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ASYMPTOTIC EXPANSIONS OF FUNCTIONS OF STATISTICS

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1. INTRODUCTION

For deriving the approximate value of expectation or variance of some function of random variables, there is often used the method of statistical differentials, or δ-method, cf. [6] 6a. 2. Such approximate values are usually given in the form of asymptotic expansions where the asymptotic is meant with respect to the size of random sample. Cramér [2], derived the asymptotic expansions of a function of two sample moments. In his theorem (27.7), the remainder term in the expansion of the expectation is $O(n^{-1})$ and that of the variance is $O(n^{-3/2})$. Later, Lomnicki and Zaremba [5] investigated the behavior of the first two moments of functions of vector statistics with the same order of approximation as Cramér.

Sometimes, formulas of higher order as well as formulas expanding functions depending on n, the size of the underlying random sample, are needed. In the present paper, the asymptotic formulas for $Eg(T_n, n)$ and var $g(T_n, n)$ are derived, where g is a sufficiently smooth function (which may depend on n, as indicated), and T_n is a (multidimensional) statistic. The order of the remainder term depends both on the "smoothness" of g and on the behaviour of the moments of T_n.

Presented formulas are general enough to be applied to a large variety of statistical problems, e.g., in estimation theory. Ångström [1] pointed out how the asymptotic expansions may be utilized to calculate the bias of a non-linear function of sample characteristics. A further possible application is the comparison of two efficient estimates using the concept of deficiency, cf. [3].

2. ONE-DIMENSIONAL CASE

We begin with the simpler case when T_n is a one-dimensional statistic.

Theorem 1. *Let $g = g(t, n)$ be a function defined on $R_1 \times N$. Assume that, for all n and some $q \geqq 1$, g admits the continuous $(q + 1)$-st derivative for $t \in [\theta - \delta, \theta + \delta]$ where $\delta > 0$ is independent of n. Suppose that g is bounded on $R_1 \times N$*

and all the derivatives $g', \ldots, g^{(q+1)}$ are bounded on $[\theta - \delta, \theta + \delta] \times N$. Let $\{T_n\}$ be a sequence of statistics with finite moments up to the order $2(q + 1)$ such that $E|T_n - \theta|^{2(q+1)} = O(n^{-(q+1)})$. Then

$$(1) \qquad E[g(T_n, n) - g(\theta, n)] = \\ = \sum_{j=1}^{q} \frac{1}{j!} \left(\frac{\partial^j g}{\partial t^j}\right)_{t=\theta} E(T_n - \theta)^j + O(n^{-(q+1)/2})$$

and

$$(2) \qquad \operatorname{var}[g(T_n, n) - g(\theta, n)] = \\ = \sum_{\substack{j=1 \\ j+k \leq q+1}}^{q} \sum_{k=1}^{q} \frac{1}{j!} \frac{1}{k!} \left(\frac{\partial^j g}{\partial t^j}\right)_{t=\theta} \left(\frac{\partial^k g}{\partial t^k}\right)_{t=\theta} \times \\ \times \operatorname{cov}[(T_n - \theta)^j, (T_n - \theta)^k] + O(n^{-(q+2)/2}).$$

Remark 1. In the sequel, we shall repeatedly make use of the fact that the assumption $E|T_n - \theta|^{2(q+1)} = O(n^{-(q+1)})$ implies

$$(3) \qquad E|T_n - \theta|^j = O(n^{-j/2}), \quad 1 \leqq j \leqq 2(q + 1),$$

as follows from the familiar absolute moment inequality $\gamma_s^{1/s} \leqq \gamma_{s+1}^{1/(s+1)}$.

