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DUAL FINITE ELEMENT ANALYSIS FOR ELLIPTIC PROBLEMS WITH OBSTACLES ON THE BOUNDARY, I

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INTRODUCTION

Recently, Mosco and Strang [5] have published an error analysis for a finite element procedure applied to unilateral problems with an obstacle in the domain. Using some ideas of their approach, the dual finite analysis has been accomplished in [3] for unilateral problems with conditions of Signorini's type on the boundary, i.e., with boundary obstacles given by a zero function.

In the present paper we extend the results of [3] to some problems with non-homogeneous obstacles on the boundary. The dual finite element procedures are proposed using piecewise linear polynomials on triangulations of the given domain and $O(h)$ convergence in energy norm proved, provided the solution is sufficiently regular. Some a posteriori error estimates and two-sided bounds for the energy of the solution are also derived.

1. THE DUAL VARIATIONAL FORMULATIONS

Let us consider the following model problem

$$(1.1) \qquad -\Delta u + u = f \quad \text{in} \quad \Omega \subset R^n ,$$

$$u - g \geqq 0 , \quad \partial u/\partial \nu \geqq 0 , \quad (u - g)\, \partial u/\partial \nu = 0 \quad \text{on} \quad \partial\Omega \equiv \Gamma ,$$

where $\partial u/\partial \nu$ denotes the derivative with respect to the outward normal $\boldsymbol{\nu}$ and f, g are given functions. Let Ω be a bounded domain with Lipschitz boundary (cf. e.g. [1] for the definition). Henceforth we use the Sobolev spaces $H^k(\Omega)$ with the usual norms $\|u\|_k$, $H^0(\Omega) = L_2(\Omega)$, $x = (x_1, x_2, \ldots, x_n)$,

$$(u, v)_0 = \int_\Omega uv \, \mathrm{d}x ,$$

$$(u, v)_1 = (u, v)_0 + \sum_{i=1}^{n} (\partial u/\partial x_i, \partial v/\partial x_i)_0 .$$

Assume that $f \in L_2(\Omega)$ and that a function $G \in H^2(\Omega)$ exists such that $G = g$ on the boundary Γ.[1])

The problem (1.1) can be recast as follows. Introduce the convex set

$$\mathscr{K} = \{v \mid v \in H^1(\Omega), \quad \gamma v - g \geqq 0 \quad \text{on} \quad \Gamma\},$$

where γv denotes the trace of v on the boundary, and the functional (potential energy)

$$\mathscr{L}(v) = \tfrac{1}{2}\|v\|_1^2 - (f, v)_0 .$$

Then the problem to find $u \in \mathscr{K}$ such that

$$\mathscr{L}(u) \leqq \mathscr{L}(v) \quad \forall v \in \mathscr{K} \tag{1.2}$$

represents a variational formulation of the problem (1.1) and it will be called *primary*.

The problem can be reformulated in terms of the gradient-vector (cf. [3]). To this end we introduce the set

$$Q = \{\boldsymbol{q} \in [L_2(\Omega)]^n, \ \operatorname{div} \boldsymbol{q} \in L_2(\Omega)\},$$

where the operator

$$\operatorname{div} \boldsymbol{q} = \sum_{i=1}^{n} \partial q_i / \partial x_i$$

is defined in the sense of distributions. For $\boldsymbol{q} \in Q$, we may define the functional $\boldsymbol{q} \,.\, \boldsymbol{v} \in H^{-1/2}(\Gamma)$ by means of the relation[2])

$$\langle \boldsymbol{q} \,.\, \boldsymbol{v}, \gamma v \rangle = \int_\Omega (\boldsymbol{q} \,.\, \operatorname{grad} v + v \operatorname{div} \boldsymbol{q}) \, dx \quad \forall v \in H^1(\Omega) . \tag{1.3}$$

We write $s \geqq 0$ for an $s \in H^{-1/2}(\Gamma)$ if

$$\langle s, \gamma v \rangle \geqq 0 \quad \forall v \in \mathscr{C},$$

where

$$\mathscr{C} = \{v \in H^1(\Omega), \ \gamma v \geqq 0 \ \text{on} \ \Gamma\} .$$

Finally, introduce the set

$$\mathscr{U} = \{\boldsymbol{\lambda} \in [L_2(\Omega)]^{n+1}, \quad \boldsymbol{\lambda} = [\boldsymbol{\lambda}', \lambda_{n+1}], \quad \boldsymbol{\lambda}' \in Q, \quad \lambda_{n+1} = f + \operatorname{div} \boldsymbol{\lambda}', \quad \boldsymbol{\lambda}' \,.\, \boldsymbol{v} \geqq 0 \ \text{on} \ \Gamma\} \tag{1.4}$$

and the functional (complementary energy)

$$\mathscr{S}_g(\boldsymbol{\lambda}) = \tfrac{1}{2} \sum_{i=1}^{n+1} \|\lambda_i\|_0^2 - \langle \boldsymbol{\lambda} \,.\, \boldsymbol{v}, g \rangle . \tag{1.5}$$

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[1]) See e. g. [2], where some sufficient conditions for the existence of G are presented.

