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APPROXIMATE METHODS FOR SOLVING DIFFERENTIAL EQUATIONS ON INFINITE INTERVAL

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INTRODUCTION

We shall consider some approximate methods of solving the differential equation on the infinite interval $(0, \infty)$

$$-u''(t) + a(t)\,u'(t) + b(t)\,u(t) = f(t)\,, \tag{1.1}$$

with the conditions

$$u(0) = 0\,, \quad u \in L^2_\eta(0, \infty)\,. \tag{1.2}$$

$L^2_\eta(0, \infty)$ denotes the space of square integrable functions on the interval $(0, \infty)$ with the weight function $\eta(t)$.

There are several articles dealing with singular boundary value problems [7], [9] concerning two-point boundary value problems on a finite interval with a singular coefficient. The articles mentioned above include implicitly the solution of our problem only in some particular cases.

Two approximate methods of solving the problem (1.1), (1.2) will be presented. The first method consists in approximating the solution of the problem (1.1), (1.2) for $\eta \equiv 1$ by a sequence of solutions of boundary value problems on finite intervals. The second method (for $\eta = e^{-t}$) is a modified collocation method. The collocation method for problems on finite intervals is discussed in [4], [8], [10].

We shall prove existence of a solution of the problem (1.1), (1.2) and convergence of the methods presented.

PART I

The aim of this part is to construct an integral equation that is equivalent to the differential problem (1.1), (1.2).

1.1 Auxiliary problem

It will be convenient to introduce the equation

(1.3) $$l\,u(t) \equiv -u''(t) + k\,u(t) = v(t)\,, \quad 0 \leqq t < \infty$$

where k is a positive constant,
and the boundary conditions

(1.4) $$u(0) = 0\,, \quad u \in L^2(0, \infty)\,.$$

Lemma 1. *The boundary value problem* (1.3), (1.4) *is selfadjoint in* $L^2(0, \infty)$.

Proof. If u, v satisfy conditions (1.4), and if there exist u', v', u'', v'' and $u'', v'' \in L^2(0, \infty)$, then

$$\int_0^\infty [-u''(t) + k\,u(t)]\,v(t)\,dt =$$
$$= \lim_{t\to\infty} [u(t)\,v'(t) - u'(t)\,v(t)] + \int_0^\infty u(t)\,[-v''(t) + k\,v(t)]\,dt\,.$$

It suffices to prove

$$\lim_{t\to\infty} [u(t)\,v'(t) - u'(t)\,v(t)] = 0\,,$$

i.e.,

$$\forall\varepsilon > 0,\ \exists N,\ \forall\tau, s > N \quad \varrho_{s\tau} \stackrel{\mathrm{df}}{=} |u(\tau)\,v'(\tau) - u'(\tau)\,v(\tau) - u(s)\,v'(s) + u'(s)\,v(s)| \leqq \varepsilon\,.$$

We have

$$\varrho_{s\tau} \leqq \int_s^\tau \left|\frac{d}{dt}[u(t)\,v'(t) - u'(t)\,v(t)]\right| dt = \int_s^\tau |u(t)\,v''(t) - u''(t)\,v(t)|\,dt\,.$$

Using the Schwartz inequality we obtain

$$\varrho_{s\tau} \leqq \left(\int_s^\tau |u(t)|^2\,dt\right)^{1/2} \left(\int_s^\tau |v''(t)|^2\,dt\right)^{1/2} + \left(\int_s^\tau |u''(t)|^2\,dt\right)^{1/2} \left(\int_s^\tau |v(t)|^2\,dt\right)^{1/2}\,.$$

If we recall that for an arbitrary function f belonging to $L^2(0, \infty)$

$$\lim_{s,\tau\to\infty} \int_s^\tau f^2(t)\,dt = 0$$

we can conclude that $\varrho_{s\tau} \to 0$ if $s, \tau \to \infty$, q.e.d.

Let L_η mean the differential operator defined by the differential expression lu (1.3) and the boundary conditions (1.2). The domain of this operator is the following set:

$$Q_\eta = \{u \in L^2_\eta(0, \infty),\ u'' \in L^2_\eta(0, \infty) \text{ and } u(0) = 0\}\,.$$

Let $L \stackrel{\mathrm{df}}{=} L_1$.

Lemma 2. *The domain of the inverse operator L^{-1} is the whole space $L^2(0, \infty)$. In the space $L^2(0, \infty)$ the operator L^{-1} is bounded.*

Proof. The proof follows from the following theorem:

Let us consider the differential expression

$$lu \equiv -(p_0 u')' + p_1 u$$

with additional conditions $\lim_{t \to \infty} p_1(t) = p$ and $p_0(t) > 0$. The selfadjoint operator in $L^2(0, \infty)$ generated by the expression lu has on the interval $(-\infty, p)$ only a point spectrum (cf. [5], § 24). The above theorem together with the fact that $\lambda = 0$ is not an eigenvalue of L proves Lemma 2.