Proof. Denote

$$(4) \qquad I = E[g(T_n, n) - g(\theta, n)] = \int_{-\infty}^{+\infty} [g(t, n) - g(\theta, n)] \, dF_n$$

where F_n is the distribution function of T_n. Fix an ε, $0 < \varepsilon < \delta$. Denote $M = \{t : |t - \theta| < \varepsilon\}$, $M^c = R_1 - M$. Then

$$(5) \qquad I = \int_M [g(t, n) - g(\theta, n)] \, dF_n + \int_{M^c} [g(t, n) - g(\theta, n)] \, dF_n = I_1 + I_2,$$

say. The boundedness of g and Chebyschev inequality imply

$$|I_2| \leqq \text{const} \int_{M^c} dF_n \leqq \text{const } E|T_n - \theta|^{q+1} = O(n^{-(q+1)/2}),$$

hence $I = I_1 + O(n^{-(q+1)/2})$. For $t \in M$ using Taylor formula we obtain

$$g(t, n) - g(\theta, n) = \sum_{j=1}^{q} \frac{1}{j!} \left(\frac{\partial^j g}{\partial x^j}\right)_{x=\theta} (t - \theta)^j + \\ + \frac{1}{(q + 1)!} \left(\frac{\partial^{q+1} g}{\partial x^{q+1}}\right)_{x=\theta+\xi(t-\theta)} (t - \theta)^{q+1}$$

where $\xi = \xi(\theta, t, n) \in (0, 1)$. Throughout the proof we shall use the notation

$$b_j = \frac{1}{j!}\left(\frac{\partial^j g}{\partial x^j}\right)_{x=\theta}, \quad j = 1, \ldots, q\,,$$

$$b_{q+1} = \frac{1}{(q+1)!}\left(\frac{\partial^{q+1} g}{\partial x^{q+1}}\right)_{x=\theta+\xi(t-\theta)},$$

$$I_{1j} = \int_M (t-\theta)^j \, dF_n\,, \quad j = 1, \ldots, 2q\,.$$

We have

$$I_1 = \sum_{j=1}^{q} b_j I_{1j} + \int_M b_{q+1}(t-\theta)^{q+1} \, \mathrm{d}F_n\,. \tag{6}$$

The last integral is $O(n^{-(q+1)/2})$ since b_{q+1} is bounded on $M \times N$. Further,

$$I_{1j} = E(T_n - \theta)^j - \int_{M^c} (t-\theta)^j \, \mathrm{d}F_n\,. \tag{7}$$

Applying both Cauchy and Chebyschev inequalities we obtain

$$\left|\int_{M^c} (t-\theta)^j \, \mathrm{d}F_n\right|^2 \leqq E|T_n - \theta|^{2j} \, P(|T_n - \theta|^{2q} \geqq \varepsilon^{2q}) = O(n^{-(q+1)})\,,$$

hence

$$I_{1j} = E(T_n - \theta)^j + O(n^{-(q+1)/2})\,.$$

Altogether we have

$$I = \sum_{j=1}^{q} b_j \, E(T_n - \theta)^j + O(n^{-(q+1)/2}) \tag{8}$$

which is the desired formula (1).

Concerning the variance, denote

$$J = \operatorname{var}[g(T_n, n) - g(\theta, n)] =$$
$$= E[g(T_n, n) - g(\theta, n)]^2 - [Eg(T_n, n) - g(\theta, n)]^2 = J_1 - J_2\,,$$

say. Applying (1),

$$J_2 = \sum_{j=1}^{q} \sum_{k=1}^{q} b_j b_k \, E(T_n - \theta)^j \, E(T_n - \theta)^k +$$
$$+ O(n^{-(q+1)/2}) \sum_{j=1}^{q} b_j \, E(T_n - \theta)^j + O(n^{-(q+1)})\,.$$

Thus

$$J_2 = \sum_{k=1}^{q} \sum_{j=1}^{q} b_j b_k \, E(T_n - \theta)^j \, E(T_n - \theta)^k + O(n^{-(q+2)/2}) \tag{9}$$

since $E(T_n - \theta)^j = O(n^{-1/2})$ for $j \geqq 1$.

