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# LJUSTERNIK ACCELERATION AND THE EXTRAPOLATED S.O.R. METHOD

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## 1. INTRODUCTION

The purpose of this paper is to extend an extrapolation procedure used first by L. A. Ljusternik [1]. This Ljusternik procedure accelerates the convergence of successive approximations solving linear algebraic systems obtained by discretizing the Laplace equation by finite differences. This method was also used for accelerating the convergence of some particular iterative procedures in both linear [2] and nonlinear problems [3].

In our paper a general result is derived and it is then used for accelerating a successive over-relaxation (S.O.R.) scheme with a non-optimal relaxation factor. An iterative procedure is obtained, the convergence of which is faster than that of the optimal S.O.R. method.

Let $\mathscr{X}$ be a complex Banach space, $\mathscr{X}'$ the corresponding dual space and $[\mathscr{X}]$ the space of all bounded linear transformations of $\mathscr{X}$ into itself. Hence, $\mathscr{X}'$ and $[\mathscr{X}]$ are Banach spaces.

The following class of operator equations will be considered,

$$x = Tx + b\ ,$$

where $T \in [\mathscr{X}]$ and its spectrum $\sigma(T)$ has the following structure: There exists a sequence $\{\lambda_k\}$ (finite of infinite) such that each $\lambda_k$ is an isolated pole of the resolvent operator and

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_k| \geqq \ldots \geqq \tau\ ,$$

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and if $\lambda \in \sigma(T)$, $\lambda \notin \{\lambda_k\}$ then $|\lambda| < \tau$. Such a structure of the spectrum is typical for Radon-Nikolskii operators [4]. We call an operator $T \in [\mathscr{X}]$ a Radon-Nikolskii operator, if $T$ can be written as, $T = U + V$, where $U$, $V \in [\mathscr{X}]$ with $U$ compact, and for the spectral radii the relation $r(V) < r(T)$ holds. This is the case of compact operators with positive spectral radii and of all non nilpotent finite dimensional operators, i.e. non nilpotent square matrices.

**Definition.** *Let $u \in \mathscr{X}$ and $\{y_k\}_{k=0}^{\infty} \subset \mathscr{X}$. Let us assume that there exists a function $\phi = \phi(k)$, $k = 0, 1, \ldots$, a constant $\varkappa > 0$ and a sequence $\{z_k\}_{k=0}^{\infty} \subset \mathscr{X}$ such that the following conditions are fulfilled*:

$$1)\ \lim_{k\to\infty} \phi(k) = 0$$

$$2)\ 1/\kappa < \limsup_{k\to\infty} \|z_k\| < \kappa$$

$$3)\ y_k - u = \phi(k)\, z_k$$

*for $k \geqq k_0$, where $k_0$ is a positive integer, we then say that the rate of convergence of $\{y_k\}$ to $u$ equals $\phi$ and express this symbolically by writting*

$$y_k \overset{\phi}{\longrightarrow} u\,.$$

*If $y_k^{(1)} \overset{\phi_1}{\longrightarrow} u$ and $y_k^{(2)} \overset{\phi_2}{\longrightarrow} u$ and $|\phi_1(k)| < |\phi_2(k)|$ for $k \geqq k_0$, where $k_0$ is some positive integer, then we say that $\{y_k^{(1)}\}$ converges to $u$ faster than $\{y_k^{(2)}\}$.*

Let $T \in [\mathscr{X}]$ be convergent, i.e. let $r(T) < 1$. Let $x_0 \in \mathscr{X}$ be a suitable element and $b \in \mathscr{X}$ be a fixed one. Let

$$x_{k+1} = Tx_k + b\,.$$

Then it follows that

$$x_k \overset{\phi_1}{\longrightarrow} x^* = Tx^* + b\,,$$

where $\phi_1(k) = [r(T)]^k k^{q-1}$ with $q = \max[q_1, \ldots, q_p]$, $q_j$ is the multiplicity of $\lambda_j \in \sigma(T)$ as a pole of the resolvent operator $R(\lambda, T) = (\lambda I - T)^{-1}$, $|\lambda_j| = r(T)$, $j = 1, \ldots, p$, and, assuming that for $\mu \in \sigma(T)$, $\mu \neq \lambda_j$, $j = 1, \ldots, p$, $|\mu| < r(T)$.

If

$$r(T) = |\lambda_1| > |\lambda_2| \geqq \ldots \text{ and } q_1 = 1\,,$$

then the sequence $\{y_k\}$ defined by

$$y_k = \frac{1}{1 - \lambda_1}(x_k - \lambda_1 x_{k-1})\,,$$

converges to $x^* = Tx^* + b$ faster than $\{x_k\}$, see [1, 2]. More precisely

$$y_k \overset{\phi_2}{\longrightarrow} x^*$$

where $\phi_2(k) = |\lambda_2|^k k^{s-1}$ and $s = \max\{q_j;\ |\lambda_j| = |\lambda_2|\}$. A generalization of this result is given in Theorem 1.

## 2. GENERAL THEOREM

Let us consider an equation

$$x = Tx + b\,,$$

and let $\lambda_1, \ldots, \lambda_t$, $t > 1$, be mutually different elements of the spectrum $\sigma(T)$. Let

$$r(T) = |\lambda_1| \geqq \ldots \geqq |\lambda_t| > |\lambda|$$

for any $\lambda \in \sigma(T)$, $\lambda \neq \lambda_j$, $j = 1, \ldots, t$.

