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Aplikace matematiky, Vol. 22 (1977), No. 4, 256–271

Persistent URL: http://dml.cz/dmlcz/103702

ON THE SOLUTION OF A PLATE WITH RIBS

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(Received April 20, 1976)

In the present paper the problem of a plate with ribs is solved. Introducing the functional of potential energy $F(v)$ of the form (1.5), we may formulate the minimum potential energy principle

$$F(u) = \min_{v \in V(G,n,m)} F(v),$$

where $V(G, n, m)$ is a Hilbert space, defined in (1.2).

For the numerical approach we use the finite element method. We prove the convergence of Ritz-Galerkin approximations to the exact solution u. The main problem consists in the fact that we know nothing about the regularity of u. This is why we have to examine the problem of density of a space of sufficiently smooth functions in $V(G, n, m)$. This paper extends results of [2]. First, we study the case when the plate is supported by a finite number of parallel ribs, then the case of two perpendicular systems of ribs. In the last chapter some practical results are presented.

§ 1. FORMULATION OF THE PROBLEM

The plate in the underformed state occupies the region $G = (-1, 1) \times (-1, 1) \subset \subset R^2$, and two systems of segments $I = \{I_i\}_{i=1}^n$, $J = \{J_j\}_{j=1}^m$, characterizing ribs, are defined in G in the following way:

(1.1a) $$I_i = \{[x, y] \in R_2,\ x = x_i,\ y \in (-1, 1)\},$$

(1.1b) $$J_j = \{[x, y] \in R^2,\ y = y_j,\ x \in (-1, 1)\},$$

where $-1 < x_i < x_j < 1$, $-1 < y_i < y_j < 1$ for $i < j$, x_i, y_j are given real numbers.

In order to formulate the problem, it is necessary to introduce, for the same reason as in [2], the following space:

$$(1.2) \qquad V(G, n, m) = \{u \in H_0^2(G),\ \ \bar{u}^i \in H_0^2(I_i),\ \bar{\bar{u}}^j \in H_0^2(J_j),\ i = 1, \dots, n;\ j = 1, \dots, m\}$$ [1])

where $\bar{u}^i = \bar{u}^i(y) = u(x_i, y)$, $y \in I_i$ is the trace of u on I_i and similarly $\bar{\bar{u}}^j = \bar{\bar{u}}^j(x) = u(x, y_j)$. One can show very easily that $V(G, n, m)$ is a Hilbert space with the scalar product

$$(1.3) \qquad (((u, v))) = \int_G (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy})\, \mathrm{d}x\, \mathrm{d}y + \sum_{i=1}^{n} \int_{I_i} \bar{u}^{i''}\bar{v}^{i''}\, \mathrm{d}y + \sum_{j=1}^{m} \int_{J_j} \bar{\bar{u}}^{j''}\bar{\bar{v}}^{j''}\, \mathrm{d}x$$ [2])

where $\bar{u}^{i''} = \mathrm{d}^2\bar{u}^i/\mathrm{d}y^2$, $\bar{\bar{u}}^{j''} = \mathrm{d}^2\bar{\bar{u}}^j/\mathrm{d}x^2$, $u_{xx} = \partial^2 u/\partial x^2$ etc. We denote by $|||v||| = (((v, v)))^{1/2}$ the corresponding norm.

The problem which is to solve, is defined as follows:

(1.4) *to find* $u \in V(G, n, m)$ *such that*

$$F(u) = \min_{v \in V(G,n,m)} F(v)\,,$$

where

$$(1.5) \qquad F(v) = |||v|||^2 - 2(f, v)\,.$$

Here $f \in (V(G, n, m))'$, i.e. f is a given linear functional on $V(G, n, m)$ and (f, v) is the duality pairing between $V(G, n, m)$ and $(V(G, n, m))'$. With the functional (1.5) we associate the bilinear form $a(u, v)$:

$$(1.6) \qquad a(u, v) = (((u, v)))\,.$$

Using (1.4), (1.6) we can give an equivalent formulation of the problem (1.4):

(1.7) *to find* $u \in V(G, n, m)$ *such that*

$$a(u, v) = (f, v) \quad \forall v \in V(G, n, m)\,.$$

[1]) $H^k(G)$ and $H^k(I_i)$ ($k \geqq 0$ integer) denote the Sobolev space on G and I_i, respectively. For their definitions and characterizations see [3], [4]. $H_0^k(G)$ and $H_0^k(I_i)$ are defined as completions of $D(G)$ and $D(I_i)$ respectively, with respect to the Sobolev norm. $D(G)$ and $D(I_i)$ are the sets of infinitely differentiable functions with compact supports in G and I_i, respectively.

[2]) If one of the systems is empty, the corresponding sum si omitted.