[2]) Henceforth $\boldsymbol{a} \,.\, \boldsymbol{b}$ denotes the scalar product $\sum_{i=1}^{n} a_i b_i$. See e.g. [1] for the definition of $H^{-1/2}(\Gamma)$.

The problem to find $\boldsymbol{\lambda}^0 \in \mathscr{U}$ such that

$$(1.6) \qquad \mathscr{S}_g(\lambda^0) \leqq \mathscr{S}_g(\boldsymbol{\lambda}) \quad \forall \boldsymbol{\lambda} \in \mathscr{U} ,$$

will be called *dual* to the primary problem (1.2).

It is easy to prove that both the primary and the dual problem possesses a unique solution. Moreover, there is an interpretation of the solution to the dual problem in terms of the solution to the primary problem.

Theorem 1.1. *If u is the solution to the primary problem* (1.2) *and $\boldsymbol{\lambda}^0$ the solution to the dual problem* (1.6), *then*

$$(1.7) \qquad \lambda_i^0 = \partial u / \partial x_i , \quad i = 1, \ldots, n , \quad \lambda_{n+1}^0 = u .$$

Proof. First we rewrite the dual problem into an equivalent one. Setting

$$(1.8) \qquad \lambda_i = p_i + \partial G / \partial x_i , \quad i = 1, \ldots, n ,$$

$$\lambda_{n+1} = p_{n+1} + G ,$$

we may write for $\boldsymbol{\lambda} \in \mathscr{U}$

$$(1.9) \qquad \mathscr{S}_g(\boldsymbol{\lambda}) = \mathscr{S}(\boldsymbol{p}) + (G, f)_0 - \tfrac{1}{2}\|G\|_1^2 ,$$

where

$$\mathscr{S}(\boldsymbol{p}) = \tfrac{1}{2} \sum_{i=1}^{n+1} \|p_i\|_0^2 .$$

It is readily seen that $\boldsymbol{\lambda} \in \mathscr{U}$ if and only if $\boldsymbol{p} \in \mathscr{U}_G$, where

$$\mathscr{U}_G = \{\boldsymbol{p} = [\boldsymbol{p}', p_{n+1}] , \quad \boldsymbol{p}' \in Q , \quad p_{n+1} = f + \Delta G - G + \operatorname{div} \boldsymbol{p}' ,$$

$$\boldsymbol{p}' . \boldsymbol{v} + \partial G / \partial v = 0 \quad \text{on} \quad \Gamma\} .$$

Consequently, the problem to find $\boldsymbol{p}^0 \in \mathscr{U}_G$ such that

$$(1.10) \qquad \mathscr{S}(\boldsymbol{p}^0) \leqq \mathscr{S}(\boldsymbol{p}) \quad \forall \boldsymbol{p} \in \mathscr{U}_G$$

is equivalent with the dual problem (1.6).

We can prove the following

Lemma 1.1. *There exists $w \in H_+^{1/2}(\Gamma)$ such that*

$$(1.11) \qquad \mathscr{S}(\boldsymbol{p}^0) - \langle \boldsymbol{p}^0 . \boldsymbol{v} + \partial G / \partial v, \mu \rangle \leqq \mathscr{S}(\boldsymbol{p}^0) - \langle \boldsymbol{p}^0 . \boldsymbol{v} + \partial G / \partial v, w \rangle \leqq$$

$$\leqq \mathscr{S}(\boldsymbol{p}) - \langle \boldsymbol{p} . \boldsymbol{v} + \partial G / \partial v, w \rangle$$

holds for any $\mu \in H_+^{1/2}(\Gamma)$, $\boldsymbol{p} \in Q_{fG}$, where

$$H_+^{1/2} = \{v \in H^{1/2}(\Gamma), v \geqq 0\} ,$$

$$Q_{fG} = \{\mathbf{p} \in [L_2(\Omega)]^{n+1},\ \mathbf{p} = [\mathbf{p}', p_{n+1}],\ \mathbf{p}' \in Q,\ p_{n+1} = f + \Delta G - G + \operatorname{div} \mathbf{p}'\} .$$

Moreover,

$$\langle \mathbf{p}^0 \,.\, \mathbf{v} + \partial G/\partial v, w\rangle = 0 .$$

Proof of this Lemma is based on a Corollary of Hahn-Banach theorem being parallel to that of Lemma 1.1 in [3].