Let

$$p(z) = z^N + \tau_1 z^{N-1} + \ldots + \tau_N$$

be a polynomial with complex coefficients such that $p(1) \neq 0$. We then put

(1) $$f_k^{(m)}(\tau_1, \ldots, \tau_N) = \frac{1}{p(1)} \{x_k + \tau_1 x_{k-m} + \ldots + \tau_N x_{k-Nm}\}\,,$$

where $m$ is a positive integer and the $x_k$ are defined by

(2) $$x_{k+1} = Tx_k + b\,.$$

For a fixed $j \in [1, t]$ let $C_j$ be a circumference with center $\lambda_j$ and radius $\varrho_j > 0$ such that

$$\{\lambda : |\lambda - \lambda_j| \leqq \varrho_j\} \cap \sigma(T) = \{\lambda_j\}\,.$$

If $\lambda_j$ is a pole of order $q_j$ of the resolvent operator $R(\lambda, T)$, we put

$$B_{j,1} = \frac{1}{2\pi i}\int_{C_j} R(\lambda, T)\,\mathrm{d}\lambda\,,$$

and

$$B_{j,k+1} = (T - \lambda_j I)^k B_{j,1}\,, \quad k = 1, 2, \ldots.$$

It is easy to see that

$$x_{k+1} = \sum_{j=1}^{t} \frac{1}{2\pi i}\int_{C_j} \lambda^{k+1} R(\lambda, T)\, x_0 \,\mathrm{d}\lambda + \frac{1}{2\pi i}\int_{C} \lambda^{k+1} R(\lambda, T)\, x_0 \,\mathrm{d}\lambda + \sum_{s=0}^{k} T^s b\,.$$

Here $C = \{\lambda : |\lambda| = \varrho, \varrho > 0\}$ is such that $\{\lambda : |\lambda| \leqq \varrho\} \cap \sigma(T)$ contains $\sigma(T)$ but not $\lambda_1, \ldots, \lambda_t$ and on $C$ there are no singularities of $R(\lambda, T)$.

**Theorem 1.** *We suppose that m, s, t, N, are positive integers, the operator T in the equation*

$$x = Tx + b\,, \quad b \in \mathscr{X}\,,$$

*is convergent, the eigenvalues* $\lambda_1, \ldots, \lambda_t, \ldots, \lambda_{t+s}$ *are poles of* $R(\lambda, T)$ *of order* $q(1) = q_1, \ldots, q(t+s) = q_{t+s}$ *respectively, and*

$$r(T) = |\lambda_1| \geqq \ldots \geqq |\lambda_t| > |\lambda_{t+1}| = \ldots = |\lambda_{t+s}| > |\lambda|$$

for any $\lambda \in \sigma(T)$, $\lambda \neq \lambda_j$, $j = 1, \ldots, t + s$ and $q(t+1) \geqq q(t+k)$, $k = 2, \ldots, s$. *Let* $x_0 \in \mathscr{X}$ *satisfy*

$$B_{t+1, q(t+1)}\left[x_0 - \frac{1}{1 - \lambda_{t+1}} b\right] \neq 0\,.$$

*We then define a sequence* $\{y_k\} \subset \mathscr{X}$ *as follows:*

$$y_k = x_k \quad \textit{for} \quad k = 0, 1, \ldots, Nm - 1\,,$$

*and*

$$y_k = f_k^{(m)}(\sigma_1^{(m)}, \ldots, \sigma_N^{(m)}) \quad \text{for} \quad k \geqq Nm\,,$$

*where* $x_k$ *is given by* (2) *and* $f_k^{(m)}$ *by* (1) *with the* $\tau_j^{(m)} = \sigma_j^{(m)}$ *being defined by*

$$p(z) = (z - \lambda_1^m)^{q_1} \ldots (z - \lambda_t^m)^{q_t} = z^N + \sigma_1^{(m)} z^{N-1} + \ldots + \sigma_N^{(m)}\,, \quad N = \sum_{j=1}^{t} q_j\,.$$

*Then*

$$y_k \xrightarrow{\phi_{t+1}} x^* = Tx^* + b\,,$$

*where*

$$\phi_{t+1}(k) = |\lambda_{t+1}|^k\, k^{q(t+1)-1}\,.$$

*Moreover,*

(3) $$y_{k+1} = Ty_k + b\,, \quad k > Nm\,,$$

*and*

$$y_k - x^* = \frac{1}{p(1)}\, p(T^m)\, (x_{k-Nm} - x^*)\,.$$

Remark. It is easy to see that the formulas for $\sigma_j^{(m)}$ are very simple if all the poles of $R(\lambda, T)$ are assumed to be of multiplicity one. This is the case of normal operators and those similar to normal ones. It should be noted that some of the results of this paper were obtained in [7] under the assumption that $T$ is a *normalizable* $n \times n$ matrix, i.e. an $n \times n$ matrix similar to a normal one. However, the S.O.R. iteration matrices are non-normalizable for certain choices of relaxation parameters [6, p. 238]. It will be shown that the assumption that $T$ is normalizable can be omitted and the failure of $T$ to be normalizable actually does not complicate the considerations essentially. The main results of [7] thus remain valid in general.

Proof of Theorem 1. By definition we have

$$y_{k+1} = \frac{1}{p(1)} \{(Tx_k + b) + \sigma_1^{(m)}(Tx_{k-m} + b) + \ldots + \sigma_N^{(m)}(Tx_{k-Nm} + b)\} = = Ty_k + b\,.$$

Further

$$x_k - x^* = T(x_{k-1} - x^*) = \ldots = T^j(x_{k-j} - x^*)\,, \quad j \geqq k\,,$$

and hence

$$x_k - x^* = T^{Nm}(x_{k-Nm} - x^*)$$

$$x_{k-m} - x^* = T^{m(N-1)}(x_{k-Nm} - x^*)\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_{k-Nm+N} - x^* = T^m(x_{k-Nm} - x^*)\,,$$

$$x_{k-Nm} - x^* = T^0(x_{k-Nm} - x^*)\,.$$

It follows that

$$p(1)\,[y_k - x^*] = p(T^m)\,[x_{k-Nm} - x^*]\,,$$

which gives

$$y_k - x^* = \frac{1}{p(1)}\, p(T^m)\, T^{k-Nm}[x_0 - x^*] = = \frac{1}{p(1)}\, p(T^m)\, T^{k-Nm}[x_0 - (I - T)^{-1}\, b]\,.$$

Using the method of functional calculus [5, p. 287] we can write the error vector $y_k - x^*$ in the form

$$y_k - x^* = \frac{1}{p(1)} \sum_{j=1}^{t} \frac{1}{2\pi i} \int_{C_j} p(\lambda^m)\, \lambda^{k-Nm}\, R(\lambda, T) \left[x_0 - \frac{1}{1-\lambda}\, b\right] \mathrm{d}\lambda + + \frac{1}{p(1)} \frac{1}{2\pi i} \int_C p(\lambda^m)\, \lambda^{k-Nm}\, R(\lambda, T) \left[x_0 - \frac{1}{1-\lambda}\, b\right] \mathrm{d}\lambda\,.$$