As $a(u, v)$ is continuous on $V(G, n, m)$ and $V(G, n, m)$-elliptic, there exists a unique solution u of (1.7). Integrating by parts, it is easily seen for u sufficiently smooth that u satisfies:

$$\Delta^2 u = f \quad \text{in} \quad G - \left(\bigcup_{i=1}^{n} I_i \cup \left(\bigcup_{j=1}^{m} J_j\right)\right)$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial G \left(\frac{\partial u}{\partial n} \text{ denotes the normal derivative of } u\right)$$

with the following interface conditions along ribs:

$$[Tu]_{I_i} = (\bar{u}^i)^{(4)}, \quad i = 1, \ldots, n;$$

$$[Tu]_{J_j} = (\bar{\bar{u}}^j)^{(4)}, \quad j = 1, \ldots, m,$$

where

$$Tu = 2\frac{\partial}{\partial n}(\Delta u) - \frac{\partial}{\partial t}(u_{xx}n_x n_y - u_{xy}(n_x^2 - n_y^2) - u_{yy}n_x n_y),$$

$n = (n_x, n_y)$ is the unit outward normal to I_i, J_j respectively, $\partial/\partial t$ denotes the tangential derivative (the derivative in the direction of segments I_i, J_j respectively) and $[Tu]_{I_i}$ is the jump of Tu on I_i, i.e. $[Tu]_{I_i} = T\bar{u}^i|_+ - T\bar{u}^i|_-$ (analogously for $[Tu]_{J_j}$).

For the numerical approach to (1.7) we shall use the finite-element method. Let V_h, $h \in (0, 1\rangle$ be a system of finite-dimensional subspaces of $V(G, n, m)$. We say that $u_h \in V_h$ is a *Galerkin approximation of u* iff

$$(1.8) \qquad a(u_h, v) = (f, v) \quad \forall v \in V_h.$$

To prove the convergence of u_h to u, we suppose that there exist a subspace $\mathscr{V} \subset V(G, n, m)$ dense in $V(G, n, m)$ and a mapping $r_h : \mathscr{V} \to V_h$ such that $|||v - r_h v||| \to 0$, $h \to 0+$, $\forall v \in \mathscr{V}$. It is well known that under these conditions $|||u_h - u||| \to 0$, $h \to 0+$ (see [1]).

In the next section we prove that we can take $\mathscr{V} = D(G)$ and $r_h v$ will be taken as an Hermite interpolation of $v \in D(G)$. Unfortunately, as we know nothing about the regularity of u, we can prove nothing about the rate of convergence.

§ 2. PROBLEM OF DENSITY

For our next considerations, we shall use some lemmas.

Lemma 2.1. *For any region $\Omega \subset R^n$ and any number $\varepsilon > 0$* there *exists a function $\eta \in D(R^n)$, such that*:

$$-0 \leqq \eta(x) \leqq 1,$$

$$-\eta(x) = 1 \quad \forall x \in \Omega_\varepsilon\,,$$

$$-\eta(x) = 0 \quad \forall \notin \Omega_{3\varepsilon}\,,$$

where $\Omega_\varepsilon(\varepsilon > 0)$ denotes the ε-neighbourhood of Ω.

Proof: see [5].

Lemma 2.2. *Let a region $\Omega \subset R^n$ be covered by a finite number of open spheres $U(x_k, r_k)$, $k = 1, \ldots, N$ ($r_k > 0$ are diameters, $x_k \in \Omega$). Then there exist functions $h_k \in D(U(x_k, r_k))$ such that*

$$\sum_{k=1}^{N} h_k(x) = 1 \quad \textit{for} \quad \forall x \in \bar{\Omega}\,. \tag{2.1}$$

Proof. For each $k = 1, \ldots, N$ we can construct $U(x_k, r_k')$ $(r_k' < r_k)$ such that $\bigcup_{k=1}^{N} U(x_k, r_k') \supset \bar{\Omega}$. According to Lemma 2.1. there exist functions $\eta_k \in D(U(x_k, r_k))$:

$$\eta_k(x) = 1 \quad \text{for} \quad \forall x \in U(x_k, r_k')\,. \tag{2.2}$$

Let us set

$$h(x) = \sum_{k=1}^{N} \eta_k(x)\,, \quad h_k(x) = \frac{\eta_k(x)}{h(x)}\,. \tag{2.3}$$

It is easy to see that the functions $h_k(x)$ satisfy the statement of our lemma.

First, let us consider the case when one of the system (e.g. J) is *empty*. We start with the simpliest case – with the space $V(G, 1, 0)$. Let us recall that

$$V(G, 1, 0) = \{v \in H_0^2(G),\ \bar{v} \in H_0^2(I_1)\}\,,$$

$$I_1 = \{[x, y],\ x = x_1 = 0,\ y \in (-1, 1)\}\,.$$

Let us define

$$D_0(G) = \{u \in D(G),\ \bar{u} = 0\}\,,$$

$$V_0(G) = \{u \in H_0^2(G),\ \bar{u} = 0\}\,.$$

Lemma 2.3. *$D_0(G)$ is dense in $V_0(G)$ with respect to $V(G, 1, 0)$-norm.*

Proof. Let $u \in V_0(G)$ be arbitrary. We can write

$$u = u_S + u_A\,, \tag{2.4}$$

where

$$u_S(x, y) = 1/2(u(x, y) + u(-x, y)) \quad \text{(symmetric part of } u\text{)} \tag{2.5}$$

$$u_A(x, y) = 1/2(u(x, y) - u(-x, y)) \quad \text{(antisymmetric part of } u\text{).}$$