Theorem 1. *If the weight function $\eta(t)$ is continuous and there exist constants c_1, c_2, c_3 that*

1° $$0 < \eta(t) \left(\min_{0 \leqq \xi \leqq t} \eta(\xi) \right)^{-1} \leqq c_1 < \infty ,$$

2° $$\exists s \in [0, \sqrt{k}) \quad \textit{such that} \quad 0 < c_2 \leqq e^{2st} \eta(t) \leqq c_3 < \infty ,$$

then the inverse operator L_η^{-1} is defined on the whole space $L_\eta^2(0, \infty)$ in the following manner:

$$L_\eta^{-1} f(t) = \int_0^\infty G(t, \tau) f(\tau) \, d\tau , \quad f \in L_\eta^2(0, \infty) , \tag{1.5}$$

where

$$G(t, \tau) = \begin{cases} \dfrac{1}{\sqrt{k}} e^{-\sqrt{(k)}t} \operatorname{sh} \sqrt{(k)}\, \tau & \tau < t \\ \dfrac{1}{\sqrt{k}} e^{-\sqrt{(k)}\tau} \operatorname{sh} \sqrt{(k)}\, t & \tau > t \end{cases} \tag{1.6}$$

Corollary 1. *For example, the assumptions 1°, 2° are satisfied by the functions*

$$\eta(t) = \varrho(t) \quad \textit{or} \quad \eta(t) = e^{-2mt} \varrho(t)$$

where the function $\varrho(t)$ is continuous and bounded and $m \in [0, \sqrt{k})$.

Proof of Theorem 1.

1. The proof has two parts. The first part consists in showing that the theorem is true for the weight function $\eta \equiv 1$.

From Lemma 2 we know that the inverse operator L^{-1} is bounded and defined on the whole space $L^2(0, \infty)$. In this case we can apply the general theorem about the form of the resolvent of differential operators ([1], Part XIII, 3.4).

From this theorem we have

$$L^{-1} f(t) = \int_0^\infty G(t, \tau) f(\tau)\,\mathrm{d}\tau\,, \quad f \in L^2(0, \infty)\,.$$

The function $G(t, \cdot)$ belongs to $L^2(0, \infty)$ for every $t \in (0, \infty)$ and satisfies the jump equations.

With regard to the fact that the operator L is selfadjoint we can use the general theorem about the form of the Green function ([1], Part XIII, 3,10).

From this it follows that the Green function has the form (1.6). Thus, Theorem 1 is proved for $\eta \equiv 1$.

2. If $\eta \not\equiv 1$ then Lemmas 1 and 2 do not hold. We must prove Theorem 1 in another way.

Therefore, in the second part we must show that the function

$$(1.7) \quad u(t) = \frac{1}{\sqrt{k}} \left\{ e^{-\sqrt{(k)}t} \int_0^t (\operatorname{sh} \sqrt{(k)}\,\tau) f(\tau)\,\mathrm{d}\tau + \operatorname{sh} \sqrt{(k)}\,t \int_t^\infty e^{-\sqrt{(k)}\tau} f(\tau) \mathrm{d}\tau \right\}$$

satisfies the equation (1.3) and the conditions (1.2). We can easily verify that the equation (1.3) and the first condition $u(0) = 0$ are satisfied.

Now it is sufficient to prove that if $f \in L^2_\eta(0, \infty)$ then $L_\eta^{-1} f$ belongs to $L^2_\eta(0, \infty)$.

We shall show that the first as well as the second component in the formula (1.7) belongs to the space $L^2_\eta(0, \infty)$. It is convenient to denote these components by w_1 and w_2 respectively. Without loss of generality we may assume that f is non-negative.

For the function w_1 we have the following estimate of the norm:

$$\|w_1\|^2_{L_\eta^2(0,\infty)} \leqq c_1 \int_0^\infty e^{-2\sqrt{(k)}t} \left[\int_0^t (\operatorname{sh} \sqrt{(k)}\,\tau) \sqrt{(\eta(\tau))} f(\tau)\,\mathrm{d}\tau \right]^2 \mathrm{d}t \leqq$$
$$\leqq c \,. \left\| \int_0^\infty G(t, \tau)\, v(\tau)\,\mathrm{d}\tau \right\|^2_{L^2(0,\infty)}\,,$$

where $v(\tau) = \sqrt{(\eta(\tau))} f(\tau)$, i.e., $v \in L^2(0, \infty)$, and c is a positive constant.

Now, we consider the norm of the second component w_2. From the assumption 2° we have evidently

$$\|w_2\|^2_{L_\eta^2(0,\infty)} = \int_0^\infty \eta(t)\, e^{2st} e^{-2st}\, w_2^2(t)\,\mathrm{d}t \leqq c_3 \int_0^\infty e^{-2st}\, w_2^2(t)\,\mathrm{d}t\,,$$

and

$$(e^{-st} \operatorname{sh} \sqrt{(k)}\,t)^2 = \operatorname{sh} (\sqrt{(k)} - s)\, t + e^{-(\sqrt{(k)}-s)t} (1 - e^{-2st}) \leqq$$
$$\leqq 2 \operatorname{sh}^2 (\sqrt{(k)} - s) + 2e^{-2(\sqrt{(k)}-s)t}\,.$$

Thus we find

$$\|w_2\|^2_{L_\eta^2(0,\infty)} \leqq 2 \frac{c_3}{c_2} \int_0^\infty \left\{ \operatorname{sh} (\sqrt{(k)} - s)\, t \int_t^\infty e^{-(\sqrt{(k)}-s)\tau} \sqrt{(\eta(\tau))} f(\tau)\,\mathrm{d}\tau \right\}^2 \mathrm{d}t + M\,,$$

where

$$M = 2\frac{c_3}{c_2}\int_0^\infty e^{-2(\sqrt{(k)}(-s)t}\left(\int_t^\infty e^{-(\sqrt{(k)}-s)\tau}\sqrt{(\eta(\tau))}\,f(\tau)\,\mathrm{d}\tau\right)^2 \mathrm{d}t < \infty\,.$$

If we take into account the assumption 2° and the additional condition $f(t) \geqq 0$ we obtain

$$\|w_2\|^2_{L_\eta^2(0,\infty)} \leqq c\left\|\int_0^\infty \tilde{G}(t,\tau)\,v(\tau)\,\mathrm{d}\tau\right\|^2_{L^2(0,\infty)} + M\,,$$

where c is a positive constant, and $\tilde{G}(t,\tau)$ is defined by the formula (1.6)

$$\text{for}\quad \tilde{k} = (\sqrt{(k)} - s)^2 > 0\,.$$

The results obtained in the first part and the above estimates of the norm w_1 and w_2 imply that

$$\|u\|_{L_\eta^2(0,\infty)} < \infty \quad \text{if} \quad f \in L^2_\eta(0,\infty)\,,$$

q.e.d.