Let us define the vectors

$$u_j = \frac{1}{p(1)} \frac{1}{2\pi i} \int_{C_j} \frac{p(\lambda^m)}{\lambda^{Nm}}\, R(\lambda, T) \left[x_0 - \frac{1}{1-\lambda}\, b\right] \mathrm{d}\lambda\,, \quad j = 1, \ldots, t$$

and assume that the eigenvalues $\lambda_{t+1}, \ldots, \lambda_{t+s}$ can be arranged in such a way that the corresponding multiplicities $q_{t+1}, \ldots, q_{t+s}$ satisfy

$$q = q_{t+1} = \ldots = q_{t+r} > q_{t+r+1} \geqq \ldots \geqq q_{t+s}\,.$$

According to our assumptions, $u_j = 0, j = 1, \ldots, t$. It is easy to see that there exists a number $\gamma$ independent of $k$ such that

$$\left\| \frac{1}{k^{q-1}} \frac{1}{p(1)} \frac{1}{2\pi i} \int_C \frac{p(\lambda^m)}{\lambda^{Nm}} \left( \frac{\lambda}{\lambda_{t+1}} \right)^k R(\lambda, T)\, \mathrm{d}\lambda \right\| \leqq \gamma\,;$$

more precisely

$$\frac{1}{2\pi i} \int_C \frac{p(\lambda^m)}{\lambda^{Nm}} \lambda^k R(\lambda, T)\, \mathrm{d}\lambda = \sum_{j=t+1}^{t+s} \frac{1}{2\pi i} \int_{C_j} \frac{p(\lambda^m)}{\lambda^{Nm}} \lambda^k R(\lambda, T)\, \mathrm{d}\lambda + z_k\,,$$

where

$$\|z_k\| \leqq \gamma_1 |\alpha \lambda_{t+1}|^k$$

with $\gamma_1$ and $\alpha$, $0 < \alpha < 1$, both independent of $k$. It follows that

$$\left\| \frac{1}{2\pi i} \int_C \frac{p(\lambda^m)}{\lambda^{Nm}} \lambda^k R(\lambda, T)\, \mathrm{d}\lambda \right\| = \left\| \sum_{j=t+1}^{t+r} \frac{1}{2\pi i} \int_{C_j} \frac{p(\lambda^m)}{\lambda^{Nm}} \lambda^k R(\lambda, T)\, \mathrm{d}\lambda \right\| + o(|\lambda_{t+1}|^k)\,.$$

Similarly,

$$\left\| \frac{1}{k^{q-1}} \frac{1}{p(1)} \frac{1}{2\pi i} \int_{C_j} \frac{p(\lambda^m)}{\lambda^{Nm}} \left( \frac{\lambda}{\lambda_{t+1}} \right)^k R(\lambda, T)\, \mathrm{d}\lambda \right\| \leqq \gamma_2 \frac{1}{k}$$

for $j = t + r + 1, \ldots, t + s$, where $\gamma_2$ does not depend on $k$.

We deduce that

$$\left\| \frac{1}{k^{q-1}} \frac{1}{p(1)} \frac{1}{2\pi i} \int_C \frac{p(\lambda^m)}{\lambda^{Nm}} \left( \frac{\lambda}{\lambda_{t+1}} \right)^k R(\lambda, T)\, \mathrm{d}\lambda \right\| \geqq$$

$$\geqq \left\| \frac{1}{k^{q-1}} \frac{1}{p(1)} \frac{1}{2\pi i} \sum_{j=t+1}^{t+r} \int_{C_j} \frac{p(\lambda^m)}{\lambda^{Nm}} \left( \frac{\lambda}{\lambda_{t+1}} \right)^k R(\lambda, T)\, \mathrm{d}\lambda \right\| + o(1).$$

In order to show that

$$\omega = \limsup_{k \to \infty} \left\| \frac{1}{k^{q-1}} \frac{1}{p(1)} \sum_{j=t+1}^{t+r} \frac{1}{2\pi i} \int_{C_j} \frac{p(\lambda^m)}{\lambda^{Nm}} \left( \frac{\lambda}{\lambda_{t+1}} \right)^k R(\lambda, T) \left[ x_0 - \frac{1}{1-\lambda} b \right] \mathrm{d}\lambda \right\| >$$

$$> 0,$$

we set

$$\phi_{t+1} = 0\,, \quad \frac{\lambda_j}{\lambda_{t+1}} = e^{i\phi_j}\,, \quad 0 < \phi_j < 2\pi\,, \quad j = t+2, \ldots, t+r\,.$$

Defining $W_j$ as

$$W_j \equiv \frac{1}{2\pi i} \int_{Cj} \frac{p(\lambda^m)}{\lambda^{Nm}} \left( \frac{\lambda}{\lambda_{t+1}} \right)^k R(\lambda, T)\, \mathrm{d}\lambda$$

we show that

$$W_j = \frac{p(\lambda_j^m)}{\lambda_j^{Nm}} e^{ik\phi_j} B_{j,1} + \frac{h_k'(\lambda_j)}{1!} B_{j,2} + \ldots + \frac{h_k^{(q-1)}(\lambda_j)}{(q-1)!} B_{j,q}\,,$$

where

$$h_k(\lambda) = \frac{p(\lambda^m)}{\lambda^{Nm}}\left(\frac{\lambda}{\lambda_{t+1}}\right)^k, \quad \text{and} \quad h_k^{(a)}(\lambda) = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^a h_k(\lambda).$$

For sufficiently large values of $k$ the main term of $W_j$ is given by

$$\frac{1}{(q-1)!}\frac{p(\lambda_j^m)}{\lambda_j^{Nm}}\frac{k^{q-1}}{\lambda_{t+1}^{q-1}}e^{i\phi_j(k-q+1)}B_{j,q}.$$