$\partial u_S/\partial x \in H_0^1(G)$ and according to (2.5), $\partial u_S/\partial x$ is an odd function with respect to x. By Beppo-Levi definition of Sobolev spaces (see [4]), we obtain $(\partial u_S/\partial x)(0, y) = 0$ a.e. on I_1, hence $u_S|_{G_i} \in H_0^2(G_i)$ $(i = 1, 2)$, where $G_1 = (-1, 0) \times (-1, 1)$, $G_2 = (0, 1) \times (-1, 1)$. By the definition of $H_0^2(G_i)$, there exist functions $\Phi_{S_i}^h \in D(G_i)$, $\Phi_{S_i}^h \to u_S|_{G_i}$ in $H_0^2(G_i)$ for $h \to 0+$ (the symbol $u_S|_{G_i}$ denotes the restriction of u to G_i). Setting

$$\varphi_{S_i}^h(x, y) = \begin{cases} \Phi_{S_i}^h(x, y) & \text{for } [x, y] \in G_i\,, \\ 0 & \text{for } [x, y] \in \bar{G} - G_i\,, \quad i = 1, 2\,, \end{cases} \tag{2.6}$$

we obtain

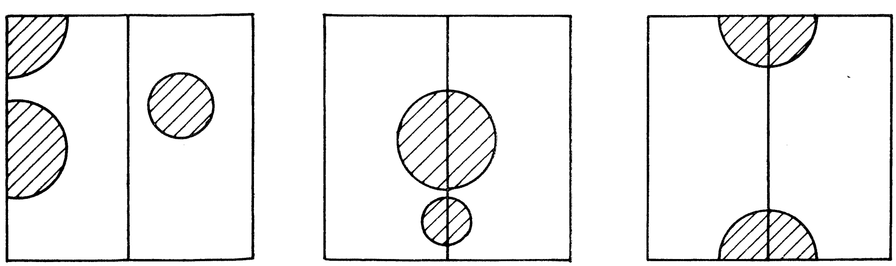
$$\varphi_S^h = \varphi_{S_1}^h + \varphi_{S_2}^h \to u_S \ (h \to 0+) \tag{2.7}$$

with respect to $H_0^2(G)$-norm and even with respect to $V(G, 1, 0)$-norm, because $\bar{u} = \bar{\varphi}_S^h = 0$.

Let us examine the antisymmetric part u_A of u. Let $\{K_i\}_{i=1}^N$ be a "symmetric" covering of G, i.e.:

- $\bigcup_{i=1}^N K_i \supset \bar{G}$
- for any $i \in \{1, \ldots, N\}$ there exists $j \in \{1, \ldots, N\}$ such that for $[x_0, y_0] \in K_i$, the point $[-x_0, y_0] \in K_j$.

From the proof of Lemma 2.1 it is clear (in our case) that the functions $h_k(x)$ giving the partition of unity corresponding to $\{K_i\}_{i=1}^N$, are symmetric with respect to x. As usual, we can write $u_A = \sum_{i=1}^N u_j$, $u_j = uh_j$ and $u_j \in H_0^2(G)$. The open sets $K_j \in \{K_i\}_{i=1}^N$ are of three types (see Fig. 1).



Type I. Type II. Type III.

Fig. 1.

If K_j is of the first type, $u_j|_{G_i} \in H_0^2(G_i)$, then there exist $\varphi_{h,i}^j \in D(G_i)$ $(i = 1, 2)$ such that

$$\varphi_{h,i}^j \to u_j|_{G_i}\,, \quad h \to 0+$$

in $H_0^2(G_i)$. In the same way as in (2.6) we can construct $\varphi_h^j \in D_0(G)$ such that

$$\varphi_h^j \to u_j\,, \quad h \to 0+ \tag{2.8}$$

with respect to $V(G, 1, 0)$-norm.

Let $K_j \in \{K_i\}_{i=1}^N$ be of the second type. Because of $\bar{\Phi}_h^j = 0$, the regularizations Φ_h^j of u_j (see [4]) belong to $D_0(G)$. Indeed,

$$\Phi_h^j(x) = \int_{E_2} \omega(x - y, h)\, u_j(y)\, \mathrm{d}y\,, \quad (\mathrm{d}y = \mathrm{d}y_1\, \mathrm{d}y_2, \quad x = [x_1, x_2])$$

u_j is antisymmetric and the kernel $\omega(x - y, h)$ is a symmetric function with respect to x. Hence

$$\Phi_h^j \to u_j\,, \quad h \to 0+ \tag{2.9}$$

with respect to $V(G, 1, 0)$-norm. Finally, let K_j be of the third type. We define $U_{j,L}(x, y) = u_j(x, y + aL)$, where $a = \pm 1$, $L > 0$ sufficiently small. The sign of a is chosen in such a way that $U_{j,L} \in H_0^2(G)$, i.e. $U_{j,L}$ has a compact support in G. From the mean value convergence theorem we obtain $U_{j,L} \to u_j$, $L \to 0+$ with respect to $V(G, 1, 0)$-norm. The regularizations $\Phi_{L,h}^j$ of $U_{j,L}$ belong to $D_0(G)$ for the same reason as in the previous case and

$$\Phi_{L,h}^j \to U_{j,L}\,, \quad h \to 0+ \tag{2.10}$$

in $V(G, 1, 0)$. Let us set

$$\varphi_{A,L}^h = \sum_{j \in \iota_1} \varphi_h^j + \sum_{j \in \iota_2} \Phi_h^j + \sum_{j \in \iota_3} \Phi_{L,h}^j \in D_0(G)\,, \tag{2.11}$$