The integral J_1 may be expressed as

$$(10) \qquad J_1 = \int_M [g(t, n) - g(\theta, n)]^2 \, dF_n + \int_{M^c} [g(t, n) - g(\theta, n)]^2 \, dF_n = \\ = J_{11} + J_{12} ,$$

say. Obviously $J_{12} = O(n^{-(q+2)/2})$. Expanding g we observe that

$$(11) \qquad J_{11} = \sum_{k=1}^{q} \sum_{j=1}^{q} b_j b_k I_{1j+k} + 2 \sum_{j=1}^{q} b_j \int_M b_{q+1}(t - \theta)^{q+j+1} \, dF_n + \\ + \int_M b_{q+1}^2 (t - \theta)^{2(q+1)} \, dF_n = J_{111} + J_{112} + J_{113} ,$$

say. Recalling that b_{q+1} is bounded, we have

$$J_{113} = O(n^{-(q+1)}) ,$$

$$\int_M b_{q+1}(t - \theta)^{q+j+1} \, dF_n = O(n^{-(q+2)/2})$$

since $j \geqq 1$, and thus

$$J_{112} + J_{113} = O(n^{-(q+2)/2}) .$$

Write $p = j + k$. Then

$$(12) \qquad I_{1p} = E(T_n - \theta)^p - \int_{M^c} (t - \theta)^p \, dF_n .$$

Suppose first $2 \leqq p \leqq q + 1$. Then

$$(13) \qquad \left| \int_{M^c} (t - \theta)^p \, dF_n \right|^2 \leqq E|T_n - \theta|^{2p} \, P(|T_n - \theta|^{2q} \geqq \varepsilon^{2q}) = O(n^{-(q+2)}) .$$

If $q + 2 \leqq p \leqq 2q$,

$$(14) \qquad \int_{M^c} (t - \theta)^p \, dF_n = O(n^{-(q+2)/2})$$

holds. Summarizing (13) and (14) we get

$$I_{1p} = E(T_n - \theta)^p + O(n^{-(q+2)/2})$$

for $p = 2, \ldots, 2q$.

Now J_1 reduces to

$$(15) \qquad J_1 = \sum_{j=1}^{q} \sum_{k=1}^{q} b_j b_k \, E(T_n - \theta)^{j+k} + O(n^{-(q+2)/2})$$

which, together with (9) provides the formula (2) in the theorem except that the summation is now not restricted to $j + k \leqq q + 1$. Since, however, the b_j's are bounded and the moments $E(T_n - \theta)^{j+k}$ are $O(n^{-(q+2)/2})$ for $j + k > q + 1$, the summands with $j + k > q + 1$ may be included in the remainder term. Q.E.D.

Note that the covariances are usually calculated as

$$(16) \qquad \operatorname{cov}\left[(T_n - \theta)^j, (T_n - \theta)^k\right] = E(T_n - \theta)^{j+k} - E(T_n - \theta)^j \, E(T_n - \theta)^k \, .$$

Remark 2. In practice, we often need an asymptotic expansion of the expected squared error. If $w(\theta)$ is the parametric function involved then a recommendable formula is

$$(17) \qquad E[g(T_n, n) - w(\theta)]^2 = \operatorname{var} g(T_n, n) + \left[Eg(T_n, n) - w(\theta)\right]^2 .$$

This formula is highly suitable in situations when $Eg(T_n, n) = w(\theta) + O(n^{-1})$ which is often the case.

Example 1. Suppose that $X_1, \ldots, X_n$ is a random sample from the rectangular parent population on $(0, \theta)$ where θ is an unknown parameter. The best unbiased estimate of θ is $\hat{\theta} = (n + 1/n) X_{(n)}$ where $X_{(n)}$ is the maximum of the observations. For the parametric function $w(\theta; y) = \exp(-2y/\theta)$, y fixed, we can use the estimate

$$\operatorname{est} w(\theta, y) = \hat{w}(\theta, y) = \exp\left(-\frac{2yn}{n+1}\frac{1}{X_{(n)}}\right).$$

We shall investigate the bias and mean squared error of such estimate up to the order $O(n^{-3})$, Put $T_n = X_{(n)}$ and

$$g(t, n) = \exp\left(-\frac{2yn}{n+1}\frac{1}{t}\right)$$

in our Theorem. For the moments of T_n we have $ET_n^j = n\theta^j/(n + j)$; hence

$$E(T_n - \theta)^j = (-\theta)^j \sum_{k=0}^{j} \binom{j}{k} (-1)^k \frac{n}{n+k} = (-\theta)^j \binom{n+j}{j}^{-1},$$