Similarly, for

$$U_j \equiv \frac{1}{2\pi i}\int_{C_j}\frac{p(\lambda^m)}{\lambda^{Nm}}\left(\frac{\lambda}{\lambda_{t+1}}\right)^k\frac{1}{1-\lambda}R(\lambda, T)\,\mathrm{d}\lambda\,,$$

we can write

$$U_j = \frac{p(\lambda_j^m)}{\lambda_j^{Nm}}e^{ik\phi_j}\frac{1}{1-\lambda_j}B_{j,1} + \frac{g_k'(\lambda_j)}{1!}B_{j,2} + \ldots + \frac{g_k^{(q-1)}(\lambda_j)}{(q-1)!}B_{j,q}\,,$$

where

$$g_k(\lambda) = \frac{p(\lambda^m)}{\lambda^{Nm}}\left(\frac{\lambda}{\lambda_{t+1}}\right)^k\frac{1}{1-\lambda}\,; \quad g_k^{(a)}(\lambda_j) = \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^a[g(\lambda)]_{\lambda=\lambda_j}.$$

The main term in this expression is given by

$$\frac{1}{(q-1)!}\frac{p(\lambda_j^m)}{\lambda_j^{Nm}}\frac{1}{1-\lambda_j}\frac{k^{q-1}}{\lambda_{t+1}^{q-1}}e^{i\phi_j(k-q+1)}B_{j,q}.$$

It follows that

$$\omega = \limsup_{k\to\infty}\frac{1}{|p(1)|}\left\|\frac{p(\lambda_{t+1}^m)}{\lambda_{t+1}^{Nm-q+1}}B_{t+1,q}\left[x_0 - \frac{1}{1-\lambda_{t+1}}b\right] + \right.$$
$$\left. + \frac{1}{\lambda_{t+1}^{q-1}}\sum_{j=t+2}^{t+r}\frac{p(\lambda_j^m)}{\lambda_j^{Nm}}e^{i\phi_j(k-q+1)}B_{j,q}\left[x_0 - \frac{1}{1-\lambda_j}b\right]\right\|.$$

Let us put

$$\lambda_{t+1}^{q-1}w_j = \frac{1}{p(1)}\frac{p(\lambda_j^m)}{\lambda_j^{Nm}}\left[x_0 - \frac{1}{1-\lambda_j}b\right].$$

We then see that

$$\omega = \limsup_{k\to\infty}\left\|B_{t+1,q}w_{t+1} + \sum_{j=t+2}^{t+r}e^{i\phi_j(k-q+1)}B_{j,q}w_j\right\|.$$

Let $k_1 < k_2 < \ldots \to +\infty$ be positive integers such that

$$\left|e^{i\phi_{j+t}(k_v-q+1)} - 1\right| \to 0 \quad \text{for} \quad v \to +\infty\,.$$

Then

$$\|B_{t+1,q}w_{t+1} + \sum_{j=2}^{r} e^{i\phi_{t+j}(k_v-q+1)} B_{t+j,q}w_{t+j}\| \geqq$$

$$\geqq \|\sum_{j=1}^{r} B_{t+j,q}w_{t+j}\| - \sum_{j=2}^{r} |e^{i\phi_{t+j}(k_v-q+1)} - 1| \, \|B_{t+j,q}w_{t+j}\| \geqq$$

$$\geqq \|\sum_{j=1}^{r} B_{t+j,q}w_{t+j}\| - o(1),$$

and we deduce that

$$\omega \geqq \tfrac{1}{2}\|\sum_{j=1}^{t} B_{t+j,q}w_{t+j}\| > 0 .$$

This proves that

$$y_k - x^* = \lambda_{t+1}^k \, k^{q-1} z_k ,$$

where

$$z_k = \frac{1}{k^{q-1}} \frac{1}{p(1)} \sum_{j=t+1}^{t+s} \frac{1}{2\pi i} \int_{C_j} \frac{p(\lambda^m)}{\lambda^{Nm}} \left(\frac{\lambda}{\lambda_{t+1}}\right)^k R(\lambda, T) \left[x_0 - \frac{1}{1-\lambda} b\right] d\lambda +$$

$$+ \frac{1}{k^{q-1}} \frac{1}{p(1)} \frac{1}{2\pi i} \int_{C} \frac{p(\lambda^m)}{\lambda^{Nm}} \left(\frac{\lambda}{\lambda_{t+1}}\right)^k R(\lambda, T) \left[x_0 - \frac{1}{1-\lambda} b\right] d\lambda .$$

We have also shown that

$$\|z_k\| \leqq \gamma$$

and

$$\limsup_{k\to\infty} \|z_k\| \geqq \tfrac{1}{2}\omega > 0 ,$$

where the bounds for $\gamma$ and $\omega$ do not depend on $k$. As a consequence we have obtained the relation

$$y_k \xrightarrow{\phi_{t+1}} x^* ,$$

where $\phi_{t+1}(k) = \lambda_{t+1}^k \, k^{q-1}$, and this completes the proof of Theorem 1.

## 3. APPROXIMATE CONSTRUCTION OF EIGENVALUES

When the method considered in the previous section is used, it is important that the appropriate eigenvalues of the iteration operator are known. In practical calculations information is usually available concerning the structure of the spectrum. This may be of some use for deciding which method of approximating the eigenvalues to choose. This is the case for certain finite dimensional problems, where some well tried methods can be used, e.g. the QR-method, QD-method, Jacobi method etc.

If no special information concerning the operator is available, the following simple power method can be used. However, we must be very careful because of the numerical instabilities. We remark that the acceleration method described in Section 2 gives practically useful results if a limited number of eigenvalues appear in the acceleration formula. We will discuss this matter in the concluding section.