where ι_k are sets of all indices i of $K_i \in \{K_j\}_{j=1}^N$ of the k-th type ($k = 1, 2, 3$). From the triangle inequality and (2.8), (2.9), (2.10) we conclude

$$\|\varphi_{A,L}^h - u_A\|_{H_2{}^0(G)} = |||\varphi_{A,L}^h - u_A||| \to 0 \tag{2.12}$$

for $h, L \to 0+$. Setting

$$\varphi_{h,L} = \varphi_S^h + \varphi_{A,L}^h\,,$$

using (2.7), (2.12) and the triangle inequality, we obtain

$$|||u - \varphi_{h,L}||| \leqq |||u_S - \varphi_S^h||| + |||u_A - \varphi_{A,L}^h||| \to 0$$

for $h, L \to 0+$. Lemma is proved.

Theorem 1.2. *$D(G)$ is dense in $V(G, 1, 0)$ with respect to the $V(G, 1, 0)$-norm.*

Proof. It can be carried out with the aid of the previous lemmas analogously as in [2].

Immediately, one can extend the previous results to the case of a plate with a finite number of parallel ribs (not necessarily equidistant). Let $I = \{I_i\}_{i=1}^n$ be a system (1.1a). Then one can prove the following

Theorem 2.2. *$D(G)$ is dense in $V(G, n, 0)$ with respect to the norm of $V(G, n, 0)$.*

Proof. Using the technique of the partition of unity and Theorem 1.2, we obtain the statement of Theorem 2.2.

Until now we have supposed that the system J is empty. Now, let us consider the case when both *I and J are nonempty*. We start again with the simpliest case. Let

$$\begin{aligned} I_1 &= \{[x, y] \in R^2,\ x = 0,\ y \in (-1, 1)\}, \\ J_1 &= \{[x, y] \in R^2,\ x \in (-1, 1),\ y = 0\}, \\ D_{00}(G) &= \{u \in D(G),\ \bar{u} = 0,\ \bar{\bar{u}} = 0\}, \\ V_{00}(G) &= \{u \in H_0^2(G),\ \bar{u} = 0, \bar{\bar{u}} = 0\}. \end{aligned}$$

Lemma 2.4. *$D_{00}(G)$ is dense in $V_{00}(G)$ with respect to the $V(G, 1, 1)$-norm.*

Proof. Let $u \in V_{00}(G)$ be arbitrary. The function u can be written in the following way:

$$u = u_{s_1} + u_{s_2} + u_{A_1} + u_{A_2} \tag{2.13}$$

where

$$\begin{aligned} u_{s_1}(x, y) &= \tfrac{1}{4}(u(x, y) + u(x, -y) + u(-x, y) + u(-x, -y)), \\ u_{s_2}(x, y) &= \tfrac{1}{4}(u(x, y) + u(-x, -y) - u(-x, y) - u(x, -y)), \\ u_{A_1}(x, y) &= \tfrac{1}{4}(u(x, y) - u(-x, -y) + u(x, -y) - u(-x, y)), \\ u_{A_2}(x, y) &= \tfrac{1}{4}(u(x, y) - u(-x, -y) - u(x, -y) + u(-x, y)), \quad [x, y] \in G. \end{aligned}$$

It is readily seen that all terms on the right hand side of (2.13) belong to $V_{00}(G)$.

u_{s_1} is symmetric with respect to x as well as to y. Hence $\partial u_{s_1}/\partial x$, $\partial u_{s_1}/\partial y$ are odd functions with respect to x and y, respectively. By Beppo-Levi definition of Sobolev spaces and using the same considerations as in Theorem 1.2 we obtain $u_{s_1}|_{G_i} \in H_0^2(G_i)$ ($i = 1, 2, 3, 4$), where

$$\begin{aligned} G_1 &= (0, 1) \times (0, 1), & G_2 &= (-1, 0) \times (0, 1), \\ G_3 &= (-1, 0) \times (-1, 0), & G_4 &= (0, 1) \times (-1, 0). \end{aligned}$$

Let

$$u_{s_1}^i = \begin{cases} u_{s_1}|_{G_i} & \text{on } G_i, \\ 0 & \text{on } \bar{G} - G_i. \end{cases}$$

Then $u^i_{s_1} = \sum_{i=1}^{4} u^i_{s_1}$. By the definition of $H^2_0(G_i)$ there exist functipns $\chi^i_h \in D(G_i)$ $(i = 1, \ldots, 4)$ such that $\chi^i_h \to u_{s_1}|_{G_i}$ for $h \to 0+$ in $H^2_0(G_i)$.