1.2. Integral equation

The equation (1.1) can be written in the following form:

$$-u''(t) + k\,u(t) + [b(t) - k]\,u(t) + a(t)\,u'(t) = f(t)$$

where k is a positive constant for which $\eta(t)$ satisfies the assumption 2° of Theorem 1.

If the conditions of Theorem 1 are satisfied then we can substitute

$$L_\eta\,u(t) = v(t)\,.$$

It is easy to find that the problem (1.1), (1.2) takes an equivalent form

$$v(t) + a(t)\int_0^\infty \frac{\partial G(t,\tau)}{\partial t}\,v(\tau)\,\mathrm{d}\tau + [b(t) - k]\int_0^\infty G(t,\tau)\,v(\tau)\,\mathrm{d}\tau = f(t)\,,$$

where $v \in L^2_\eta(0,\infty)$.

This equation can be expressed in a shorter form

$$[I + K]\,v = f\,, \tag{1.8}$$

where I is the identity operator in $L^2_\eta(0,\infty)$ and K is the integral operator with the kernel

$$K(t,\tau) = [b(t) - k]\,G(t,\tau) + a(t)\frac{\partial G(t,\tau)}{\partial t}\,. \tag{1.9}$$

Under the hypothesis of Theorem 1 and the assumption that the functions

a, $[b(\cdot) - k]$ belong to $L^2_\eta(0, \infty)$, the operator K transforms the space $L^2_\eta(0, \infty)$ in $L^2_\eta(0, \infty)$.

Thus we have the integral equation (1.8) in the Hilbert space $L^2_\eta(0, \infty)$.

PART II

This part is devoted to the approximation of boundary value problem (1.1), (1.2) by a sequence of boundary value problems on finite intervals.

We shal consider the problem (1.1), (1.2) only for $\eta \equiv 1$. In this case the conditions of Theorem 1 are satisfied for $s = 0$. According to the results of Part I, the problem (1.1), (1.2) is equivalent to the integral equation

$$[I + K]\, v = f \tag{2.1}$$

in the space $L^2(0, \infty)$.

2.1 Projection methods

Let us consider an equation of the form (2.1) in a Banach space X. We can separate a class of approximate methods solving this equation — the so called projection methods.

A projection method is defined if we have a sequence of subspaces $\{X_n\}$ of the space X and a sequence $\{P_n\}$ of continuous projection operators from X onto X_n. The approximate equations have the form

$$[I + P_n K]\, v_n = P_n f \tag{2.2}$$

where $v_n \in X_n$.

It is well-known [4], [10] that the following theorem holds.

Theorem 2. *Suppose that*

1° *the homogeneous equation $[I + K]\, v = 0$ has only the trivial solution in the space X,*

2° *the operators P_n converge strongly to the identity operator $I : X \to X$,*

3° *the operator K is completely continuous.*

Then for all sufficiently large n there exists a unique solution $v_n \in X_n$ of the equation (2.2). *Moreover, $v_n(t)$ converge in the norm to $v_0(t)$ and the convergence satisfies*

$$\|v_0 - v_n\| \leqq c\|P_n v_0 - v_0\|$$

where c is a constant independent of n.

2.2 Approximation by boundary value problems on finite intervals

Let the projection operators P_n be defined in the following way:

$$P_n : L^2(0, \infty) \to L^2(0, n)$$

and

$$P_n f(t) = \begin{cases} f(t) & t \leqq n \\ 0 & t > n . \end{cases} \tag{2.3}$$

Defining the subspaces $X_n = L^2(0, n)$ and the operators P_n we obtain a projection method.

It turns out that the approximate equation (2.2) for P_n and X_n defined above is equivalent to a certain boundary value problem on a finite interval $(0, n)$ as we have the following lemma:

Lemma 3. *If $a \in L^2(0, \infty)$ and $\exists k > 0\ [b(\cdot) - k] \in L^2(0, \infty)$ and if the operators P_n are defined by the formula (2.3), then the equation (2.2) is equivalent to the following boundary value problem:*

$$-u_n''(t) + a(t)\, u_n'(t) + b(t)\, u_n(t) = f(t)\,, \quad t \in (0, n)\,, \tag{2.4}$$

$$u_n(0) = 0\,, \quad \sqrt{(k)}\, u_n(n) = -u_n'(n)\,. \tag{2.5}$$

Proof. We show that if v_n is a solution of the equation (2.2), then $u_n = L^{-1} v_n$ is a solution of the problem (2.4), (2.5). For $t < n$

$$f(t) = P_n f(t) = [I + P_n K]\, v_n = [I + K]\, v_n\,.$$

This means that $u_n = L^{-1} v_n$ satisfies the equation (2.4) for $t < n$ because the hypotheses of Theorem 1 are satisfied. The function u_n satisfies also the condition at the point 0. With regard to the equality

$$v_n(t) = 0 \quad \text{for} \quad t > n$$

we have

$$u_n(t) = \frac{1}{\sqrt{k}} \left\{ e^{-\sqrt{(k)} t} \int_0^t (\mathrm{sh} \sqrt{(k)}\, \tau)\, v_n(\tau)\, \mathrm{d}\tau + \mathrm{sh} \sqrt{(k)}\, t \int_t^n e^{-\sqrt{(k)} \tau}\, v_n(\tau)\, \mathrm{d}\tau \right.$$