**Theorem 2.** *Suppose that $a$, $m$, $p$, $t$, $N$, are positive integers, $T \in [\mathscr{X}]$, and $\lambda_j \in \sigma(T)$ fulfil the relations*

$$|\lambda_1| \geqq \ldots \geqq |\lambda_{t+1}| > |\lambda_{t+2}| \geqq \ldots \geqq |\lambda_p| \geqq \ldots .$$

*Further let $\mu \in \sigma(T)$, $\mu \neq \lambda_j$, $j = 1, \ldots, p$, not necessarily an eigenvalue, satisfy*

$$|\mu| < |\lambda_{t+1}| ,$$

*and assume that the eigenvalues $\lambda_1, \ldots, \lambda_t$ are known. Let $x'_k$ and $x'$ be in $\mathscr{X}'$ and*

$$\lim_{k \to \infty} x'_k(x) = x'(x)$$

*for all $x \in \mathscr{X}$. Finally let $x^{(0)} \in \mathscr{X}$ be such that*

$$x'(B_{t+1,a} x^{(0)}) \neq 0 \quad \text{and} \quad B_{t+1,a+1} x^{(0)} = 0.$$

*We construct vectors $x^{(k)}$ and $y^{(k)}$ from*

$$x^{(k+1)} = T x^{(k)} ,$$

*and*

$$y^{(k)} = \begin{cases} x^{(k)} & \text{for} \quad k = 0, \ldots, Nm - 1 , \\ g_k^{(m)}(\sigma_1^{(m)}, \ldots, \sigma_N^{(m)}) & \text{for} \quad k \geqq Nm , \end{cases}$$

*where*

$$g_k^{(m)}(\sigma_1^{(m)}, \ldots, \sigma_N^{(m)}) = x_k + \sigma_1^{(m)} x_{k-m} + \ldots + \sigma_N^{(m)} x_{k-Nm}$$

*with the $\sigma_j^{(m)}$ defined by*

$$p(z) = (z - \lambda_1^m)^{q_1} \ldots (z - \lambda_t^m)^{q_t} = z^N + \sigma_1^{(m)} z^{N-1} + \ldots + \sigma_N^{(m)} .$$

*If we put*

$$v_k = \frac{x'_k(y^{(k+1)})}{x'_k(y^{(k)})} ,$$

*then*

$$\lim_{k \to \infty} v_k = \lambda_{t+1} .$$

Note that if $\mathscr{X}$ is a Hilbert space with an inner product $(x, y)$, $x, y \in \mathscr{X}$, then the following choice of $x'_k$ is advantageous $x'_k(x) = (x, y^{(k)})$, i.e.

$$v_k = \frac{(y^{(k+1)}, y^{(k)})}{(y^{(k)}, y^{(k)})} .$$

Proof. Using the same machinery as in the proof of Theorem 1, we show easily that

$$y^{(k)} = \sum_{j=1}^{t+1} \frac{1}{2\pi i} \int_{C_j} h_k(\lambda)\, R(\lambda, T)\, x^{(0)}\, d\lambda + \frac{1}{2\pi i} \int_C h_k(\lambda)\, R(\lambda, T)\, x^{(0)}\, d\lambda\,,$$

where

$$h_k(\lambda) = \lambda^k + \sigma_1^{(m)} \lambda^{k-m} + \ldots + \sigma_N^{(m)} = p(\lambda^m)\, \lambda^{k-Nm}\,.$$

It follows that

$$y^{(k)} = h_k(\lambda_{t+1})\, B_{t+1,1} x^{(0)} + \ldots + h^{(a-1)}(\lambda_{t+1})\, B_{t+1,a}\, x^{(0)} + W_k\,,$$

with

$$\lim_{k\to\infty} \frac{1}{h_k^{(a-1)}(\lambda_{t+1})}\, W_k = 0\,,$$

where

$$h_k^{(j)}(\lambda) = \left[\left(\frac{d}{dz}\right)^j h_k(z)\right]_{z=\lambda}.$$

Consequently

$$v_k = \frac{1}{h_k^{(a-1)}(\lambda_{t+1})}\, y^{(k)} \to B_{t+1,a} x^{(0)} = v\,.$$

Obviously

$$v_{k+1} = T v_k$$

and thus

$$\nu_k - \lambda_{t+1} = \frac{x_k'(v_{k+1})}{x_k'(v_k)} - \frac{x'(Tv)}{x'(v)}\,.$$

Furthermore,

$$\left|\nu_k - \lambda_{t+1}\right| \leqq \left|\frac{x_k'(Tv_k)}{x_k'(v_k)} - \frac{x'(Tv)}{x_k'(v_k)}\right| + \left|\frac{x'(Tv)}{x_k'(v_k)} - \frac{x'(Tv)}{x'(v)}\right|,$$

and there is a positive number $\delta$ independent of $k$ such that for $k$ large enough

$$\inf\{|x_k'(v_k)|,\ |x_k'(v)| : k \geqq k_0\} = \delta > 0\,.$$

This implies that

$$\left|\nu_k - \lambda_{t+1}\right| \leqq \frac{1}{\delta}\left|x_k'(Tv_k) - x'(Tv)\right| + \frac{|x'(Tv)|}{\delta^2}\left|x_k'(v_k) - x'(v)\right| \leqq$$

$$\leqq \frac{1}{\delta}\{|x_k'(T[v_k - v])| + |x_k'(Tv) - x'(Tv)|\} +$$

$$+ \frac{|x'(Tv)|}{\delta^2}\{|x_k'(v_k - v)| + |x_k'(v) - x'(v)|\}\,.$$

Since $\{x'_k\}$ is a convergent sequence, it is uniformly norm-bounded with respect to $k$. Finally we obtain that

$$|v_k - \lambda_{t+1}| \leqq \frac{1}{\delta}\{\tau\|Tv_k - Tv\| + |x'_k(Tv) - x'(Tv)|\} +$$

$$+ \frac{|x'(Tv)|}{\delta^2}\{\tau\|v_k - v\| + |x'_k(v) - x'(v)|\} .$$

with

$$\tau = \sup_k \|x'_k\| < +\infty .$$

Since all the right hand side terms tend to zero, the proof is complete.

Remark. It should be mentioned that in practice one can recommend using the results of Theorem 2 only if $q_j = 1$ for $j \leqq t$ and $q_{t+1} \leqq 2$. As we shall see in Section 4 this is also the case for the S.O.R. iterations.

## 4. EXTRAPOLATED S.O.R.

In this section we show how the preceding results can be applied to accelerate the S.O.R. iterations.