If

$$\zeta^i_h = \begin{cases} \chi^i_h & \text{on } G_i , \\ 0 & \text{on } \bar{G} - G_i , \end{cases}$$

we obtain $\zeta^i_h \to u^i_{s_1}$, $h \to 0+$ in $V(G, 1, 1)$ and from the triangle inequality:

$$\varphi^h_{s_1} \to u_{s_1} , \quad h \to 0+ \quad \text{in} \quad V(G, 1, 1) , \tag{2.14}$$

where

$$\varphi^h_{s_1} = \sum_{i=1}^{4} \zeta^i_h \in D_{00}(G) .$$

Now, let us examine u_{A_1} (analogously u_{A_2}). u_{A_1} is symmetric with respect to y and odd with respect to x. Using the same considerations as in the previous case, we deduce that

$$u_{A_i}|_{F_i} \in V_0(F_i) \quad (i = 1, 2) ,$$

where

$$F_1 = (-1, 1) \times (0, 1) , \quad F_2 = (-1, 1) \times (-1, 0) .$$

Using the statement of Theorem 1.2, we find that there exist $\varphi^h_{A_1,i} \in D_0(F_i)$:

$$\varphi^h_{A_1,i} \to u_{A_1}|_{F_i} \text{ in } V_0(F_i) .$$

Let us set

$$\varphi^h_{A_1} = \begin{cases} \varphi^h_{A_1,1} & \text{on } F_1 , \\ \varphi^h_{A_1,2} & \text{on } F_2 . \end{cases}$$

Then $\varphi^h_{A_1} \in D_{00}(G)$ and

$$|||u_{A_1} - \varphi^h_{A_1}||| \to 0 \quad \text{for} \quad h \to 0+ .$$

In the same way we obtain

$$|||u_{A_2} - \varphi^h_{A_2}||| \to 0 \quad \text{for} \quad h \to 0+ .$$

The situation is more complicated for u_{s_2}.

Let the system $\{K_j\}_{j=1}^N$ of open circles cover G and let the following condition, concerning their mutual position, be satisfied:

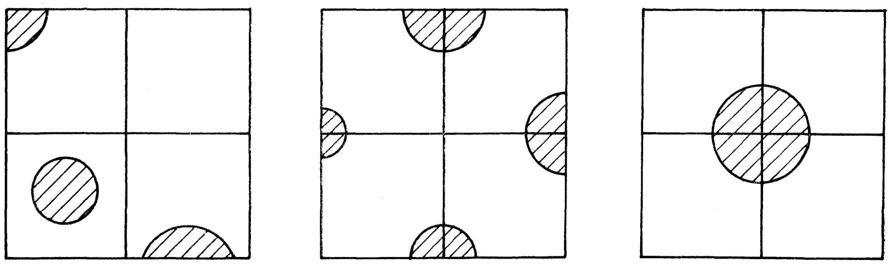
(2.15) *for each* $i \in \{1, \ldots, N\}$ *there exist* $i_1, i_2, i_3 \in \{1, \ldots, N\}$ *(not necessarily different), such that for any* $x = [x_0, y_0] \in K_i$, *the points* $x_1 = [-x_0, y_0] \in K_{i_1}$, $x_2 = [x_0, -y_0] \in K_{i_2}$ and $x_3 = [-x_0, -y_0] \in K_{i_3}$.

$K_i \in \{K_j\}_{j=1}^N$ are of three types (see Fig. 2).

Using the technique of the partition of unity corresponding to $\{K_i\}_{i=1}^N$, we can write $u_{s_2} = \sum_{j=1}^N u_j$, where $u_j = h_j \,.\, u_{s_2}$, and $h_j \in D(K_j)$ satisfy (2.1). From the proof of Lemma 2.2 one can see immediately that

$$(2.16) \qquad h_i(x_0, y_0) = h_{i_1}(-x_0, y_0) = h_{i_2}(x_0, -y_0) = h_{i_3}(-x_0, -y_0),$$

where $i, i_1, i_2, i_3 \in \{1, \ldots, N\}$ satisfy (2.15).



Type I. Type II. Type III.

Fig. 2.

The function u_{s_2} is antisymmetric with respect to x and y. From this and (2.16) we can reach the assertion of lemma 2.4 in the same way as in the proof of Lemma 2.3.

Lemma 2.5. *Let*

$$V_1 = \{u \in H_0^2((-1, 1)),\ u(0) = 0\},$$
$$D_1 = \{u \in D((-1, 1)),\ u(0) = 0\}.$$

Then D_1 is dense in V_1 with respect to $H_0^2((-1, 1))$-norm.

Proof. We can express again u as a sum of a symmetric and an antisymmetric part:

$$u = u_S + u_A.$$

The rest of the proof is analogous to that of Lemma 2.3.

The main result is contained in

Theorem 2.3. *$D(G)$ is dense in $V(G, 1, 1)$ with respect to $V(G, 1, 1)$-norm.*

Proof. Let $u \in V(G, 1, 1)$ be an arbitrary function. First, we introduce some auxiliary functions:

$$(2.17) \quad \varphi \in D(G), \quad 0 \leqq \varphi(x, y) \leqq 1, \quad \varphi(x, y) = 1 \quad \text{for} \quad x^2 + y^2 \leqq 1/4,$$

(2.18) $\varphi^x \in D(I_1)\,,\quad \varphi^y \in D(J_1)\,,\quad \varphi^x(x) = 1 \quad \text{for} \quad x \in \langle -1 + \alpha;\ 1 - \alpha \rangle\,,$

$$\varphi^y(y) = 1 \quad \text{for} \quad y \in \langle -1 + \alpha,\ 1 - \alpha \rangle\ (\alpha \in (0, 1))\,,$$