Using Lemma 1.1 and following the proof of Theorem 1.1 in [3], we show that

$$\mathbf{p}^0 = [\mathbf{p}^{0\prime}, p^0_{n+1}] , \quad \mathbf{p}^{0\prime} = \operatorname{grad} \tilde{u} , \quad p^0_{n+1} = \tilde{u} , \tag{1.12}$$

where $\tilde{u}$ solves the problem

$$-\Delta\tilde{u} + \tilde{u} = f + \Delta G - G \quad \text{in} \quad \Omega , \quad \gamma\tilde{u} = w \quad \text{on} \quad \Gamma . \tag{1.13}$$

Finally, let us prove that $\tilde{u} = u - G$, which will complete the proof of Theorem 1.1, by virtue of (1.8), (1.10) and (1.12). Setting $V = v - G$, $U = u - G$, we have $V \in \mathscr{C}$, $U \in \mathscr{C}$ (see (1.3)). The function u is a solution of (1.2), precisely if

$$(u, v - u)_1 \geqq (f, v - u)_0 \quad \forall v \in \mathscr{K} .$$

Thus for U we obtain an equivalent version:

$$(U, V - U)_1 \geqq (f, V - U)_0 - (G, V - U)_1 \quad \forall V \in \mathscr{C} . \tag{1.14}$$

Inserting $V = 0$ and $V = 2U$, we derive

$$(U, U)_1 = (f, U)_0 - (G, U)_1 . \tag{1.15}$$

Consequently, (1.15) and (1.14) result in

$$(U, V)_1 = (f, V)_0 - (G, V)_1 \quad \forall V \in \mathscr{C} . \tag{1.16}$$

U is a solution of (1.14) if and only if it satisfies (1.15), (1.16). Let us verify (1.15), (1.16) for $\tilde{u}$. In fact, we have

$$0 \leqq \left\langle \frac{\partial}{\partial v} (\tilde{u} + G), \gamma V \right\rangle = \int_\Omega [\operatorname{grad} (\tilde{u} + G) \,.\, \operatorname{grad} V + V \operatorname{div} \operatorname{grad} (\tilde{u} + G)] \, dx \quad \forall V \in \mathscr{C} ,$$

where (1.12) and the definition of $\mathscr{U}_G$ has been used.

On the other hand, from (1.13) we obtain that

$$\operatorname{div} \operatorname{grad} (\tilde{u} + G) = \tilde{u} + G - f .$$

Consequently,

$$0 \leqq \int_\Omega [\operatorname{grad} (\tilde{u} + G) \,.\, \operatorname{grad} V + V(\tilde{u} + G - f)] \, dx = (\tilde{u} + G, V)_1 - (V, f)_0 \quad \forall V \in \mathscr{C} ,$$

i.e., (1.16) is satisfied for $U = \tilde{u}$.

Making use of Lemma (1.1), we may write

$$0 = \left\langle \frac{\partial}{\partial \nu} (\tilde{u} + G), \gamma u \right\rangle = (\tilde{u} + G, \tilde{u})_1 - (f, \tilde{u})_0 ,$$

which is (1.15). Q.E.D.

2. FINITE ELEMENT APPROXIMATIONS OF THE PRIMARY PROBLEM

To propose a consistent dual finite element procedure, we restrict ourselves to plane polygonal domains (multiply connected, in general). Thus let Ω be a polygonal bounded domain. We carve it into triangles T, generating a triangulation $\mathscr{T}_h$. Denote h the maximal side of all triangles in $\mathscr{T}_h$ and S_h the space of continuous (in Ω) piecewise linear functions on $\mathscr{T}_h$. Henceforth we shall consider only α-β-regular families of triangulations $\{\mathscr{T}_h\}$, $0 < h \leqq 1$, i.e. such that positive parameters α, β exist, independent of h, and such that (i) no angle of all the triangles in $\mathscr{T}_h$ is less than α, (ii) the ratio of any two sides in $\mathscr{T}_h$ is less than β.

Let us define g_h as the linear interpolate of g on Γ with the nodes determined by the vertices of the triangulation $\mathscr{T}_h$.