and

$$u_n'(t) = -e^{-\sqrt{(k)} t} \int_0^t (\mathrm{sh} \sqrt{(k)}\, \tau)\, v_n(\tau)\, \mathrm{d}\tau + \mathrm{ch} \sqrt{(k)}\, t \int_t^n e^{-\sqrt{(k)} \tau}\, v_n(\tau)\, \mathrm{d}\tau\,.$$

thus for $t = n$

$$u_n'(n) = -\sqrt{(k)}\, u_n(n)\,.$$

Now it suffices to prove that on the interval $(0, \infty)$ there exists an extension $\tilde{u}_n$ of the solution u_n of the problem (2.4), (2.5), such that $\tilde{u}_n \in L^2(0, \infty)$ and $v_n = L\tilde{u}_n$ is a solution of (2.2).

Let for $t > n$

$$\tilde{u}_n(t) = u_n(n)\, e^{(n-t)\sqrt{k}} . \tag{2.6}$$

Then it is easy to find that for $t > n$

$$-\tilde{u}''(t) + k\, \tilde{u}_n(t) = 0 .$$

Hence $v_n = L\tilde{u}_n$ is a solution of (2.2). q.e.d.

2.3 Convergence

The convergence of the method defined in 2.2 follows from the general Theorem 2. Thus it suffices to show when the hypotheses of Theorem 2 are satisfied.

Lemma 4. *Let us suppose that $a \in L^2(0, \infty)$ and that there exists a positive constant k such that $[b(\cdot) - k] \in L^2(0, \infty)$.*

Then the operator K is completely continuous.

Proof. First we shall prove the following estimate:

$$\int_0^\infty \int_0^\infty K^2(t, \tau)\, d\tau\, dt < \infty . \tag{2.7}$$

The Schwartz inequality permits us to obtain

$$\int_0^\infty \int_0^\infty K^2(t, \tau)\, d\tau\, dt \leqq \int_0^\infty \left[|b(t) - k| \left(\int_0^\infty G^2(t, \tau)\, d\tau \right)^{1/2} + \right.$$

$$\left. + |a(t)| \left(\int_0^\infty \left(\frac{\partial G(t, \tau)}{\partial t} \right)^2 d\tau \right)^{1/2} \right]^2 dt .$$

It is easy to find that

$$\int_0^\infty G^2(t, \tau)\, d\tau = \frac{1}{4k} \left\{ \frac{1}{\sqrt{k}} - \left(2t + \frac{1}{\sqrt{k}} \right) e^{-2\sqrt{(k)}t} \right\}$$

and

$$\int_0^\infty \left[\frac{\partial G(t, \tau)}{\partial t} \right]^2 d\tau = \frac{1}{4} \left\{ \frac{1}{\sqrt{k}} + \left(\frac{1}{\sqrt{k}} - 2t \right) e_\varphi^{-2\sqrt{(k)}t} \right\} .$$

Hence

$$\int_0^\infty \int_0^\infty K^2(t, \tau)\, d\tau\, dt \leqq c \int_0^\infty \{ |b(t) - k| + |a(t)| \}^2\, dt < \infty .$$

This implies the complete continuity of K because, as is well-known [1], every integral operator in $L^2(0, \infty)$ with a kernel satisfying the estimate (2.7) is completely continuous. q.e.d.

It remains to prove that the operators P_n defined by the formula (2.3) satisfy the condition 2° of Theorem 2.

We have evidently

$$\|f - P_n f\|_{L^2(0,\infty)} = \left(\int_n^\infty f^2(t)\, dt\right)^{1/2}$$

and if $n \to \infty$ then for any $f \in L^2(0, \infty)$

$$\lim_{n\to\infty} \int_n^\infty f^2(t)\, dt = 0\,.$$

Thus if we assume that the homogeneous equation (2.1) has only the trivial solution in $L^2(0, \infty)$ and that the hypotheses of Lemma 4 are satisfied, then we obtain convergence in the norm of $L^2(0, \infty)$ of the sequence $\{v_n\}$ to a solution of the equation (2.1).

Returning to the differential form of the problem (2.1) we obtain the following theorem:

Theorem 3. Let $\tilde{u}_n$ be the extension of u_n defined by the formula (2.6). Suppose that $a \in L^2(0, \infty)$ and $\exists k > 0$, $[b(\cdot) - k] \in L^2(0, \infty)$. Also suppose that the homogeneous problem (1.1), (1.2) for $\eta \equiv 1$ has a unique solution.

Then for any $f \in L^2(0, \infty)$ there exists a solution u_0 of the problem (1.1), (1.2) and

$$\|\tilde{u}_n - u_0\|_{L^2(0,\infty)} \xrightarrow[n\to\infty]{} 0$$

and there exist constants independent of n and f such that

$$\sup_{0<t<\infty} |\tilde{u}_n^i(t) - u_0^i(t)| \leqq c_i\left(\int_n^\infty (-u_0'' + ku_0)^2\, dt\right)^{1/2}, \quad i = 0, 1\,.$$

Proof. This theorem is an easy consequence of Theorem 2.

With regard to the continuity of the operator L^{-1} we obtain the convergence of u_n to u_0 in the norm of $L^2(0, \infty)$.