$0 \leqq \varphi^x, \varphi^y \leqq 1.$

Further, we define

$$\begin{cases} u_2 = \bar{u} - u(0, 0)\, \varphi(0, y) & \text{on} \quad I_1\,, \\ u_3 = \bar{\bar{u}} - u(0, 0)\, \varphi(x, 0) & \text{on} \quad J_1\,. \end{cases} \tag{2.19}$$

It is easy to see that $u_2 \in H_0^2(I_1)$, $u_3 \in H_0^2(J_1)$, $u_2(0) = u_3(0) = 0$. By Lemma 2.5 there exist functions χ_2^n, χ_3^n from $D(I_1)$, $D(J_1)$ respectively, $\chi_2^n(0) = \chi_3^n(0) = 0$,

$$\chi_2^n \to u_2\,, \quad \chi_3^n \to u_3 \quad \text{in} \quad H_0^2(I_1)\,, \quad H_0^2(J_1) \tag{2.20}$$

respectively. Let

$$\begin{aligned} \Phi^x(x, y) &= \varphi^x(x)\,, \\ \Phi^y(x, y) &= \varphi^y(y)\,, \\ \tilde{\chi}_2^n(x, y) &= \chi_2^n(y)\,, \\ \tilde{\chi}_3^n(x, y) &= \chi_3^n(x)\,, \\ \tilde{U}_2(x, y) &= u_2(y)\,, \\ \tilde{U}_3(x, y) &= u_3(x)\,, \\ \chi_2^n(x, y) &= \tilde{\chi}_2^n(x, y)\, \Phi^x(x, y)\,, \\ \chi_3^n(x, y) &= \tilde{\chi}_3^n(x, y)\, \Phi^y(x, y)\,, \\ U_2(x, y) &= \tilde{U}_2(x, y)\, \Phi^x(x, y)\,, \\ U_3(x, y) &= \tilde{U}_3(x, y)\, \Phi^y(x, y)\,. \end{aligned} \tag{2.21}$$

It is readily verified that $\chi_2^n, \chi_3^n \in D(G)$ and

$$\chi_2^n \to U_2\,, \quad \chi_3^n \to U_3\,, \quad n \to \infty \quad \text{in} \quad V(G, I, I)\,. \tag{2.22}$$

Indeed,

$$\frac{\partial^2 \tilde{U}_1}{\partial x^2} = \frac{\partial^2 \tilde{U}_2}{\partial x\, \partial y} = \frac{\partial^2 \tilde{U}_3}{\partial y^2} = \frac{\partial^2 \tilde{U}_3}{\partial x\, \partial y} = \frac{\partial^2 \tilde{\chi}_2^n}{\partial x^2} = \frac{\partial^2 \tilde{\chi}_2^n}{\partial x\, \partial y} = \frac{\partial^2 \tilde{\chi}_3^n}{\partial y^2} = \frac{\partial^2 \tilde{\chi}_3^n}{\partial x\, \partial y} = 0 \tag{2.23}$$

from (2.21) and by Fubini's theorem

$$\frac{\partial^2 \tilde{\chi}_2^n}{\partial y^2} \to \frac{\partial^2 \tilde{U}_2}{\partial y^2}\,, \quad \frac{\partial^2 \tilde{\chi}_3^n}{\partial x^2} \to \frac{\partial^2 \tilde{U}_3}{\partial x^2}\,, \quad n \to \infty$$

in $L^2(G)$.

By definition of U_2, U_3, χ_2^n, χ_3^n we have

(2.24)
$$\begin{aligned} \chi_2^n &\to U_2 \\ \chi_3^n &\to U_3 \quad \text{in} \quad H_0^2(G) . \end{aligned}$$

Further,

$$\begin{aligned} \bar{\chi}_2^n &= \tilde{\chi}_2^n(0, y)\, \Phi^x(0, y) = \chi_2^n(y) , \\ \bar{\bar{\chi}}_3^n &= \tilde{\chi}_3^n(x, 0)\, \Phi^y(x, 0) = \chi_3^n(x) , \\ \bar{U}_2 &= u_2 , \\ \bar{\bar{U}}_3 &= u_3 \end{aligned}$$

and from (2.20) we obtain

(2.25)
$$\begin{aligned} \bar{\chi}_2^n &\to \bar{U}_2 \quad \text{in} \quad H_0^2(I_1) , \\ \bar{\bar{\chi}}_3^n &\to \bar{\bar{U}}_3 \quad \text{in} \quad H_0^2(J_1) . \end{aligned}$$

Moreover,

(2.26)
$$\bar{\bar{\chi}}_2^n = \bar{\bar{U}}_2 = \bar{\chi}_3^n = \bar{U}_3 = 0 \quad \forall n \in N .$$

Thus (2.24), (2.25), (2.26) yield (2.22).