Introduce the following sets:

$$\mathscr{K}_h = \{v \in S_h, \gamma v - g_h \geqq 0 \text{ on } \Gamma\} ,$$

$$\mathscr{C}_h = \{v \in S_h, \gamma v \geqq 0 \text{ on } \Gamma\} = \mathscr{C} \cap S_h .$$

We say that $u_h \in \mathscr{K}_h$ is a finite element approximation of the primary problem (1.2) if

$$\mathscr{L}(u_h) \leqq \mathscr{L}(v) \quad \forall v \in \mathscr{K}_h . \tag{2.1}$$

Since $\mathscr{K}_h$ is a closed convex subset of $H^1(\Omega)$, it is easy to see that (2.1) has a unique solution. To find it, we can apply e.g. the algorithm of Gauss-Seidel with constraints (cf. [4] Chpt., 4, § 1.4 or [3]-Section 2).

Next let us derive an error estimate for $u - u_h$. First we prove the following (cf. an analogous result of [5])

Lemma 2.1. *Let a function $W_h \in \mathscr{C}_h$ exist such that $2(u - G) - W_h \in \mathscr{C}$. Then*

$$\|u - u_h\|_1 \leqq \|u - G - W_h\|_1 + \|G - G_I\|_1 , \tag{2.2}$$

where G_I denotes the linear interpolate of G on the triangulation $\mathscr{T}_h$.

Proof. Denote $u = G + U$ and set $v = G + W_h$. Then $v \in \mathscr{K}$ and $2u - v = G + (2U - W_h) \in \mathscr{K}$. We have

$$(u, w - u)_1 - (f, w - u)_0 \geqq 0 \quad \forall w \in \mathscr{K} . \tag{2.3}$$

Consequently, inserting $w = v$ and $w = 2u - v$, we derive the equation

$$(2.4) \qquad (u, W_h - U)_1 = (f, W_h - U)_0 .$$

Denoting $U_h = u_h - G_I$ and setting $v = G + U_h$, we have $\gamma U_h = \gamma u_h - g_h \geqq 0$, consequently $v \in \mathscr{K}$. If $w = v$ is inserted in (2.3), it follows that

$$(2.5) \qquad (u, U_h - U)_1 \geqq (f, U_h - U)_0 .$$

Third, choosing $v_h = G_I + W_h$, we have $v_h \in \mathscr{K}_h$. From (2.1) we obtain that

$$(2.6) \qquad (u_h, v_h - u_h)_1 = (u_h, W_h - U_h)_1 \geqq (f, W_h - U_h)_0 .$$

Then using (2.4), (2.5) and (2.6), we may write

$$(u - u_h, U_h - W_h)_1 = (u, U - W_h + U_h - U)_1 - (u_h, U_h - W_h)_1 \geqq$$
$$\geqq (f, U - W_h)_0 + (f, U_h - U)_0 + (f, W_h - U_h)_0 = 0 .$$

Consequently,

$$\|u - u_h\|_1^2 = (u - u_h, G - G_I + U - U_h)_1 \leqq$$
$$\leqq (u - u_h, G - G_I)_1 + (u - u_h, U - U_h + U_h - W_h)_1 \leqq$$
$$\leqq \|u - u_h\|_1 \{\|G - G_I\|_1 + \|U - W_h\|_1\} , \qquad \text{Q.E.D.}$$

According to Lemma 2.1, it remains to show the existence of a function $W_h \in \mathscr{C}_h$, sufficiently close to $U = u - G$ and such that $2U - W_h \in \mathscr{C}$. The answer to this question is contained in the following

Theorem 2.1. *Assume that $u \in H^2(\Omega)$ and $u - g \in H^2(\Gamma_m)$, $m = 1, \ldots, M$, where Γ_m denotes any side of the polygonal boundary Γ.*

Then there exists $W_h \in S_h$ such that

$$0 \leqq W_h \leqq u - g \quad on \quad \Gamma$$

and

$$\|u - G - W_h\|_1 \leqq Ch(\|u - G\|_2 + \sum_{m=1}^{M} \|u - g\|_{H^2(\Gamma_m)}) ,$$

where C is independent of h, u and G.

For the proof – see [3] Section 2.

Corollary 2.1. Let the assumption of Theorem 2.1 be satisfied. Then

$$\|u - u_h\|_1 = O(h) .$$

The proof follows from Lemma 2.1, Theorem 2.1 and the inequalities

$$2(u - g) - W_h \geqq u - g - W_h \quad \text{on} \quad \Gamma ,$$
$$\|G - G_I\|_1 \leqq Ch\|G\|_2 .$$