The estimate of the term $\sup\limits_{0<t<\infty} |u_n^i(t) - u_0^i(t)|$ follows from the following inequalities:

$$|\tilde{u}_n(t) - u_0(t)| \leqq \left(\int_0^\infty G^2(t, \tau)\, d\tau\right)^{1/2} . \|v_n - v_0\|_{L^2(0,\infty)}\,,$$

$$|\tilde{u}_n'(t) - u_0(t)| \leqq \left(\int_0^\infty \left(\frac{\partial G(t, \tau)}{\partial t}\right)^2 d\tau\right)^{1/2} . \|v_n - v_0\|_{L^2(0,\infty)}$$

and from the fact that

$$\|v_n - v_0\|_{L^2(0,\infty)} \leqq c\,.\, \|P_n v_0 - v_0\| \leqq c\left(\int_n^\infty (v_0(t))^2\, dt\right)^{1/2}.$$

The proof is complete.

Let us consider the equation

$$[I + K]\, v = f \tag{3.1}$$

in the space $L^2_\eta(0, \infty)$ for $\eta(t) = e^{-t}$. The operators I and K are defined in 1.2.

From the Theorem 1 it follows that the equation (3.1) is equivalent to the differential problem (1.1), (1.2) if the constant k from the formula (1.9) satisfies $k > \frac{1}{4}$.

3.1 Description of collocation method

It is possible to define such subspaces X_n and projection operators P_n so that the method obtained is simple and has direct numerical applications.

Let $C[0, \infty]$ mean the space of continuous functions on $[0, \infty)$ which have finite limits in infinity.

Suppose a partition π_n on the interval $(0, \infty)$ is given:

$$\pi_n : 0 < t_0^n < t_1^n < \ldots < t_n^n . \tag{3.2}$$

Let $t_i^n = -\ln x_i^n$, $i = 0, \ldots, n$, where x_i^n are zeros of orthogonal polynomials on the interval $[0, 1]$ with a positive and continuous weight function $\varrho(x)$. Define P_n as the projection operator from $C[0, \infty]$ onto the space X_n which is generated by the basis $(1, e^{-t}, \ldots, e^{-nt})$.

More precisely,

$$P_n v = v_n = \alpha_0 + \alpha_1 e^{-t} + \ldots + \alpha_n e^{-nt} ,$$

where the coefficients $\alpha_0, \ldots, \alpha_n$ are such that

$$v(t_i^n) = v_n(t_i^n) , \quad i = 0, \ldots, n .$$

According to the definition of a projection method (point 2.1) we have an approximate equation in the form:

$$[I + P_n K]\, v_n = P_n f . \tag{3.3}$$

Let us return to the differential problem (1.1), (1.2) for $\eta(t) = e^{-t}$.

It is clear that the equalities

$$P_n f(t_i^n) = P_n\, g(t_i^n) , \quad i = 0, \ldots, n$$

for $f,\ g \in C[0, \infty]$ imply that

$$P_n f \equiv P_n g .$$

Thus the equation (3.3) is equivalent to the system of equations

$$v_n(t_i^n) + K\,v(t_i^n) = f(t_i^n)\,, \quad i = 0, \ldots, n$$

which with regard to Theorem 1 can be written as

$$(3.4) \qquad -u_n''(t_i^n) + a(t_i^n)\,u_n'(t_i^n) + b(t_i^n)\,u_n(t_i^n) = f(t_i^n)\,, \quad i = 0, \ldots, n$$

where

$$(3.5) \qquad u_n(t) = \int_0^\infty G(t, \tau)\,v_n(\tau)\,\mathrm{d}\tau\,.$$

Since

$$\int_0^\infty G(t, \tau)\,e^{-s\tau}\,\mathrm{d}\tau = \frac{1}{k - s^2}\left[e^{-st} - e^{-\sqrt{(k)}t}\right] \quad \text{for} \quad s \neq \sqrt{k}$$

and

$$\int_0^\infty G(t, \tau)\,e^{-s\tau}\,\mathrm{d}\tau = \frac{1}{2\sqrt{k}}\,e^{-\sqrt{(k)}t}t \quad \text{for} \quad s = \sqrt{k}\,,$$

the method defined above (3.3) is the so-called collocation method which consists in finding a function of the form

$$(3.6) \qquad u_n(t) = \sum_{i=0}^{n} \beta_i \varphi_i$$

where

$$\varphi_i(t) = e^{-it} - e^{-\sqrt{(k)}t} \quad \text{if} \quad i \neq \sqrt{k} \quad \text{and} \quad \varphi_i(t) = e^{-it}t \quad \text{if} \quad i = \sqrt{k}\,,$$

which satisfies the differential equation (1.1) exactly at the collocation points, i.e., at t_i^n, $i = 0, \ldots, n$.

3.2 Properties of operators P_n and K

Lemma 5. *For each function $v \in [0, \infty]$ the sequence $\{P_n v\}$ converges to v in the norm of the space $L_\eta^2(0, \infty)$.*

Proof. We shall base our proof on the Erdös-Turan theorem [6] about a convergence of interpolation polynomials.

Let us consider a function f continuous on the interval $[0, 1]$. Suppose that $L_n f$ is its interpolation polynomial of degree n based on nodes $x_0^n, \ldots, x_n^n$ which are the zeros of orthogonal polynomials with a continuous weight function $\varrho(x) > 0$. The Erdös-Turan theorem implies that

$$\lim_{n \to \infty} \int_0^1 \varrho(x)\,[L_n f(x) - f(x)]^2\,\mathrm{d}x = 0\,.$$

Let us substitute $x = e^{-t}$. On account of

$$L_n f(e^{-t}) = P_n v(t) \quad \text{where} \quad v(t) \stackrel{\text{df}}{=} f(e^{-t})$$

we obtain

$$\int_0^1 \varrho(x) \left[L_n f(x) - f(x)\right]^2 \mathrm{d}x = \int_0^\infty e^{-t} \varrho(e^{-t}) \left[P_n v(t) - v(t)\right]^2 \mathrm{d}t .$$

It follows that

$$\lim_{n\to\infty} \int_0^\infty e^{-t}\left[P_n v(t) - v(t)\right]^2 \mathrm{d}t = 0 ,$$

q.e.d.