using the result in [7], Ex. 1, p. 47. Thus we have

$$E(T_n - \theta) = -\frac{\theta}{n+1}$$

$$E(T_n - \theta)^2 = \frac{2\theta^2}{(n+2)(n+1)} .$$

It is obvious that $E(T_n - \theta)^j = O(n^{-j})$ so that to achieve the order $O(n^{-3})$ in the expansions of $E\hat{w}$ and $\operatorname{var} \hat{w}$ we need $q = 5$ in the expansion of $E\hat{w}$ and $q = 4$

in that of var $\hat{w}$. All the derivatives of g are bounded however, and $E(T_n - \theta)^j = O(n^{-j})$; hence it suffices to consider the terms with $j \leqq 2$ in the expansion of expectation and with $j = 1$, $k = 1$ in that of variance. The derivatives of g are

$$g'(t, n) = \frac{2yn}{n+1} t^{-2} \exp\left(-\frac{2yn}{n+1} t^{-1}\right),$$

$$g''(t, n) = \frac{2yn}{n+1} t^{-3} \exp\left(-\frac{2yn}{n+1} t^{-1}\right)\left(\frac{2yn}{n+1} t^{-1} - 2\right).$$

Denote $\varkappa = 2y/\theta$. Applying the theorem we obtain

$$E \exp\left(-\frac{2yn}{n+1}\frac{1}{X_{(n)}}\right) = \exp\left(-\frac{\varkappa}{1+1/n}\right)\left[1 - \frac{\varkappa}{n(1+1/n)^2} + \right.$$

$$\left. + \frac{\varkappa}{n^2(1+1/n)}\left(\frac{\varkappa}{1+1/n} - 2\right)\right] + O(n^{-3}).$$

This expression may be substantially simplified using the fact

$$\exp\left(-\frac{\varkappa}{1+1/n}\right) = e^{-\varkappa}\left[1 + \frac{\varkappa}{n} + \frac{\varkappa}{2n^2}(\varkappa - 2)\right] + O(n^{-3}).$$

After some calculations we have

$$E \exp(-2y/\hat{\theta}) = \exp(-2y/\theta)\left[1 + \frac{1}{n^2}\frac{y}{\theta}\left(\frac{2y}{\theta} - 2\right)\right] + O(n^{-3}).$$

For calculating var $\hat{w}$ we need $\operatorname{cov}[(T_n - \theta), (T_n - \theta)] = ET_n^2 - (ET_n)^2 = \theta^2[n/(n+2) - n^2/(n+1)^2] = \theta^2/n^2 + O(n^{-3})$. Taking into account the fact $E(T_n - \theta)^j = O(n^{-j})$ mentioned above, it follows from the theorem that

$$\operatorname{var}[\exp(-2y/\hat{\theta})] = \frac{1}{n^2}\exp\left(-\frac{4y}{\theta}\right)\frac{4y^2}{\theta^2} + O(n^{-3}).$$

Sometimes the expected squared error may be of a greater practical interest. Due to the fact

$$\left[E \exp(-2y/\hat{\theta}) - \exp\left(2\frac{y}{\theta}\right)\right]^2 = O(n^{-4}),$$

the expansion of the expected squared error coincides with that of variance, i.e.

$$E\left[\exp(-2y/\hat{\theta}) - \exp\left(2\frac{y}{\theta}\right)\right]^2 = \frac{1}{n^2}\exp\left(-\frac{4y}{\theta}\right)\frac{4y^2}{\theta^2} + O(n^{-3}).$$

Table 1

(By first, respectively second, approximation we mean the approximation when in (18) are considered the terms up to the order $O(n^{-2})$, respectively $O(n^{-3})$.)

$n = 4$

$-\delta$	True value	$\Phi(-\delta)$	1. approx.	2. approx.
0·5	·67264	·69146	·66946	·67324
1·0	·81442	·84135	·81110	·81488
1·5	·91014	·93319	·90891	·91005
2·0	·96318	·97725	·96375	·96291
2·5	·98732	·99379	·98831	·98720

$n = 10$

$-\delta$	True value	$\Phi(-\delta)$	1. approx.	2. approx.
0·5	·68317	·69146	·68266	·68327
1·0	·82982	·84135	·82925	·82985
1·5	·92367	·93319	·92348	·92366
2·0	·97172	·97725	·97185	·97172
2·5	·99143	·99379	·99160	·99142