3. FINITE ELEMENT APPROXIMATIONS OF THE DUAL PROBLEM

Making use of the definition (1.4), we can transform the dual problem (1.6) into an equivalent one: to find $\mathbf{q}^0 \in \mathscr{U}_0$ such that

$$I(\mathbf{q}^0) \leqq I(\mathbf{q}) \ \forall \mathbf{q} \in \mathscr{U}_0 , \tag{3.1}$$

where

$$\mathscr{U}_0 = \{\mathbf{q} \in Q, \ \mathbf{q} \cdot \mathbf{v} \geqq 0 \quad \text{on} \quad \Gamma\} ,$$

$$I(\mathbf{q}) = \tfrac{1}{2}\big(\sum_{i=1}^{n} \|q_i\|_0^2 + \|\operatorname{div} \mathbf{q}\|_0^2\big) + (f, \operatorname{div} \mathbf{q})_0 - \langle \mathbf{q} \cdot \mathbf{v}, g \rangle . \tag{3.2}$$

Then

$$\lambda_i^0 = q_i^0 , \quad (i = 1, \ldots, n) , \quad \lambda_{n+1}^0 = f + \operatorname{div} \mathbf{q}^0 .$$

Consider again the α-β-regular triangulations $\mathscr{T}_h$ of $\Omega \subset R^2$ and the spaces S_h of piecewise linear functions on $\mathscr{T}_h$. Introducing the subset

$$\mathscr{U}_{0h} = \mathscr{U}_0 \cap [S_h]^2 ,$$

we may define:
a vector $\mathbf{q}^h \in \mathscr{U}_{0h}$ will be called a finite element approximation of the dual problem (3.1), if

$$I(\mathbf{q}^h) \leqq I(\mathbf{q}) \quad \forall \mathbf{q} \in \mathscr{U}_{0h} . \tag{3.3}$$

The problem (3.3) has a unique solution (cf. an analogue in [3]-Section 3, where also some algorithm for solving (3.3) has been proposed). Note that the last term in (3.2) reduces to an integral, i.e.,

$$-\langle \mathbf{q} \cdot \mathbf{v}, g \rangle = - \int_\Gamma \mathbf{q} \cdot \mathbf{v} g \, \mathrm{d}s \quad \forall \mathbf{q} \in S_h . \tag{3.4}$$

As far as the error estimate for $\mathbf{q}^0 - \mathbf{q}^h$ is concerned, we may apply the approach of [3]-Section 3, completing only the functional I of [3] by the term (3.4). Thus we come to the following

Theorem 3.1. *Assume that $\mathbf{q}^0 \in [H^2(\Omega)]^2$ and $\mathbf{q}^0 . \mathbf{v} \in H^2(\Gamma_m)$, where Γ_m is any side of the polygonal boundary Γ. Then*

$$\sum_{i=1}^{2} \|q_i^0 - q_i^h\|_0 + \|\operatorname{div} (\mathbf{q}^0 - \mathbf{q}^h)\|_0 = O(h) .$$

Remark 3.1. If $\boldsymbol{q}^h$ is a solution of (3.3), then

$$\boldsymbol{\lambda}^h = \{q_1^h, q_2^h, f + \operatorname{div} \boldsymbol{q}^h\} \in \mathscr{U}$$

is an approximation to the solution $\boldsymbol{\lambda}^0$ of (1.6). By virtue of Theorem 1.1 and 3.1, it holds

$$\sum_{i=1}^{2} \|q_i^h - \partial u/\partial x_i\|_0 = O(h)\,, \quad \|\operatorname{div} \boldsymbol{q}^h + f - u\|_0 = O(h)\,.$$

4. A POSTERIORI ERROR ESTIMATES AND TWO-SIDED BOUNDS OF ENERGY

In this section, we derive some a posteriori error estimates, utilising the dual finite element analysis.

Since u satisfies (2.3), we may write for any $w \in \mathscr{K}$

$$(4.1) \qquad 2[\mathscr{L}(w) - \mathscr{L}(u)] = \|w\|_1^2 - \|u\|_1^2 - 2(f, w - u)_0 \geqq$$
$$\geqq \|w\|_1^2 - \|u\|_1^2 - 2(u, w - u)_1 = \|w - u\|_1^2 .$$

Let us search an upper bound for $-\mathscr{L}(u)$. From the duality theory (cf. e.g. [4] chpt. 5, § 3 for an analogous problem) it follows

$$(4.2) \qquad \mathscr{L}(u) = \operatorname*{Max}_{\mu \in H_+^{-1/2}(\Gamma)} \operatorname*{Min}_{v \in H^1(\Omega)} \{\mathscr{L}(v) - \langle \mu, \gamma v - g \rangle\}\,,$$

where

$$H_+^{-1/2}(\Gamma) = \{s \in H^{-1/2}(\Gamma)\,, \quad s \geqq 0\}\,.$$

Setting

$$V = v - G\,,$$

we can write

$$\mathscr{L}(v) = \langle \mu, \gamma v - g \rangle = \tfrac{1}{2}\|V + G\|_1^2 - (f, V + G)_0 - \langle \mu, \gamma V \rangle =$$
$$= \tfrac{1}{2}\|V\|_1^2 + (V, G)_1 - (f, V)_0 - \langle \mu, \gamma V \rangle + \tfrac{1}{2}\|G\|_1^2 - (f, G)_0\,.$$