Let $\mathscr{X}$ be the n-dimensional complex vector space. We consider the equation

$$Ax = b ,$$

where $A$ is an $n \times n$ positive definite matrix, $n \geqq 2$. Let us write

$$A = D(I - L - U) ,$$

where $D$ is diagonal with entries as in $A$, $L$ and $U$ are strictly lower and upper triangular matrices respectively corresponding to $A$. The above system is equivalent to the following

$$x = Bx + c ,$$

where $B = L + U$ and $c = D^{-1}b$.

We assume that $B$ satisfies the following conditions

(i) $B$ is weakly cyclic of index 2 (see [6, p. 162]),

(ii) $B$ is consistently ordered ([6, p. 144]),

Let

$$\mu_1 > \mu_2 > \ldots > \mu_p$$

be mutually different positive eigenvalues of $B$ and let

$$\sigma(B) - \{0\} = \{\mu_1, \ldots, \mu_p, -\mu_1, \ldots, -\mu_p\} .$$

As usual, we let

$$H(\omega) = (I - \omega L)^{-1} [(1 - \omega) I + \omega U] .$$

It is well known [6, p. 172–173] that

$$r(H(\omega)) < 1 \quad \text{for} \quad \omega \in (0, 2) ,$$

and

$$\inf \{r(H(\omega)) : \omega \in (0, 2)\} = \omega_b - 1 = r(H(\omega_b)) ,$$

where

$$\omega_b = \frac{2}{1 + \sqrt{(1 - [r(B)]^2)}} .$$

If $\omega \in (0, 2)$ satisfies the relation

$$\omega^2 \mu_j^2 - 4(\omega - 1) = 0$$

for some $\mu_j \in \sigma(B)$, then we call it *j-optimal* and denote it by $\omega_j$. If $\omega \in (0, 2)$ is not $j$-optimal for any $j \in [1, p]$, we call it *regular*. Note that $\omega_1 = \omega_b$ and

$$2 > \omega_1 > \ldots > \omega_p > 1 .$$

**Theorem 3.** *Let the Jacobi matrix B corresponding to a positive definite matrix A satisfy the conditions* (i)–(ii).

*If $\omega \in (0, 2)$ is regular, then $H(\omega)$ is normalizable and the numbers*

$$\lambda_{2j-1}(\omega) = \tfrac{1}{4}(\omega\mu_j + \sqrt{[\omega^2\mu_j^2 - 4(\omega - 1)]})^2 ,$$
$$\lambda_{2j}(\omega) \quad = \tfrac{1}{4}(\omega\mu_j - \sqrt{[\omega^2\mu_j^2 - 4(\omega - 1)]})^2 ,$$

*for $j = 1, \ldots, p$, and*

$$\lambda = 1 - \omega \quad \text{if} \quad 0 \in \sigma(B) ,$$

are eigenvalues of $H(\omega)$.

*Let r be a positive integer, $1 \leqq r \leqq p$. Then the matrix $H(\omega_r)$ is not normalizable, more precisely, $H(\omega_r)$ possesses $d_r$ principal vectors each of grade* 2, *where $d_r$ is the dimension of the eigenspace of B corresponding to the eigenvalue $\mu_r$. All the other eigenvalues of $H(\omega_r)$ are simple poles of the resolvent matrix $R(\lambda, H(\omega_r))$. The eigenvalues of $H(\omega_r)$ fulfil the following relations*

$$\lambda_1(\omega_r) > \lambda_3(\omega_r) > \ldots > \lambda_{2r-1}(\omega_r) = \lambda_{2r}(\omega_r) > \lambda_{2r-2}(\omega_r) > \ldots > \lambda_2(\omega_r) .$$

*Moreover,*

$$|\lambda_{2r-1}(\omega_r)| = |\lambda_{2r}(\omega_r)| = \omega_r - 1$$

*and*

$$|\lambda_{2j-1}(\omega_r)| = |\lambda_{2j}(\omega_r)|$$

*for* $j = r + 1, \ldots, p$ *and*

$$|\lambda_{2j-1}(\omega_r)| = \omega_r - 1$$

*for* $j = r, \ldots, p$, *and*

$$\lambda_j(\omega_r) = 1 - \omega_r$$

*for the remaining indices.*

Theorem 3 contains results which are actually proved in [6, pp. 234–238]. We only summarize these and state them in a form•suitable for our purposes.

As direct applications of the Theorems 1 and 3 we obtain the following results.

**Theorem 4.** *Let* $r \in [1, p]$ *and let* $\omega \in (\omega_{r-1}, \omega_r)$,

$$\Lambda_j(\omega) = \tfrac{1}{4}(\omega\mu_j + \sqrt{(\omega^2\mu_j^2 - 4(\omega - 1))^2})$$

*for* $j = 1, \ldots, p$ *and*

$$S_{r,j} = S_{r,j}^{(m)} = (-1)^j \sum_{\substack{a(1),\ldots,a(j)=1 \\ a(1)<\ldots<a(j)}}^{r-1} \Lambda_{a(1)}^m, \ldots \Lambda_{a(j)}^m$$

*for* $j = 1, \ldots, r - 1$. *Define*

$$S = 1 + S_{r,1} + \ldots + S_{r,r-1} \,.$$

*Let* $P_{j,1}$ *be the eigenprojection onto the eigenspace corresponding to* $\Lambda_j$ *and let* $x_0$ *be such that*

$$P_{r,1}x_0 - \frac{1}{1 - \Lambda_r} P_{r,1}\,\mathrm{d}(\omega) \neq 0\,,$$

*where*

$$d(\omega) = \omega(I - \omega L)^{-1}\, D^{-1}b\,.$$

*If we put*

$$y_k = x_k \quad \text{for} \quad k = 0, \ldots, (r - 1)\, m - 1$$

*and*

$$y_k = \frac{1}{S}\,(x_k + S_{r,1}x_{k-m} + \ldots + S_{r,r-1}x_{k-(r-1)m}) \quad \text{for} \quad k \geqq (r - 1)\, m$$