$n = 30$

$-\delta$	True value	$\Phi(-\delta)$	1. approx.	2. approx.
0·5	·68859	·69146	·68853	·68859
1·0	·83738	·84135	·83731	·83738
1·5	·92998	·93319	·92996	·92997
2·0	·97543	·97725	·97545	·97543
2·5	·99304	·99379	·99306	·99304

The problem of estimation of the function $\exp\left(-2y/\theta\right)$ arises e.g. in reliability theory. A system consists of two parts with the same average life-time $\theta/2$, say. Suppose that the distribution of the time to failure of the first part is exponential and that of the second part is rectangular on $(0, \theta)$. When the second part fails down, the failure of the system is instantly observable. If the first part fails down, however, the system operates mistakenly but its wrong function is not immediately observable. We want to estimate the reliability of the first part assuming that the observations of the times to failure $X_1, \ldots, X_n$, of the second part are available.

Example 2. Let us study the reliability function in the normal case, i.e.

$$R(x; \mu, \sigma^2) = \Phi[(\mu - x)/\sigma]$$

where $\Phi(u)$ is the standard normal distribution function. We suppose that μ is an unknown parameter and that the standard deviation σ is known. The maximum likelihood estimator of R is

$$\hat{R}(x) = \Phi[(\bar{X} - x)/\sigma]$$

which has the expectation

$$E\,\hat{R}(x) = \Phi\left(-\delta\sqrt{\frac{n}{n+1}}\right)$$

after denoting $\delta = -(\mu - x)/\sigma$, and the mean squared error

$$E(\hat{R} - R)^2 = \Phi\left(\delta\sqrt{\frac{n}{n+1}},\ \delta\sqrt{\frac{n}{n+1}};\ \frac{1}{n+1}\right) -$$
$$- 2\Phi(\delta)\left[\Phi\left(\delta\sqrt{\frac{n}{n+1}}\right) - \tfrac{1}{2}\Phi(\delta)\right]$$

where $\Phi(\cdot, \cdot; \varrho)$ is the distribution function of the bivariate standard normal variable with the correlation coefficient ϱ. For the last results see [8]. In this case, we have an opportunity to compare the asymptotic formulas for $E\hat{R}$ with the true values. Applying the theorem with $g(t, n) = \Phi[t - x)/\sigma]$ (in fact, g is independent of n), $T_n = \bar{X}$ and using $E(\bar{X} - \mu)^k = 0$ for k odd, $E(\bar{X} - \mu)^k = \sigma^k n^{-k/2}(k-1)(k-3)\ldots$ $\ldots 1$ for k even, we have

$$E\,\hat{R}(x) = \Phi(-\delta) + \frac{1}{2n}\varphi'(-\delta) + \frac{1}{8n^2}\varphi'''(-\delta) + O(n^{-3}), \tag{18}$$

where φ is normal density function.

Some numerical results for $n = 4$, 10, and 30 are given in Table 1. One can see that the approximation is rather good even in the case $n = 4$. To express var $\hat{R}$, we need the following covariances:

$$\operatorname{cov}[(\bar{X} - \mu),\ (\bar{X} - \mu)] = \sigma^2/n$$
$$\operatorname{cov}[(\bar{X} - \mu),\ (\bar{X} - \mu)^2] = 0$$
$$\operatorname{cov}[(\bar{X} - \mu),\ (\bar{X} - \mu)^3] = 3\sigma^4/n^2$$
$$\operatorname{cov}[(\bar{X} - \mu)^2,\ (\bar{X} - \mu)^2] = 2\sigma^4/n^2\,.$$

Remaining covariances either vanish or are $O(n^{-3})$. Thus, after an easy calculation, we obtain

$$\operatorname{var} \hat{R} = \frac{1}{n} \varphi^2(\delta) + \frac{1}{n^2} [\varphi(-\delta)\, \varphi''(-\delta) + \tfrac{1}{2}\varphi'^2(-\delta)] + O(n^{-3}) .$$

Using the well-known formulas $\varphi'(y) = -y\, \varphi(y)$, $\varphi''(y) = (y^2 - 1)\, \varphi(y)$ the last expression may be reduced to

$$\operatorname{var} \hat{R} = \varphi^2(\delta) \left[\frac{1}{n} + \frac{1}{n^2}\left(\frac{3}{2}\delta^2 - 1\right)\right] + O(n^{-3}) .$$