Finally, let us set

(2.27)
$$\begin{aligned} U(x, y) &= u(0, 0)\, \varphi(x, y) + U_2(x, y) + U_3(x, y) , \\ \chi_n(x, y) &= u(0, 0)\, \varphi(x, y) + \chi_2^n(x, y) + \chi_3^n(x, y) . \end{aligned}$$

Then $\chi_n \in D(G), \bar{U} = \bar{u}, \bar{\bar{U}} = \bar{\bar{u}}$ and

(2.28)
$$\chi_n \to U \text{ with respect to } V(G, 1, 1)\text{-norm.}$$

The function u can be written in the form

(2.29)
$$u = U + Z ,$$

where U is defined by (2.27). Hence it follows that $Z \in V_{00}(G)$. Using Lemma 2.4 we see that there exist $\zeta_n \in D_{00}(G)$ such that

(2.30)
$$\zeta_n \to Z \quad \text{in} \quad V(G, 1, 1) .$$

It is readily seen that

$$\varphi_n = \chi_n + \zeta_n \in D(G)$$

and

$$|||u - \varphi_n||| \leqq |||U - \chi_n||| + |||\zeta_n - Z||| \to 0 , \quad n \to \infty .$$

Thus the theorem is proved.

Using the partition of unity and the statement of Theorem 2.3, one can prove

Theorem 2.4. *$D(G)$ is dense in $V(G, n, m)$ with respect to the $V(G, n, m)$-norm.*

§ 3. NUMERICAL EXPERIENCE

As has been explained above the analysis of bending of a plate with ribs can be regarded as a combination of a plate flexure analysis and a beam flexure analysis (see also the type of the energy functional (1.5) or (3.3) below). Under Kirchhoff's and Bernoulli's hypotheses, the bending energies of the plate and the beam have the following form, respectively:

$$F_p(w) = \frac{Et^3}{12(1 - \nu^2)} \int_G \{\nu(\Delta w)^2 + (1 - 2\nu) \times (w_{xx}^2 + 2w_{xy}^2 + w_{yy}^2)\} \, dx \, dy \,, \tag{3.1}$$

$$F_b(w) = \sum_{i=1}^{n} \int_{I_i} EJ(w'')^2 \, d\gamma \tag{3.2}$$

where E is Young's modulus, J the inertia momentum, w is the deflection function and the summation in (3.2) is taken over the ribs. The internal energy of construction can be expressed as follows:

$$F_i(w) = F_p(w) + F_b(w) \,. \tag{3.3}$$

The explicit form of the stiffness matrix of a beam with degrees of freedom w, w' and of an element of Ahlin's type is shown in many papers. For this reason it is sufficient to give the Hooke law, corresponding to (3.3), in the matrix form:

$$\{\sigma\} = [D] \{\varepsilon\}$$

where

$$[D] = \frac{Et^3}{12(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1 - \nu) \end{bmatrix},$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix},$$

σ_x, σ_y, τ_{xy} are normal and shear stresses, respectively, ε_x, ε_y, γ_{xy} deflections, t is the thickness of the plate, E Young's modulus and ν is Poisson's ratio.

In order to test the convergence of our numerical approach in the sense of the finite element method in practice, two model examples were solved. The results have been compared with solutions that were obtained by the folded plate method — see [6].[1]) The geometry of the specimens under consideration has been chosen with

[1]) Thanks are due to Doc. Křístek and Ing. Kvasnička, ČVUT, Prague for providing the results needed.

respect to the extraordinary precision of the folded plate method in this case. Error analysis with respect to the different mesh size is given in Tables 1 and 2. The errors are computed according to the following formula:

$$\frac{\text{finite element method} - \text{folded plate method}}{\text{folded plate method}} \; 100\%$$

where plus or minus indicates over- and underestimation respectively.

In the first example a square plate $2 \times 2\ m^2$ is stiffened by one rib $2 \times 12\ cm^2$ and subjected to the uniformly distributed load $p = 1\ kp/cm^2$. The edges which are parallel to the ribs are clamped and the remaining edges are simply supported (see Fig. 3, where various grids used are shown as well).

From Table 1 it is readily seen that the finite element results are in a good agree-

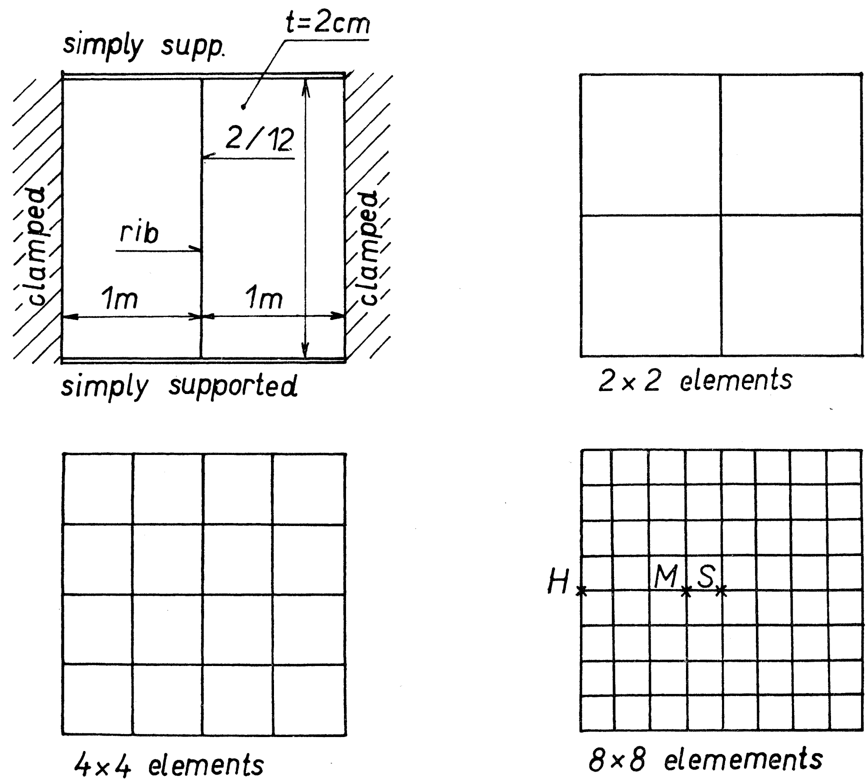


Fig. 3.

