$$\Sigma(v^*, I_1) = \{x \in \mathbb{R}^n / g_r(x) = v_r^*, \ r \in I_1, \ g_r(x) < v_r^*, \ r \notin I_1\},$$

$$\Sigma(v^*, I_2) = \{x \in \mathbb{R}^n / g_r(x) = v_r^*, \ r \in I_2, \ g_r(x) < v_r^*, \ r \notin I_2\}.$$

Therefore, $g_s(x^*) = v_s^*$; $g_s(x^*) < v_s^*$ for at least one $s \in \{1, 2, \ldots, l\}$ if $I_1 \neq I_2$, and then $I_1 = I_2$ which is a contradiction. Hence the result.

Remark 4. Lemma 8 gives a decomposition of the set $\mathfrak{B}$ into nonempty stability sets of the second kind corresponding to certain index subsets of $\{1, 2, \ldots, l\}$.

In order to have more properties concerning the stability set of the second kind, let us concentrate our attention to the problem

$(I)_q$
$$\min \sum_{i,j=1}^{n} \tfrac{1}{2} c_{ij} x_i x_j$$

subject to the restriction set $\mathfrak{M}(v)$,

where $[c_{ij}]$, $i; j = 1, 2, \ldots, n$ is a real symmetric positive semidefinite matrix.

Lemma 9. *If problem* (I) *assumes the form* $(I)_q$, *then the set* $q(\bar{v}, I_L)$ *is convex where* $(\bar{v}, I_L)$ *characterizes either a linear side of* $\mathfrak{M}(\bar{v})$ *or int* $\mathfrak{M}(\bar{v})$.

Proof. The proof will be done for the case of int $\mathfrak{M}(\bar{v})$. The proof for the case of a linear side of $\mathfrak{M}(\bar{v})$ is similar.

Suppose that $(\bar{v}, I_L)$ characterizes int $\mathfrak{M}(\bar{v})$ and v^1, v^2 are two points in $q(\bar{v}, I_L)$, then there exist $x^1 \in$ int $\mathfrak{M}(v^1)$, and $x^2 \in$ int $\mathfrak{M}(v^2)$ such that $(x^1, 0) \in \mathbb{R}^{n+l}$, $(x^2, 0) \in \mathbb{R}^{n+l}$ solve the following Kuhn-Tucker problems (23), (24) respectively:

(23)
$$\sum_{j=1}^{n} c_{\alpha j} x_j^1 = 0, \qquad \alpha = 1, 2, \ldots, n,$$
$$g_r(x^1) < v_r^1, \qquad r = 1, 2, \ldots, l,$$
$$u_r^1(g_r(x^1) - v_r^1) = 0, \quad r = 1, 2, \ldots, l,$$
$$u_r^1 = 0, \qquad r = 1, 2, \ldots, l,$$

and

(24)
$$\sum_{j=1}^{n} c_{\alpha j} x_j^2 = 0, \qquad \alpha = 1, 2, \ldots, n,$$
$$g_r(x^2) < v_r^2, \qquad r = 1, 2, \ldots, l,$$
$$u_r^2(g_r(x^2) - v_r^2) = 0, \quad r = 1, 2, \ldots, l,$$
$$u_r^2 = 0, \qquad r = 1, 2, \ldots, l.$$

Therefore

$$\sum_{j=1}^{n} c_{\alpha j}\left[(1 - \omega) x_j^1 + \omega x_j^2\right] = 0 , \qquad\qquad \alpha = 1, 2, \ldots, n ,$$

$$g_r\left[(1 - \omega) x^1 + \omega x^2\right] < (1 - \omega) v_r^1 + \omega v_r^2 , \quad r = 1, 2, \ldots, l ,$$

$$u_r^*\left(g_r\left[(1 - \omega) x^1 + \omega x^2\right]\right) = 0 , \qquad\qquad r = 1, 2, \ldots, l ,$$