Simple calculation gives

$$(E\hat{R} - R)^2 = \frac{1}{4n^2} \delta^2 \varphi^2(\delta) + O(n^{-3})$$

so that from (17) it follows that

$$E(\hat{R} - R)^2 = \varphi^2(\delta) \left[\frac{1}{n} + \frac{1}{n^2}\left(\frac{7}{4}\delta^2 - 1\right)\right] + O(n^{-3}) .$$

The last expression is in fact more convenient for calculation than the exact one because it does not contain the two-dimensional normal probability integral.

3. MULTI-DIMENSIONAL CASE

Lemma 1. (Generalized Hölder inequality) *Let $\beta_1, \ldots, \beta_r$ be positive real numbers such that $1/\beta_1 + \ldots + 1/\beta_r = 1$. Suppose that $X_1, \ldots, X_r$ are random variables. Then*

$$E|X_1 \ldots X_r| \leqq [E|X_1|^{\beta_1}]^{1/\beta_1} \ldots [E|X_r|^{\beta_r}]^{1/\beta_r}$$

assuming only that the moments exist.

Proof. The proof is a straightforward generalization of Hölder inequality for $r = 2$.

Lemma 2. *Let $i_1, \ldots, i_r$ be nonnegative real numbers, $\sum_{k=1}^r i_k = j$, $j > 0$, and $T_1, \ldots, T_r$ be random variables. Then*

$$E[|T_1|^{i_1} \ldots |T_r|^{i_r}] \leqq \{[E|T_1|^j]^{i_1} \ldots [E|T_r|^j]^{i_r}\}^{1/j}$$

assuming only that the moments exist.

Proof. Without loss of generality we may suppose that all the $i_1, \ldots, i_r$ are positive. Then in the preceeding lemma put $X_k = |T_k|^{i_k}$, $\beta_k = j/i_k$. Q.E.D.

Theorem 2. *Let $g = g(t_1, \ldots, t_r, n)$ be a function defined on $R_r \times N$. Assume that,*
1) *for all n, g is $(q + 1)$-times totally differentiable with respect to t_i's in the interval*
$$K = \mathop{\times}_{i=1}^{r} [\theta_i - \delta_i, \theta_i + \delta_i],\ \delta_i > 0,\ \delta_i \text{ independent of } n,$$
2) *g is bounded on $R_r \times N$,*
3) *all the derivatives up to the order $q + 1$ are bounded on $K \times N$,*
4) *$\{(T_{1n}, \ldots, T_{rn})\}_{n=1}^{\infty}$ is a sequence of multidimensional statistics such that*
5) *there exist absolute moments of T_{in} up to the order $2(q + 1)$*
6) *for $i = 1, \ldots, r$ $\quad E|T_{in} - \theta_i|^{2(q+1)} = O(n^{-(q+1)})$.*

Then

$$E[g(T_{1n}, \ldots, T_{rn}, n) - g(\theta_1, \ldots, \theta_r, n)] = \tag{19}$$
$$= \sum_{j=1}^{q} \frac{1}{j!} \sum_{\substack{i_1 + \ldots + i_r = j \\ i_1 + \ldots + i_r \geqq 0}} \!\!\cdots \sum \left(\frac{\partial^j g}{\partial t_1^{i_1} \ldots \partial t_r^{i_r}}\right)_{t=\theta} \times$$
$$\times\ E[(T_{1n} - \theta_1)^{i_1} \ldots (T_{rn} - \theta_r)^{i_r}] + O(n^{-(q+1)/2}),$$