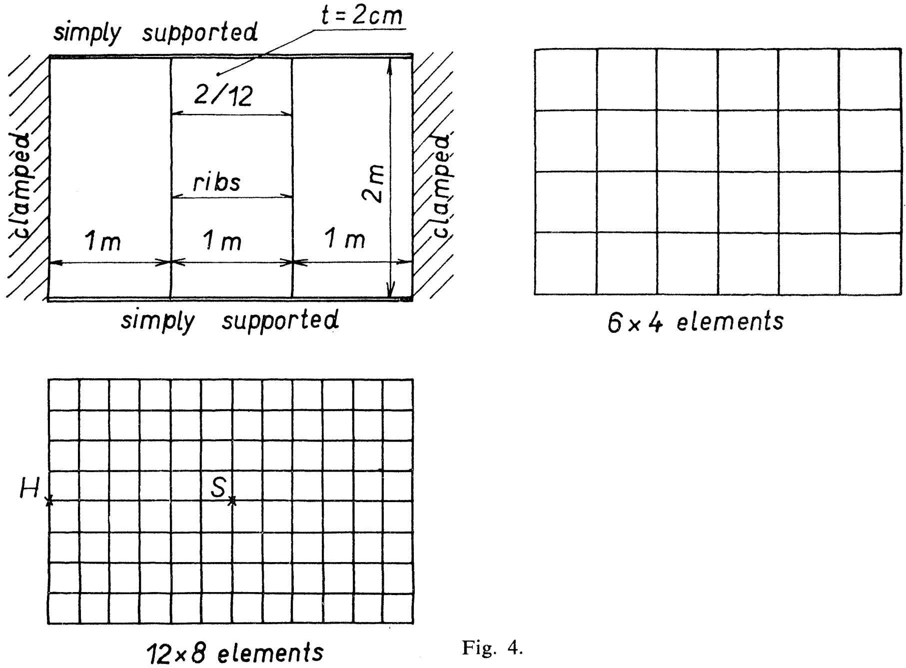


Fig. 4.

ment with the folded plate solution and that they do converge towards the exact values. The bending moments are defined as usual:

$$M_x = -D(w^2_{xx} + \nu w^2_{yy}),$$

$$M_y = -D(w^2_{yy} + \nu w^2_{xx}),$$

$$D = \frac{Et^3}{12(1 - \nu^2)}.$$

In order to investigate the situation when more ribs are taken into account, we have considered a problem of a rectangular plate $3 \times 2\ \mathrm{m}^2$ with two ribs, $2 \times 12\ \mathrm{cm}^2$. The boundary conditions are the same as in the first example. The plate is subjected to the uniformly distributed load $p = 1\ \mathrm{kp/cm}^2$. The geometry and the grids used are shown in Fig. 4. The results obtained for this problem are compared with the folded plate solution in Table 2. The agreement between the finite element method and the folded plate method is reasonably good.

Table 1

Number elements	Central deflection		Central bending noments				Maximal positive bending moments		Minimal negative bending moments	
	w_S		M_{xS}		M_{yS}		M_M		M_H	
	value	error	value	error	value	error	value	error	value	error
2 × 2	1·22	−4·05	926·116	55·4	1282·286	113·6	—	—	−1125·76	−46
4 × 4	1·244	−2·16	802·857	34·6	714·658	19·02	—	—	−1870·785	− 9·8
8 × 8	1·268	−0·28	617·685	3·54	639·823	6·64	818·709	6·95	−1922·526	− 7·75
folded plate	1·272		596·550		600·090		765·681		−2079·749	

Table 2

Number of elements	Central deflections		Maximal positive bending moments				Minimal negative bending moments	
	w_S		M_{xS}		M_{yS}		M_H	
	value	error	value	error	value	error	value	error
6 × 4	1·973	−18·5	830·811	−23·27	777·441	−22	−2228·447	−2·61
12 × 8	2·281	− 5·74	993·378	− 8·29	895·733	−10·2	−2290·687	−0·356
folded plate	2·42		1083·29		996·72		−2298·968	

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Souhrn

ŘEŠENÍ PROBLÉMU DESKY S ŽEBRY

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V této práci je studován problém desky, podepřené žebry. K numerickému řešení tohoto problému užíváme metodu konečných prvků. Protože však není nic známo o hladkosti řešení daného problému, je třeba zkoumat problém hustoty dostatečně hladkých funkcí ve výchozím energetickém prostoru. Tomuto problému je věnován § 2. Zkoumáme jednak případ navzájem rovnoběžných žeber, jednak případ k sobě kolmých žeber (roštu). Tato práce doplňuje a rozšiřuje výsledky obsažené v [2].

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