Lemma 6. *Suppose that*

1° $a, b \in C[0, \infty]$,
2° $\exists k > \frac{1}{4} \lim_{t\to\infty} (b(t) - k)\, e^{t/2} = 0$,
3° $\lim_{t\to\infty} a(t)\, e^{t/2} = 0$.

Then the operator K maps the space $L^2_\eta(0, \infty)$ into $C[0, \infty]$.

Proof. By the definition of K (1.9),

$$K\, v(t) = (b(t) - k) \int_0^\infty G(t, \tau)\, v(\tau)\, \mathrm{d}\tau + a(t) \int_0^\infty \frac{\partial G(t, \tau)}{\partial t}\, v(\tau)\, \mathrm{d}\tau .$$

It is obvious that Kv is a continuous function on the interval $[0, \infty)$.

It remains to prove that the function $K\, v(t)$ has a finite limit in infinity.

The function Kv may also be written as

$$K\, v(t) = (b(t) - k)\, e^{t/2}(w_1 + w_2) + \sqrt{(k)}\, a(t)\, e^{t/2}(w_2 - w_1) +$$
$$+ \frac{1}{2}\left[\frac{1}{\sqrt{k}}(b(t) - k) - a(t)\right] e^{-\sqrt{(k)}t} \int_t^\infty e^{-\sqrt{(k)}\tau}\, v(\tau)\, \mathrm{d}\tau ,$$

where

$$w_1 = \frac{1}{2\sqrt{k}}\, e^{-(\sqrt{(k)}+1/2)t} \int_0^t \left(e^{\sqrt{(k)}\tau} - e^{-\sqrt{(k)}\tau}\right) v(\tau)\, \mathrm{d}\tau ,$$

$$w_2 = \frac{1}{2\sqrt{k}}\, e^{(\sqrt{(k)}-1/2)t} \int_t^\infty e^{-\sqrt{(k)}\tau}\, v(\tau)\, \mathrm{d}\tau .$$

By the Schwartz inequality it is easy to find that

$$|w_1(t)| \leqq \alpha_1 \|v\|_{L_\eta{}^2(0,\infty)} , \tag{3.7}$$

$$|w_2(t)| \leqq \alpha_2 \|v\|_{L_\eta{}^2(0,\infty)} , \tag{3.8}$$

where α_1, α_2 are independent of t.

The above fact together with the asumptions $1°-3°$ implies Lemma 6.

Lemma 7. *Under the hypotheses of Lemma 6 the operator $K : L^2_\eta(0, \infty) \to C[0, \infty]$ is completely continuous.*

Proof. As we know from the Arzela theorem we must prove that the operator K maps the unit sphere from $L^2_\eta(0, \infty)$ into a set of equicontinuous and uniformly bounded functions from $C[0, \infty]$.

From the proof of Lemmas 6 and 7 it evidently follows that K maps the unit sphere into a set of uniformly bounded functions. The definition of a set P of equicontinuous functions in the space $C[0, \infty]$ is as follows:

For every $\varepsilon > 0$ there exist open sets $Q_1, \ldots, Q_n$ such that

$$[0, \infty] = \bigcup_{j=1}^{n} Q_j \quad \text{and} \quad \forall t_1, t_2 \in Q_j\,, \quad j = 1, \ldots, n\,, \quad |f(t_1) - f(t_2)| < \varepsilon$$

for an arbitrary function belonging to P.

We may assume without loss of generality that $t_2 > t_1$. We shall prove that for every $\varepsilon > 0$ there exist δ and γ such that if $t_2 - t_1 < \delta$ or $t_1 > \gamma$ then for every $v\ (\|v\|_{L_\eta^2(0,\infty)} \leqq 1)$

$$|K\, v(t_1) - K\, v(t_2)| < \varepsilon\,.$$

Let us denote

$$c_\pm(t) = b(t) - k \pm \sqrt{(k)}\, a(t)\,.$$

We have

$$|K\, v(t_1) - K\, v(t_2)| \leqq A + B\,,$$

where

$$A = \left| \frac{1}{\sqrt{k}}\, c_-(t_1)\, e^{-\sqrt{(k)}t_1} \int_0^{t_1} (\operatorname{sh} \sqrt{(k)}\, \tau)\, v(\tau)\, d\tau - \right.$$

$$\left. - \frac{1}{\sqrt{k}}\, c_-(t_2)\, e^{-\sqrt{(k)}t_2} \int_0^{t_2} (\operatorname{sh} \sqrt{(k)}\, \tau)\, v(\tau)\, d\tau \right|,$$

$$B = \frac{1}{2k} \left| \left[c_+(t_1)\, e^{\sqrt{(k)}t_1} + c_-(t_2)\, e^{\sqrt{(k)}t_1}\right] \int_{t_1}^{\infty} e^{-\sqrt{(k)}\tau} v(\tau)\, d\tau - \right.$$

$$\left. - \left[c_+(t_2)\, e^{\sqrt{(k)}t_2} + c_-(t_2)\, e^{\sqrt{(k)}t_2}\right] . \int_{t_2}^{\infty} e^{-\sqrt{(k)}\tau}\, v(\tau)\, d\tau \right| .$$