$$\operatorname{var}[g(T_{1n}, \ldots, T_{rn}, n) - g(\theta_1, \ldots, \theta_r, n)] = \tag{20}$$
$$= \sum_{\substack{j=1 \\ j+k \leq q+1}}^{q} \sum_{k=1}^{q} \frac{1}{j!}\frac{1}{k!} \sum_{\substack{i_1 + \ldots + i_r = j \\ i_1, \ldots, i_r \geqq 0}} \!\!\cdots \sum \ \sum_{\substack{m_1 + \ldots + m_r = k \\ m_1, \ldots, m_r \geqq 0}} \!\!\cdots \sum \times$$
$$\times \left(\frac{\partial^j g}{\partial t_{t_1}^{i_1} \ldots \partial t_{t_r}^{i_r}}\right)_{t=\theta} \left(\frac{\partial^k g}{\partial t_{t_1}^{m_1} \ldots \partial t_{t_r}^{m_r}}\right)_{t=\theta} \times$$
$$\times \operatorname{cov}[(T_{1n} - \theta_1)^{i_1} \ldots (T_{rn} - \theta_r)^{i_r}, (T_{1n} - \theta_1)^{m_1} \ldots (T_{rn} - \theta_r)^{m_r}] +$$
$$+ O(n^{-(q+2)/2}),$$

where $\boldsymbol{t} = (t_1, \ldots, t_r)$, $\boldsymbol{\theta} = (\theta_1, , \theta_r)$.

Proof. The proof is a straightforward generalization of the one-dimensional version. We point out only some technicalities of the multi-dimensional case.

Let $0 < \varepsilon < \min(\delta_1, \ldots, \delta_r)$, fixed in the sequel. The set M should be replaced by $M = \bigcap_{\alpha=1}^{r} \{t \in E_r : |t_\alpha - \theta_\alpha| < \varepsilon\}$. The integrals of the type

$$\int \cdots \int_{M^c} (t_1 - \theta_1)^{i_1} \ldots (t_r - \theta_r)^{i_r} \, \mathrm{d}F_n$$

for $i_1 + \ldots + i_r \leqq q$ are estimated using Cauchy and Chebyshev inequalities and then applying Lemma 2. Let b_{q+1} denote a $(q + 1)$-st derivative at the point $\boldsymbol{\theta} + \xi(\mathbf{t} - \boldsymbol{\theta})$. If $i_1 + \ldots + i_r = q + 1$ then

$$\left|\int_M \ldots \int b_{q+1}(t_1 - \theta_1)^{i_1} \ldots (t_r - \theta_r)^{i_r} \, \mathrm{d}F_n\right| \leqq$$

$$\leqq \text{const}\, E\left[\left|T_{1n} - \theta_1\right|^{i_1} \ldots \left|T_{rn} - \theta_r\right|^{i_r}\right] \leqq$$

$$\leqq \text{const}\left\{\left[E\left|T_{1n} - \theta_1\right|^{q+1}\right]^{i_1} \ldots \left[E\left|T_{rn} - \theta_r\right|^{q+1}\right]^{i_r}\right\}^{1/(q+1)} =$$

$$= O\left(n^{-(q+1)/2}\right)$$

applying Lemma 2 again.

No new ideas are used when handling the variance. Q.E.D.

Example 3. (Estimation of the reliability function in the truncated exponential distribution when the point of truncation is not known.) Consider a population with the density

$$f(x; \theta, A) = \frac{1}{\theta} \exp\left[-\frac{1}{\theta}(x - A)\right] \quad \text{if} \quad x > A\,,$$

$$= 0 \qquad\qquad \text{otherwise}\,,$$

where θ and A are unknown parameters. The corresponding reliability function is

$$R(x, \theta, A) = \exp\left[-\frac{1}{\theta}(x - A)\right] \quad \text{if} \quad x > A\,,$$

$$= 1 \qquad\qquad \text{otherwise.}$$

The maximum likelihood estimators of the parameters θ, A based on the sample $X_1, \ldots, X_n$ are

$$A^* = X_{(1)} = \min\left(X_1, \ldots, X_n\right), \quad \theta^* = \bar{X} - A^*\,.$$