Then obviously

$$(4.3) \qquad \operatorname*{Min}_{v \in H^1(\Omega)} \{\mathscr{L}(v) - \langle \mu, \gamma v - g \rangle\} = \tfrac{1}{2}\|G\|_1^2 - (f, G)_0 + \operatorname*{Min}_{V \in H^1(\Omega)} \mathscr{L}_1(V)\,,$$

where

$$\mathscr{L}_1(V) = \tfrac{1}{2}\|V\|_1^2 + (V, G)_1 - (f, V)_0 - \langle \mu, \gamma V \rangle\,.$$

It is well-known that if V_μ minimizes $\mathscr{L}_1(V)$ over $H^1(\Omega)$, then

$$\boldsymbol{\lambda}(\mu) = \{\partial V_\mu/\partial x_1,\ \partial V_\mu/\partial x_2,\ V_\mu\}$$

minimizes the functional (of complementary energy — cf. e.g. [6])

$$\mathscr{S}(\boldsymbol{p}) = \tfrac{1}{2} \sum_{i=1}^{3} \|p_i\|_0^2$$

over the set Λ_μ and

$$\operatorname*{Min}_{V \in H^1(\Omega)} \mathscr{L}_1(V) = - \operatorname*{Min}_{\boldsymbol{p} \in \Lambda_\mu} \mathscr{S}(\boldsymbol{p}) ,$$

where

$$\Lambda_\mu = \{\boldsymbol{p} \in [L_2(\Omega)]^3, \sum_{i=1}^{2} (p_i, \partial V/\partial x_i)_0 + (p_3, V)_0 =$$

$$= (f, V)_0 - (V, G)_1 + \langle \mu, \gamma V \rangle \quad \forall V \in H^1(\Omega)\} .$$

The latter relation, however, can be rewritten as follows

$$\langle \mu, \gamma V \rangle = \int_\Omega [\sum_{i=1}^{2} (p_i + \partial G/\partial x_i) \, \partial V/\partial x_i + (p_3 + G - f) \, V] \, \mathrm{d}x .$$

Inserting $V = \varphi \in C_0^\infty(\Omega)$, we obtain that

$$\boldsymbol{p}' + \operatorname{grad} G \in Q ,$$

$$\operatorname{div} (\boldsymbol{p}' + \operatorname{grad} G) = p_3 + G - f ,$$

consequently

(4.4) $$p_3 = f + \operatorname{div} \boldsymbol{p}' + \Delta G - G .$$

Using also (1.3), we may write

$$\langle \mu, \gamma V \rangle = \langle \boldsymbol{p}' . \boldsymbol{v} + \partial G/\partial \nu, \gamma V \rangle \quad \forall V \in H^1(\Omega) .$$

It means that

(4.5) $$\boldsymbol{p}' . \boldsymbol{v} + \partial G/\partial \nu = \mu \geqq 0 .$$

Now (4.4), (4.5) imply that (cf. the proof of Theorem 1.1)

(4.6) $$\bigcup_{\mu \in H_+^{-1/2}(\Gamma)} \Lambda_\mu = \mathscr{U}_G .$$

Inserting (4.3) into (4.2), we obtain

$$\mathscr{L}(u) = \operatorname*{Max}_{\mu \in H_+^{-1/2}(\Gamma)} \{\tfrac{1}{2}\|G\|_1^2 - (f, G)_0 - \operatorname*{Min}_{\boldsymbol{p} \in \Lambda_\mu} \mathscr{S}(\boldsymbol{p})\} .$$

We have, by virtue of (4.6),

$$\operatorname*{Max}_{\mu \in H_+^{-1/2}(\Gamma)} (- \operatorname*{Min}_{\boldsymbol{p} \in \Lambda_\mu} \mathscr{S}(p)) = - \operatorname*{Min}_{\mu} \operatorname*{Min}_{\boldsymbol{p}} \mathscr{S}(\boldsymbol{p}) = - \operatorname*{Min}_{\boldsymbol{p} \in U_G} \mathscr{S}(\boldsymbol{p}) ,$$