To A we apply the Schwartz inequality and the estimate (3.7) which give

$$A \leqq \alpha_1 |e^{t1/2}\, c_-(t_1) - e^{t1/2}\, c_-(t_2)| + \left| \frac{1}{\sqrt{k}}\, c_-(t_2) \right| \left(\int_{t_1}^{t_2} (\operatorname{sh} \sqrt{(k)}\, \tau)^2\, d\tau \right)^{1/2} .$$

Likewise, by using the estimation (3.8) and denoting

$$\alpha_3 = \sup_{0<t<\infty} |c_+(t)|\, \frac{1}{2k}\, e^{-(\sqrt{(k)}+1/2)} \int_t^{\infty} e^{-\sqrt{(k)}\tau}\, v(\tau)\, d\tau\,,$$

we obtain

$$B \leqq \frac{1}{2k}(\alpha_2 + \alpha_3)\left|c_+(t_1)\, e^{t_1/2} - c_+(t_2)\, e^{t_2/2}\right| + \frac{1}{2k}\left|c_+(t_2)\, e^{-\sqrt{(k)}t_2} - \right.$$

$$\left. - c_-(t_2)\, e^{\sqrt{(k)}t_2}\right|\left(\int_{t_1}^{t_2} e^{-2\sqrt{(k)}\tau}\,\mathrm{d}\tau\right)^{1/2}.$$

With regard to the hypotheses of Lemma 7 it follows that if $|t_1 - t_2| \to 0$ or $t_1 \to \infty$ then $A \to 0$ and $B \to 0$. *q.e.d.*

3.3 Convergence

Theorem 4. *Suppose that*

1° $a, b \in C[0, \infty]$,
2° $\exists k > \frac{1}{4} \lim\limits_{t\to\infty} [b(t) - k]\, e^{t/2} = 0$,
3° $\lim\limits_{t\to\infty} a(t)\, e^{t/2} = 0$,
4° the homogeneous problem $[I + K]\, v = 0$ has only the trivial solution in the space $L^2_\eta(0, \infty)$.

Then for every f belonging to $L^2_\eta(0, \infty)$ there exists a unique solution $v_0 \in L^2_\eta(0, \infty)$ of the equation (3.1) and for all sufficiently large n there exists a unique solution of the equation (3.3) which satisfies

$$\|v_0 - v_n\|_{L_\eta^2(0,\infty)} \leqq c \,.\, \|P_n v_0 - v_0\|_{L_\eta^2(0,\infty)}$$

where c is a positive constant independent of n.

Proof. This proof is a certain modification of the proof of Theorem 2. For the sake of clarity we will quote it in its full form.

A convergent sequence in $C[0, \infty]$ converges also in the norm of the space $L^2_\eta(0, \infty)$ because the weight function η is integrable. Thus, taking into account Lemma 7, we can state that the operator K is completely continuous also as the operator from $L^2_\eta(0, \infty)$ into $L^2_\eta(0, \infty)$. On the other hand, the following general theorem [2] is known:

If X is a Banach space and $T = I + K : X \to X$ is the sum of the identity operator I and a completely continuous operator K and the equation $Tx = 0$ has only the trivial solution in X, then

$$T(X) = X\,.$$

Applying the Banach theorem about the inverse operator to the operator $I + K :$ $: L^2_\eta(0, \infty) \to L^2_\eta(0, \infty)$ where K is defined in 1.2 we obtain that $(I + K)^{-1}$ is continuous on $L^2_\eta(0, \infty)$, i.e.,

$$(3.9) \qquad \exists m > 0\,,\ \forall v \in L^2_\eta(0, \infty)\ \ \|(I + K)\, v\|_{L\eta^2(0,\infty)} \geqq m\|v\|_{L\eta^2(0,\infty)}\,.$$

From Lemmas 5 and 7 it follows by contradiction that

$$\|P_n K - K\|_{L_\eta^2 \to L_\eta^2} \xrightarrow[n \to \infty]{} 0 ,$$

i.e.,

$$\exists n_0 > 0 , \quad \forall n > n_0 \ \|P_n K - K\|_{L_\eta^2 \to L_\eta^2} < \frac{m}{2} . \tag{3.10}$$

The estimates (3.9) and (3.10) imply

$$\|[I + P_n K] v\| = \|[I + K] v + [P_n K - K] v\| \geqq \frac{m}{2} .$$

This means that there exists a continuous inverse operator $[I + P_n K]^{-1}$.

For each function $v \in L_\eta^2(0, \infty)$ we have

$$\|v\| = \|(I + P_n K)(I + P_n K)^{-1} v\| \geqq \frac{m}{2} \|(I + P_n K)^{-1} v\| .$$

Thus

$$\|(I + P_n K)^{-1} v\| \leqq \frac{2}{m} \|v\| ,$$

i.e., the norms of the operators $(I + P_n K)^{-1}$ are uniformly bounded.

We have the following equalities:

$$(I + P_n K) v = P_n f + (I - P_n) v ,$$
$$(I + P_n K)(v_0 - v_n) = (I - P_n) v_0 .$$

Taking into account the above proved properties of the operators $(I + P_n K)$ we can state that

$$\|v_n - v_0\|_{L_\eta^2(0,\infty)} \leqq \|(I + P_n K)^{-1}\| \, \|P_n v_0 - v_0\| \leqq \frac{2}{m} \|P_n v_0 - v_0\|$$

q.e.d.