$$u_r^* = 0 , \qquad\qquad r = 1, 2, \ldots, l ,$$

$$0 \leqq \omega \leqq 1 .$$

Then $(1 - \omega)\ x^1 + \omega x^2 \in \text{int } \mathfrak{M}((1 - \omega)\ v^1 + \omega v^2)$ for all $0 \leqq \omega \leqq 1$, and the Kuhn-Tucker sufficient optimality theorem implies that $(1 - \omega)\ x^1 + \omega x^2 \in$ $\in \mathbf{m}_{\text{opt}}((1 - \omega)\ v^1 + \omega v^2)$ for all $0 \leqq \omega \leqq 1$. Therefore $\mathbf{m}_{\text{opt}}((1 - \omega)\ v^1 + \omega v^2) \cap$ $\cap \text{ int } \mathfrak{M}((1 - \omega)\ v^1 + \omega v^2) \neq \emptyset$ for all $0 \leqq \omega \leqq 1$ and hence $(1 - \omega)\ v^1 + \omega v^2 \in$ $\in \mathbf{q}(\bar{v}, \mathsf{l}_L)$ for all $0 \leqq \omega \leqq 1$, i.e. the set $\mathbf{q}(\bar{v}, \mathsf{l}_L)$ is convex. From the assumptions of Lemma 9 it is clear that $(\bar{v}, \mathsf{l}_L)$ characterizes either a unique side of $\mathfrak{M}(\bar{v})$, or $\text{int } \mathfrak{M}(\bar{v})$. The uniqueness follows from the linearity of the functions $g_r(x)$, $r \in \mathsf{l}_L$.

Example 3. For the problem

Minimize

$$\left[x_1^2 + (x_2 - 1)^2\right] ,$$

subject to

$$x_1^2 + x_2^2 \leqq v_1 ,$$
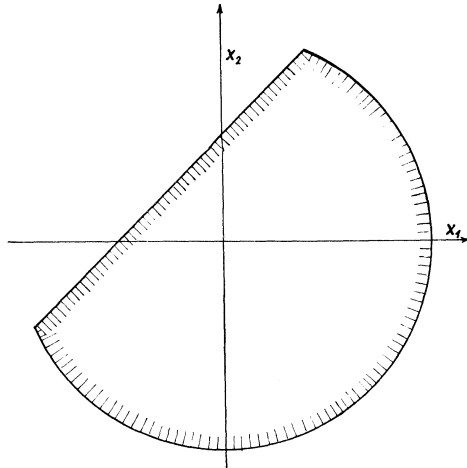
$$-x_1 + x_2 \leqq v_2$$

Fig. a. The set $\mathfrak{W}(4, 1)$.

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the following sets are obtained:

$$\mathfrak{M}(4, 1) = \left\{(x_1, x_2) \in R^2 / x_1^2 + x_2^2 \leqq 4, \quad -x_1 + x_2 \leqq 1\right\},$$

see Fig. a,

$$\mathfrak{A} = \left\{(v_1, v_2) \in {}'R^2 / v_1 \geqq 0, v_2 \geqq 0\right\} \cup \left\{(v_1, v_2) \in {}'R^2 / v_1 \geqq 0, \ v_2 \leqq 0, \ v_2^2 \leqq 2v_1\right\},$$

see Fig. b,

$$\mathfrak{B} = \mathfrak{A}$$

$$\mathfrak{S}(0, 1) = \left\{(v_1, v_2) \in {}'R^2 / v_1 \geqq 1, \ v_2 \geqq 1\right\},$$
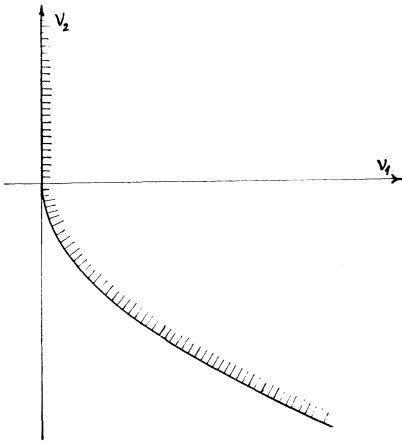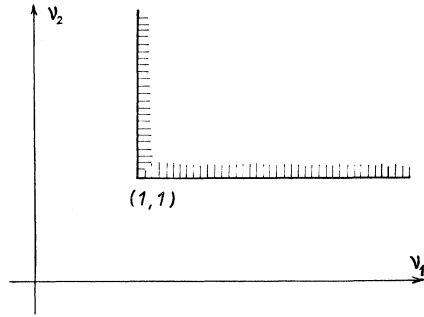
see Fig. c.

Fig. b. The set $\mathfrak{A}$.

Fig. c. The set $\mathfrak{S}(0, 1)$.