which results in the desired bound

$$(4.7) \qquad -\mathscr{L}(u) = -\tfrac{1}{2}\|G\|_1^2 + (f, G)_0 + \mathscr{S}(\boldsymbol{p}^0) \leqq$$
$$\leqq \mathscr{L}(G) + \mathscr{S}(\boldsymbol{p}) = \mathscr{S}_g(\boldsymbol{\lambda}) \quad \forall \boldsymbol{\lambda} \in \mathscr{U} ,$$

where (1.9) has been used. Thus we are led to the following

Theorem 4.1. *Let* $\tilde{u}_h$ *be any approximation of the primary problem* (1.2) *such that* $\tilde{u}_h \in \mathscr{K}$[1]). *Let* $\boldsymbol{q}^h \in \mathscr{U}_{0h}$ *be a finite element approximation of the dual problem* (3.1). *Then*

$$(4.8) \qquad \|\tilde{u}_h - u\|_1^2 \leqq \sum_{i=1}^{2} \|q_i^h - \partial u_h/\partial x_i\|_0^2 + \|f + \operatorname{div} \boldsymbol{q}^h - \tilde{u}_h\|_0^2 +$$
$$+ 2\int_\Gamma \boldsymbol{q}^h \,.\, \nu(\tilde{u}_h - g)\, \mathrm{d}s \equiv E(\boldsymbol{q}^h, \tilde{u}_h) \,.$$

Proof is parallel to that of Theorem 6.1 in [3].

Remark 4.1. Suppose that G is known explicitely. Then

$$\tilde{u}_h = u_h + G - G_I$$

can be substituted in (4.8), where $u_h \in \mathscr{K}_h$ is the finite element approximation (or any iterative solution of the problem (2.1), obtained by means of the Gauss-Seidel algorithm with constraints). Instead of $\boldsymbol{q}^h$ we may insert any $\boldsymbol{q}^{hm} \in \mathscr{U}_{0h}$.

Note that all terms in the right-hand side of (4.8) are non-negative.

Theorem 4.2. (*Two-sided bounds for the energy*). *Let* $\tilde{u}_h$ *and* $\boldsymbol{q}^h$ *be the same as in Theorem* 4.1. *Then for* $U = u - G$ *it holds*

$$(4.9) \qquad 2\mathscr{L}(G) - 2\mathscr{L}(\tilde{u}_h) \leqq \|U\|_1^2 \leqq \|q_i^h - \partial G/\partial x_i\|_0^2 +$$
$$+ \|f - G + \operatorname{div} \boldsymbol{q}^h\|_0^2 \equiv F(\boldsymbol{q}^h) ,$$

$$(4.10) \qquad 2\mathscr{L}(G) - 2\mathscr{L}(\tilde{u}_h) \leqq (f, U)_0 - (G, U)_1 \leqq F(\boldsymbol{q}^h) \,.$$

Proof. From (1.15) we know that

$$\|U\|_1^2 = (f, U)_0 - (G, U)_1 \,.$$

Then we have

$$\mathscr{L}(u) = \tfrac{1}{2}\|U + G\|_1^2 - (f, U + G)_0 = \tfrac{1}{2}\|U\|_1^2 - (f, U)_0 + (G, U)_1 +$$
$$+ \tfrac{1}{2}\|G\|_1^2 - (f, G)_0 = -\tfrac{1}{2}\|U\|_1^2 + \mathscr{L}(G) ,$$

[1]) Note that $\mathscr{K}_h \not\subset \mathscr{K}$ unless $g_h \geqq g$! Therefore, the finite element approximations $u_h \in \mathscr{K}_h$ cannot be used, in general.

consequently,

$$\|U\|_1^2 = 2\mathscr{L}(G) - 2\mathscr{L}(u) \geqq 2\mathscr{L}(G) - 2\mathscr{L}(\tilde{u}_h) .$$

Using (4.7) and (1.9), we obtain for any $\boldsymbol{p} \in U_G$

$$\|U\|_1^2 = 2[\mathscr{L}(G) - \mathscr{L}(u)] \geqq 2[\mathscr{L}(G) + \mathscr{S}_g(\lambda)] = 2\mathscr{S}(\boldsymbol{p}) .$$

Finally, if $\boldsymbol{q}^h \in \mathscr{U}_{0h}$, then $\boldsymbol{p} = [\boldsymbol{p}', p_3] \in \mathscr{U}_G$ with

(4.11) $$\boldsymbol{p}' = \boldsymbol{q}^h - \operatorname{grad} G , \quad p_3 \doteq f + \operatorname{div} \boldsymbol{q}^h - G .$$ Q.E.D.