*then*

$$y_k \overset{\psi_r}{\rightarrow} x^*\,,$$

*where* $x^*$ *is the unique solution of*

$$Ax = b\,,$$

$A = D(I - L - U)$, *and*

$$\psi_r(k) = [\Lambda_r(\omega)]^k$$

*where*

$$|\Lambda_r(\omega)| = \omega - 1\,.$$

**Theorem 5.** *Let $r \in [1, p]$, $s$ be a positive integer and, $\omega = \omega_s$,*

$$p(z) = (z - \Lambda_1(\omega)) \dots (z - \Lambda_{s-1}(\omega)) (z - \Lambda_s(\omega))^2 \,.$$
$$. (z - \Lambda_{s+1}(\omega)) \dots (z - \Lambda_r(\omega))$$

*and let $y_k$ be defined as in Theorem 1 via $f_k^{(m)}(\sigma_1^{(m)}, \dots, \sigma_{r+v}^m)$, where*

$$p(z) = z^{r+v} + \sigma_1^{(m)} z^{r-1+v} + \dots + \sigma_{r+v}^{(m)}$$

*with $v = 0$ for $r < s$ and $v = 1$ for $r \geqq s$.*

*If $x_0 \in \mathscr{X}$ is such that*

$$P_{r,1+\delta(s-r)} x_0 - \frac{1}{1 - \Lambda_r} P_{r,1+\delta(s-r)} \, \mathrm{d}(\omega) \neq 0 \,,$$

*where*

$$P_{r,k+1} = (H(\omega) - \Lambda_r(\omega) \, I) \, P_{r,k}, \; k = 1, 2 \,,$$

*and*

$$\delta(s - r) = \begin{matrix} 1 & \text{for} & s = r \\ 0 & \text{for} & s \neq r \end{matrix} \,,$$

*then*

$$y_k \xrightarrow{\psi_{r,s}} x^*$$

*with*

$$\psi_{r,s}(k) = \begin{matrix} [\Lambda_r(\omega_s)]^k & \text{for} & r \neq s \\ k[\Lambda_r(\omega_r)]^k & \text{for} & r = s \end{matrix} \,.$$

It is quite clear that the choice $r \neq s$ is not advantageous.

If $\omega = \omega_r$ and $x_0$ satisfies

$$P_{r,1} x_0 - \frac{1}{1 - \Lambda_r} P_{r,1} \, \mathrm{d}(\omega_r) \neq 0$$

while

$$P_{r,2} x_0 - \frac{1}{1 - \Lambda_r} P_{r,2} \, \mathrm{d}(\omega_r) = 0 \,,$$

we obtain the following rate of convergence

$$\hat{\psi}_{r,r}(k) = [\Lambda_r(\omega_r)]^k = (\omega_r - 1)^k \,.$$

It is obvious that this fact has only theoretical value.

## 5. CONCLUDING REMARKS

In this concluding section we give some comments concerning a practical realization of the method described. We may proceed in two different ways. According as to whether we use the definition of the acceleration iterations or (3) in Theorem 1. We found that the use of a combination of both of these approaches in the calculation is preferable. Thus the strategy is to evaluate $x_1, x_2, \ldots, x_{Nm-1}$ and the corresponding vector $y_{Nm}$, to continue by using (3) till $k \leqq k_0$, then to improve $y_{k_0}$ by putting,

$$y_{k_0+1} = f_k^{(m)}(\sigma_1^{(m)}, \ldots, \sigma_N^{(m)})$$

and again to use (3) until $2k_0$, etc.

As usual, we start with $m = 1$ and change it during the calculation. Because the method is suitable for solving many systems with the same matrix and different right hand side vectors, we determine the appropriate $m$ by solving the system with a fixed right hand side vector and keep the value of $m$ fixed for the other right hand side vectors.

Because of difficulty in obtaining the required eigenvalues and also because of possible numerical instabilities the method is effective only if a relatively small number of eigenvalue cuttings are used. Tests show that in practice $t \leqq 4$. The improvement in convergence obtained in this way may be remarkable. The extrapolation used to accelerate the S.O.R. using $t$ eigenvalue cuttings has an asymptotic rate of convergence of order $c\,h$ for the model problem

$$\Delta u = f \quad \text{on} \quad [0, 1] \times [0, 1],$$

with

$$u = 0 \quad \text{on} \quad \partial\,\{[0, 1] \times [0, 1]\},$$

when discretized by the well-known five point finite difference formula. This means that our extrapolation retains the asymptotic behaviour with respect to the mesh size as does the S.O.R. with the optimal relaxation factor.

We have shown that Ljusternik acceleration applied to $H(\omega)$ gives improvements in the convergence rate particularly, if $\omega = \omega_s$ for some $s > 1$. There is no improvement if one extrapolates $H(\omega_r)$ (see Theorem 7.2, Chapter 11, p. 375 in [6]). A similar argument shows that the extrapolation of the S.O.R. with $\omega \leqq 1$ leads to weaker results than that with $\omega > 1$.

A typical example where the extrapolated S.O.R. can effectively be used is the source iterative technique in solving reactor physics diffusion systems.

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## Souhrn

# LJUSTERNIKOVO URYCHLENÍ A EXTRAPOLOVANÁ METODA S.O.R.

Ivo Marek, Jan Zítko

Nechť $\mathscr{X}$ je Banachův prostor, $\mathscr{X}'$ odpovídající duální prostor a $[\mathscr{X}]$ prostor všech ohraničených zobrazení $\mathscr{X}$ do $\mathscr{X}$.

Nechť $u \in \mathscr{X}$ a členy posloupnosti $\{y_k\}_{k=0}^{\infty}$ leží v $\mathscr{X}$. Nechť existuje funkce $\Phi = \Phi(k)$, $k = 0, 1, \ldots$, konstanta $K > 0$ a posloupnost $\{z_k\}_{k=0}^{\infty} \in \mathscr{X}$ tak, že jsou splněny následující podmínky:

1) $\lim \Phi(k) = 0$,
2) $1/\varkappa < \limsup \|z_k\| < \varkappa$,
3) existuje přirozené číslo $k_0$ tak, že pro $k \geqq k_0$ je

$$y_k - u = \Phi(k)\, z_k$$

Jsou-li splněny tyto podmínky, pak řekneme, že rychlost konvergence posloupnosti $\{y_k\}_{k=0}^{\infty}$ k $u$ se rovná $\Phi$ a budeme to symbolicky zapisovat

$$y_k \xrightarrow{\Phi} u\,.$$

Jestliže pro posloupnosti $\{y_k^{(1)}\}_{k=0}^{\infty}$, $\{y_k^{(2)}\}_{k=0}^{\infty}$ platí

$$y_k^{(1)} \xrightarrow{\Phi_1} u\,, \quad y_k^{(2)} \xrightarrow{\Phi_2} u$$

a existuje-li přirozené číslo $k_0$ takové, že pro $k \geqq k_0$ je

$$|\Phi_1(k)| < |\Phi_2(k)|\,,$$

pak řekneme, že posloupnost $\{y_k^{(1)}\}$ konverguje k $u$ rychleji než posloupnost $\{y_k^{(2)}\}$.