Let us denote the stability sets of the second kind of our problem corresponding to the index subsets $I_i \subseteq \{1, 2\}$ by q_i, $i = 1, 2, 3, 4$, where $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \emptyset$ and $I_4 = \{1, 2\}$. Then

$$q_1 = \left\{v \in {}'R^2 / v_1 \geqq 0, \ \sqrt{(v_1)} < v_2, \ \sqrt{(v_1)} \leqq 1\right\},$$

$$q_2 = \left\{v \in {}'R^2 / 1 + v_2^2 < 2v_1, \ v_2 \leqq 1\right\},$$

$$q_3 = \left\{v \in {}'R^2 / v_1 > 1, \ v_2 > 1\right\},$$

$$q_4 = \left\{v \in {}'R^2 / v_2^2 \leqq 2v_1, \ v_2^2 \geqq 2v_1 - 1, \ v_2 \leqq 0\right\} \cup$$

$$\cup \left\{v \in {}'R^2 / v_2^2 \leqq v_1, \ v_2^2 \geqq 2v_1 - 1, \ v_2 \geqq 0\right\}.$$

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The decomposition of the set $\mathfrak{B}$ into the sets q_i, $i = 1, 2, 3, 4$ is shown in Fig. d, and it is clear that $q_i \cap q_j = \emptyset$, $i \neq j$, $i; j = 1, 2, 3, 4$. The sets q_2 and q_3 are seen to be convex (see Fig. d).
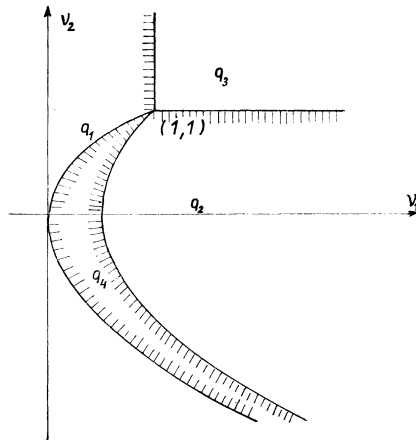

Fig. d. The nonempty stability sets of the second kind.

References

[1] *Abadie, J.:* On the Kuhn-Tucker theorem, in J. Abadie (ed.) "Nonlinear Programming", pp. 21—36, North Holland Publishing Company, Amsterdam, 1967.

[2] *Boot, J. C. G.:* On sensitivity analysis in convex quadratic programming, Op. Research, 11, 771—786 (1963).

[3] *Daniel, J. W.:* Stability of the solution of definite quadratic programs. Math. Programming, 5, 41—53 (1973).

[4] *Dantzig, G. B., Folkman, J., Shapiro, N.:* On the continuity of the minimum set of a continuous function, J. Math. Anal. and Appl. 17, 519—548 (1967).

[5] *Dieudonne, J.:* Foundations of modern analysis, New York: Academic Press 1960.

[6] *Evans, J. P., Gould, F. J.:* Stability in nonlinear programming, Op. Research, 18, 107—118 (1970).

[7] *Guddat, J.:* Stabilitätsuntersuchungen in der quadratischen parametrischen Optimierung, Dissertation. Zur Erlagung des akademischen Grades (dr. Sc. nat.), Humboldt Universität, Berlin, 1974.

[8] *Mangasarian, O. L.:* Nonlinear Programming, McGraw-Hill, Inc., New York, London, 1969.

[9] *Nožička, F., Guddat, J., Hollatz, H., Bank, B.:* Theorie der linearen parametrischen Optimierung, Akademie-Verlag, Berlin, 1974.

[10] *Rockafellar, R. T.:* Duality and Stability in Extremum Problems Involving Convex Functions, Pacific J. of Math. 21, 167—187 (1967).

[11] *Rockaffelar, R. T.:* Convex Analysis, Princeton, Princeton University Press, 1969.

[12] *Stoer, J., Witzgall, Ch.:* Convexity and Optimization in Finite Dimensions I, Berlin, Heidelberg, New York, 1970.

Souhrn

KVALITATIVNÍ ANALÝZA ZÁKLADNÍCH POJMŮ PARAMETRICKÉHO KONVEXNÍHO PROGRAMOVÁNÍ, I

(Parametry v omezujících podmínkách)

MOHAMED SAYED ALI OSMAN

V článku je podána kvalitativní analýza základních pojmů parametrického konvexního programování pro konvexní programy s parametry na pravé straně omezujících podmínek. Jsou to pojmy množiny přípustných parametrů, množina řešitelnosti a množiny stability prvního a druhého druhu. Předpokládá se, že vyšetřované funkce mají spojité parciální derivace prvního řádu v R^n a že parametry nabývají libovolných reálných hodnot. Výsledky mohou být použity pro širokou třídu konvexních programů.

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