This leads to the following estimator of reliability:

$$\hat{R}(x) = \exp\left(-\frac{x - A^*}{\bar{X} - A^*}\right) \quad \text{if} \quad A^* < x\,,$$

$$= 1 \qquad\qquad \text{otherweise.}$$

We shall investigate expectation and variance of $\hat{R}$. We have to distinguish two possible ranges of the true parameter A, namely $A \geqq x$ and $A < x$. If $A \geqq x$ then $P(X_{(1)} \geqq A) = 1$, and hence $P(X_{(1)} \geqq x) = 1$ which results in $P(\hat{R}(x) = 1) = 1$, in this case. To manage the case $A < x$ we employ Theorem 2 for $q = 1$. Introduce new parameters $\theta_1 = \theta + A$, $\theta_2 = A$ and put $T_{1n} = \bar{X}$, $T_{2n} = X_{(1)}$. Further, define for x fixed

$$\begin{aligned} g(t_1, t_2) &= \exp\left(-\frac{x - t_2}{t_1 - t_2}\right) \quad \text{if} \quad t_1 > t_2\,, \quad t_2 < x \\ &= 1 \qquad\qquad\qquad\quad\;\; \text{if} \quad t_1 > t_2\,, \quad t_2 \geqq x \\ &= 0 \qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

We can observe that, for $n \geqq 2$, $\hat{R}(x) = g(T_{1n}, T_{2n})$ with probability one because $\bar{X} > A^*$ with probability one. After some algebra we come to desired moments of T_{1n} and T_{2n}, namely

$$E(T_{1n} - \theta_1) = 0\,, \quad E(T_{1n} - \theta_1)^2 = (\theta_1 - \theta_2)^2/n\,,$$

and generally

$$E(T_{1n} - \theta_1)^{2k} = O(n^{-k})\,, \quad k \geqq 1$$

(for the last appraisal see [2], 27.3.1),

$$E(T_{2n} - \theta_2)^s = O(n^{-s})\,, \quad s \geqq 1\,.$$

From the above relations we deduce

$$\operatorname{cov}\left[(T_{1n} - \theta_1)\,, \quad (T_{2n} - \theta_2)\right] = O(n^{-3/2})\,.$$

The true value $\theta_2 < x$ so that in some neighbourhood of (θ_1, θ_2) all the derivatives of g exist and obviously fulfil the conditions of the theorem. We shall, however, need only $\partial g/\partial t_1$ because the terms standing at the other derivatives are of higher orders than considered here. We have

$$\left(\frac{\partial g}{\partial t_1}\right)_{t=\theta} = \frac{x - \theta_2}{(\theta_1 - \theta_2)^2} \exp\left(-\frac{x - \theta_2}{\theta_1 - \theta_2}\right)$$

so that

$$E\,\hat{R}(x) = \exp\left(-\frac{x - \theta_2}{\theta_1 - \theta_2}\right) + O(n^{-1}) = \exp\left(-\frac{x - A}{\theta}\right) + O(n^{-1})\,,$$

$$\begin{aligned} \operatorname{var} \hat{R}(x) &= \left(\frac{\partial g}{\partial t_1}\right)_{t=\theta} \operatorname{var}(T_{1n} - \theta_1) + O(n^{-3/2}) = \\ &= \frac{1}{n}\left(\frac{x - A}{\theta}\right)^2 \exp\left(-2\,\frac{x - A}{\theta}\right) + O(n^{-3/2}) \end{aligned}$$

after returning to the original parameters θ, A.

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Souhrn

ASYMPTOTICKÉ ROZVOJE FUNKCÍ STATISTIK

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Nechť $\{T_n\}$ je posloupnost statistik taková, že $E|T_n - \theta|^{2(q+1)} = O(n^{-(q+1)})$, $g = g(t, n)$ reálná funkce definovaná na $R \times N$. Ukazuje se, že za jistých předpokladů kladených na funkci g je $E[g(T_n, n) - g(\theta, n)] = \sum_{j=1}^{q} j!^{-1}(\partial^j g/\partial t^j)_{t=\theta} E(T_n - \theta)^j + O(n^{-(q+1)/2})$ a $\operatorname{var} g(T_n, n) = \sum_{j=1}^{q} \sum_{k=1}^{q} j!^{-1} k!^{-1}(\partial^j g/\partial t^j)_{t=\theta}(\partial^k g/\partial t^k)_{t=\theta} \times \operatorname{cov}[(T_n - \theta)^j, (T_n - \theta)^k] + O(n^{-(q+2)/2})$. Jsou uvedeny i analogické vzorce v případě, kdy T_n je vektorová statistika a θ vektorový parametr.

Uvedené rozvoje jsou aplikovány na příkladech z teorie spolehlivosti.

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