Remark 4.2 If $f = 0$, a two-sided estimate for $\|u_1^2\|$ follows easily from (4.9) and (4.10), as

$$\|u\|_1^2 = \|U + G\|_1^2 = \|U\|_1^2 + \|G\|_1^2 + 2(G, U)_1 .$$

Theorem 4.3. *Let $\tilde{u}_h$, $\boldsymbol{q}^h$ and $E(\boldsymbol{q}^h, \tilde{u}_h)$ be the same as in Theorem 4.1. Then it holds*

(4.12) $$\sum_{i=1}^{2} \|q_i^h - \partial u/\partial x_i\|_0^2 + \|f + \operatorname{div} \boldsymbol{q}^h - u\|_0^2 \leqq E(\boldsymbol{q}^h, \tilde{u}_h) .$$

Proof. The solution $\boldsymbol{p}^0$ of (1.10) satisfies the inequality

$$(\boldsymbol{p}^0, \boldsymbol{p} - \boldsymbol{p}^0) \geqq 0 \quad \forall \boldsymbol{p} \in \mathscr{U}_G ,$$

where

$$(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{3} (p_i, q_i)_0 .$$

Consequently, we may write for any $\boldsymbol{p} \in \mathscr{U}_G$

(4.13) $$2\mathscr{S}(\boldsymbol{p}) - 2\mathscr{S}(\boldsymbol{p}^0) = \|\boldsymbol{p}\|^2 - \|\boldsymbol{p}^0\|^2 \geqq \|\boldsymbol{p}\|^2 - (\boldsymbol{p}^0, \boldsymbol{p}) =$$
$$= (\boldsymbol{p}, \boldsymbol{p} - \boldsymbol{p}^0) - (\boldsymbol{p}^0, \boldsymbol{p} - \boldsymbol{p}^0) + (\boldsymbol{p}^0, \boldsymbol{p} - \boldsymbol{p}^0) \geqq \|\boldsymbol{p} - \boldsymbol{p}^0\|^2 .$$

From (1.9) and (4.7) it follows that

(4.14) $$\mathscr{S}(\boldsymbol{p}) - \mathscr{S}(\boldsymbol{p}^0) = \mathscr{S}_g(\boldsymbol{\lambda}) - \mathscr{S}_g(\boldsymbol{\lambda}^0) = \mathscr{S}_g(\boldsymbol{\lambda}) + \mathscr{L}(u) \leqq$$
$$\leqq \mathscr{S}_g(\boldsymbol{\lambda}) + \mathscr{L}(v) \quad \forall \boldsymbol{\lambda} \in \mathscr{U} , \quad v \in \mathscr{K} ,$$

if $\boldsymbol{\lambda} = [\boldsymbol{\lambda}', \lambda_3]$, where $\boldsymbol{\lambda}' = \boldsymbol{p}' + \operatorname{grad} G$, $\lambda_3 = p_3 + G$. We have $\boldsymbol{p} - \boldsymbol{p}^0 = \boldsymbol{\lambda} - \boldsymbol{\lambda}^0$ and substituting

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}^h = \{q_1^h, q_2^h, f + \operatorname{div} \boldsymbol{q}^h\} , \quad \boldsymbol{\lambda}^0 = \{\partial u/\partial x_1, \partial x/\partial x_2, u\} ,$$

from (4.13) and (4.14) we obtain that the left-hand side of (4.12) is bounded above by $2\mathscr{S}_g(\lambda^h) + 2\mathscr{L}(\tilde{u}_h)$. The latter sum, however, can be rearranged to $E(\boldsymbol{q}^h, \tilde{u}_h)$ (cf. the proof of Theorem 6.1 in [3]).

Remark 4.3. Using Corollary 2.1 and Theorem 3.1, it is easy to prove that $E(\boldsymbol{q}^h, \tilde{u}_h)$, where $\tilde{u}_h = u_h + G - G_I$ (cf. Remark 4.1), tends to zero with $h \to 0$.

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Souhrn

DUÁLNÍ ROZBOR ELIPTICKÝCH ÚLOH S PŘEKÁŽKAMI NA HRANICI METODOU KONEČNÝCH PRVKŮ, I

IVAN HLAVÁČEK

V nedávné práci [3] předložil autor duální analýzu eliptických úloh druhého řádu s okrajovými podmínkami Signoriniho typu, tj. s překážkami na hranici danými nulovou funkcí. V tomto článku se rozšiřují výsledky z [3] na jednu třídu podobných úloh, ale s nehomogenními překážkami na hranici.

Pomocí po částech lineárních polynomů na triangulaci dané oblasti jsou navrženy duální metody konečných prvků a dokazuje se jejich $O(h)$-konvergence v energetické normě, za předpokladu, že řešení je dostatečně hladké. Dále se odvozují též některé aposteriorní odhady chyb obou duálních metod a oboustranné odhady energie řešení.

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