Předmětem studia je v tomto článku operátorová rovnice $x = Tx + b$, kde $T \in [X]$, spektrální poloměr $r(T) < 1$ a spektrum operátoru $T$ má následující strukturu:

Existuje posloupnost $\{\lambda_k\}$ (konečná nebo nekonečná) tak, že

$$\lambda_k \in \sigma(T), \quad |\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_k| \geqq \ldots \geqq \tau > 0$$

a je-li $\lambda \in \sigma(T)$, $\lambda \neq \lambda_k$, pak $|\lambda| \leqq \tau$. Uvažujme nyní iterační proces

$$(*) \qquad x_{k+1} = Tx_k + b\,.$$

Pak

$$x_k \xrightarrow{\Phi_1} x^* = Tx^* + b\,,$$

kde $\Phi_1(k) = [r(T)]^k k^{q-1}$. Přitom $q = \max(q_1, \ldots, q_p)$, $q_j$ je násobnost $\lambda_j \in \sigma(T)$, kde $|\lambda_j| = r(T)$ pro $j = 1, 2, \ldots, p$, a pro každé $\mu \in \sigma(T)$, $\mu \neq \lambda_j (j = 1, 2, \ldots, p)$ je $|\mu| < r(T)$. Jestliže $\lambda_1, \lambda_2, \ldots, \lambda_t, \lambda_{t+1}$ jsou navzájem různé póly resolventy $R(\lambda, T)$ násobností $q_1, \ldots, q_{t+1}$ a platí-li

$$|\lambda_1| \geqq \ldots \geqq |\lambda_t| > |\lambda_{t+1}| \geqq |\lambda|$$

pro každé $\lambda \in \sigma(T), \lambda \neq \lambda_j (j = 1, \ldots, t + 1)$, pak můžeme iterační proces $(*)$ urychlit. Nechť

$$p(z) = (z - \lambda_1^m)^{q_1} \ldots (z - \lambda_t^m)^{q_t} = z^N + \sigma_1^{(m)} z^{N-1} + \ldots + \sigma_N^{(m)}\,, \quad N = \sum_{j=1}^{t} q_j\,,$$

kde $m$ je zvolené přirozené číslo a položme

$$y_k = x_k \quad k = 0, 1, \ldots, Nm - 1\,,$$

$$y_k = \frac{1}{p(1)} \left(x_k + \sigma_1^{(m)} x_{k-m} + \ldots + \sigma_N^{(m)} x_{k-Nm}\right).$$

Pak

$$y_k \xrightarrow{\Phi_{t+1}} x^* = Tx^* + b\,,$$

kde

$$\Phi_{t+1}(k) = |\lambda_{t+1}|^k k^{q_{t+1}-1}$$

Navíc je pro $k > Nm$ posloupnost $\{y_k\}$ možné počítat podle $(*)$, tj.

$$y_{k+1} = Ty_k + b\,.$$

Toto je podrobně popsáno a dokázáno ve Větě 1.

V další části práce je pak odvozen algoritmus pro výpočet několika prvních vlastních čísel (Věta 2).

V poslední části práce je pak ukázáno, jak je možné použít vytvořené teorie na urychlení metody S.O.R. Uvažujme systém lineárních algebraických rovnic

$$(**) \qquad Ax = b\,,$$

kde $A$ je positivně definitní $n \times n$ matice, $n \geqq 2$. Předpokládejme, že Jacobiho matice $B$ příslušná k matici $A$ je slabě cyklická s indexem 2 a shodně uspořádaná. Nechť $\mu_1, \ldots, \mu_p$ jsou všechna kladná a navzájem různá vlastní čísla matice $\boldsymbol{B}$. Řešení soustavy $(**)$ hledáme podle algoritmu

$$x_{k+1} = H(\omega)\, x_k + d\,,$$

kde

$$H(\omega) = (I - \omega L)^{-1}\left[\omega U + (1 - \omega) I\right]$$

a

$$d = \omega(I - \omega L)^{-1} (\operatorname{diag} A)^{-1}\, b\,.$$

Označme si

$$\omega_j = \frac{2}{1 + \sqrt{(1 - \mu_j^2)}}\,, \quad \omega_j \in (1, 2)\,.$$

Platí

$$2 > \omega_1 > \omega_2 > \ldots > \omega_p > 1\,.$$

Je známo, že pro $\omega = \omega_1$ je $x_k \overset{\Psi_1}{\to} x^* = A^{-1}b$, kde $\Psi_1(k) = k(\omega_1 - 1)^k$ (optimální S.O.R. (Young)). Nechť jsou známa vlastní čísla $\mu_1, \ldots, \mu_r$ Jacobiho matice $\boldsymbol{B}$ a zvolme

$$\omega_r < \omega < \omega_{r-1}$$

V práci je sestrojena podle obecného postupu posloupnost $\{y_k\}_{k=0}^{\infty}$ taková, že

$$y_{k+1} = H(\omega)\, y_k + d \quad (k > rm)$$

a

$$y_{k+1} \overset{\Psi_r}{\to} x^* = A^{-1}b\,, \quad \text{kde}$$

$$\Psi_r(k) = (\omega - 1)^k \quad \text{(Věta 4).}$$

Zvolíme-li $\omega = \omega_r$, pak

$$\Psi_r(k) = k(\omega_r - 1)^k \quad \text{(Věta 5).}$$

Poznamenejme závěrem, že tato metoda je vhodná pro řešení problémů se soustavami typu $(**)$, kdy matice soustavy je pevná a mění se vektory pravých stran.

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