Corollary 1. *If the solution v_0 of the equation (3.1) belongs to the space $C[0, \infty]$, then with regard to Lemma 5 $v_n \to v_0$ in the norm of the space $L_\eta^2(0, \infty)$.*

Corollary 2. *If v_0 belongs to the space $C[0, \infty]$ then there exists a constant $\alpha > 0$ such that the following estimate holds*

$$\|v_0 - v_n\|_{L_\eta^2(0,\infty)} \leqq \alpha \inf_{v \in X_n} \{\|v_0 - v\|_{C[0,\infty]}\} .$$

This follows directly from the Banach-Steinhaus theorem in viritue of the fact that for any function $v \in X_n$

$$\|P_n v_0 - v_0\|_{L_\eta^2(0,\infty)} = \|P_n(v_0 - v) - (v_0 - v)\|_{L_\eta^2(0,\infty)} \leqq$$
$$\leqq \|I - P_n\| \, \|v_0 - v\|_{C[0,\infty]} .$$

Let us return to the differential problem (1.1), (1.2) for $\eta(t) = e^{-t}$. Theorem 4 implies the following theorem about convergence of the collocation method.

Theorem 5. *Suppose that*

1° $a, b \in C[0, \infty]$,

2° $\lim_{t\to\infty} e^{t/2}\, a(t) = 0$, $\exists k > \frac{1}{4} \lim_{t\to\infty} (b(t) - k)\, e^{t/2} = 0$,

3° *the homogeneous problem* (1.1), (1.2) *has only the trivial solution,*

4° *the solution* u_0 *of the problem* (1.1), (1.2) *belongs to* $C^2[0, \infty]$.

If $1^\circ - 3^\circ$ *then for every* $f \in L^2_\eta(0, \infty)$ *there exists a unique solution of the problem* (1.1), (1.2).

If $1^\circ - 4^\circ$ *then there exists* n_0 *such that for* $n > n_0$ *the approximate solution* u_n *defined by* (3.4), (3.5) *is unique and the sequence* $\{u_n\}$ *converges to* u_0 *in the supremum norm on any finite interval* $[0, \xi]$; *more exactly*

$$\forall \xi < \infty\,, \quad \sup_{0 \leqq t \leqq \xi} |u_n^i(t) - u_0^i(t)| \leqq c_i e^{\xi/2} \|v_n - v_0\|_{L\eta^2(0,\infty)}\,,$$

where $i = 0, 1$.

Proof. By Theorem 4 there exists n_0 such that for $n > n_0$ there exists a uniquely defined solution of the equation (3.3). From (3.5) it follows that u_n is also well defined for $n > n_0$.

Using the Schwartz inequality we can estimate the difference between u_n^i and u_0^i, $i = 0, 1$ in the following way:

$$|u_n(t) - u_0(t)| \leqq \left(\int_0^\infty \frac{1}{\eta(\tau)}\, G^2(t, \tau)\, \mathrm{d}\tau\right)^{1/2} \|v_n - v_0\|_{L\eta^2(0,\infty)}\,,$$

$$|u_n'(t) - u_0'(t)| \leqq \left(\int_0^\infty \frac{1}{\eta(\tau)} \left(\frac{\partial G(t, \tau)}{\partial t}\right) \mathrm{d}\tau\right)^{1/2} \|v_n - v_0\|_{L_\eta^2(0,\infty)}\,.$$

On the other hand, it holds

$$\int_0^\infty e^\tau\, G^2(t, \tau)\, \mathrm{d}\tau = \frac{1}{\sqrt{k}} \cdot \frac{1}{4k-1}\, e^t + \frac{2}{4k-1}\, e^{-2\sqrt{(k)}t} - \frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{(k)} - 1}\, e^{-(2\sqrt{(k)}-1)t}\,,$$

$$\int_0^\infty e^\tau \left(\frac{\partial G(t, \tau)}{\partial t}\right)^2 \mathrm{d}\tau = \frac{\sqrt{k}}{4k-1}\, e^t + \frac{2k}{4k-1}\, e^{-2\sqrt{(k)}t} + \frac{1-\sqrt{k}}{2\sqrt{(k)}-1}\, e^{-(2\sqrt{(k)}-1)t}\,,$$

i.e., there exist constants c_0, c_1 independent of t such that

$$\int_0^\infty e^\tau\, G^2(t, \tau)\, \mathrm{d}\tau \leqq c_0 e^t$$

and

$$\int_0^\infty e^\tau \left(\frac{\partial G(t, \tau)}{\partial t}\right)^2 \mathrm{d}\tau \leqq c_1 e^t .$$

Hence, Theorem 4 yields

$$\sup_{0 \leqq t \leqq \xi} \left|u_n^i(t) - u_0^i(t)\right| \leqq c_i e^{\xi/2} \left\|v_n - v_0\right\|_{L_\eta^2(0,\infty)} \xrightarrow[n\to\infty]{} 0$$

where $i = 0, 1$.

Corollary 3. If the function $v_0 = -u_0'' + ku_0$ has derivatives up to the order r ($r \geqq 0$), $v^i \in C[0, \infty]$ ($i = 0, \ldots, r$) and v^r satisfies the Lipschitz condition with an exponent α then

$$\sup_{0 \leqq t \leqq \xi} \left|u_n^i(t) - u_0^i(t)\right| \leqq M e^{\xi/2} n^{-r-\alpha} \quad i = 0, 1 ,$$

where M is a constant independent of ξ and n.

This follows from Corollary 2 and from the Jackson theorem about the rate of approximation of a continuous function on the interval [0, 1] by polynomials [3].

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Souhrn

PŘIBLIŽNÉ METODY ŘEŠENÍ DIFERENCIÁLNÍCH ROVNIC NA NEKONEČNÉM INTERVALU

TERESA REGIŃSKA

Autorka uvádí dvě metody přibližného řešení jistého okrajového problému na nekonečném intervalu. Prvá metoda spočívá v aproximaci řešení posloupností řešení jistých okrajových úloh na konečných intervalech. Druhá metoda je modifikovaná kolokační metoda. Dokazuje se existence řešení a konvergence uvedených metod.

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