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THE NONCONFORMING FINITE ELEMENT METHOD IN THE PROBLEM OF CLAMPED PLATE WITH RIBS

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The finite element method may be viewed as an analogue of the Ritz-Galerkin procedure with a special choice of basic functions. Both of them operate in a space of approximations, i.e. in the space that is defined as the linear hull of the basic functions. The Ritz-Galerkin method has one very favourable property, namely, the admissibility of basic functions in the trial variational principle guarantees its convergence. This is not the case with the nonconforming method because of the space generated by the basic functions is not a subspace of the one, on which the minimum of the trial energy functional is to be found. If the nonconforming method converges then this approach has many advantages. The conforming method as well as the Ritz-Galerkin procedure may be too complicated, while the nonconforming method is much simpler. A nonconforming solution, especially for few elements of division, may be more exact then the conforming one and in comparison with the Ritz-Galerkin method it is not so "stiff". Hence, in many cases the nonconforming method yields results which are more secure against violation of constructions. It is necessary to point out that the use of nonconforming elements in practice is perilous. It shows, namely, that the nonconforming method need not at all converge to the right results. A process, leading to a necessary condition of the convergence has lately been proposed. This process, known as "patch test" and developed by B. B. Irons, has first been studied from a mathematical standpoint by G. Strang, then by Ciarlet and others. In the sense of this approach we prove the convergence of Ari-Adini's rectangle in the problem of a clamped plate with ribs. The ribs are assumed stiff against bending and torsion in the sense of Saint-Venant theory.

1. DIMENSION REDUCTION.

The problem of a plate with ribs is a three-dimensional problem of a theory of elasticity and hence, except a few cases, analytically untreatable. Numerical solution is too cumbersome and possible only if a top-quality computer is available. Therefore it is necessary to use some approximative hypothesis.

We originate from small displacement theory and formulate one type of energy functional. This approach has many advantages. In general complicated expressions are in our case simplified and moreover, many hypotheses are included.

The plate before deformation is situated in the three dimensional space R^3 so that its middle plane and the plane $Oxy = R^2$ coincide. In the middle plane Oxy the plate occupies a region G. For the sake of simplicity we assume:

$$G = (-a, a) \times (-b, b) \tag{1.1}$$

where a, b are positive constants. Let $I = \{I_i\}_{i=1}^n$ be the set of straight line segments in G characterizing ribs that are parallel to y-axis while $J = \{J_j\}_{j=1}^m$ is the set of similar segments that are parallel to x-axis. The endpoints of the segments lie on the boundary of G, *i.e.*

$$\begin{aligned} I_i &= \{[x, y] \in R^2;\ x = x_i,\ y \in (-b, b)\}\,, \\ J_j &= \{[x, y] \in R^2;\ x \in (-a, a),\ y = y_j\}\,, \end{aligned} \tag{1.2}$$

$-a < x_i < a$, $-b < y_j < b$ and x_i, y_j are constants, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

From the technical standpoint we have the following problem: Let $f : G \to R$ be the density of loading. Find a function $w : G \to R$ that is minimizing the energy functional of a type

$$F(w) = F_p(w) + F_I(w) + F_J(w) + F_{TI}(w) + F_{TJ}(w) + F_f(w, f) \tag{1.3}$$

on a set of admissible functions which satisfy stable boundary conditions. In (1.3) the particular terms stand for the following expressions:

$$F_p(w) = \frac{1}{2}\iint_G \left\{(\Delta w)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right\} \mathrm{d}x\, \mathrm{d}y$$

$$F_I(w) = \frac{1}{2}\sum_{I_i \in I} \int_{I_i} \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \mathrm{d}y$$

$$F_J(w) = \frac{1}{2}\sum_{J_j \in J} \int_{J_j} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x$$

$$F_{TI}(w) = \frac{1}{2}\sum_{I_i \in I} \int_{I_i} \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 \mathrm{d}y$$

$$F_{TJ}(w) = \frac{1}{2}\sum_{J_j \in J} \int_{J_j} \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 \mathrm{d}x$$

$$F_f(w, f) = -2\iint_G f w \,\mathrm{d}x\, \mathrm{d}y\,.$$

If the function w is sufficiently smooth, for example $w \in C^4(\bar{G})$, we obtain by integration by parts the Eulerian equation of the variational principle:

$$\Delta^2 w = f \quad \text{on} \quad G$$

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial G, \ n \text{ is the outward normal}$$

(1.4)
$$[Mw]_i = \frac{\partial^3 w}{\partial x\, \partial y^2}\bigg|_{I_i}, \quad [Tw]_i = \frac{\partial^4 w}{\partial y^4}\bigg|_{I_i}$$

$$Mw = \Delta w - \frac{\partial^2 w}{\partial x^2}$$

$$Tw = 2\frac{\partial \Delta w}{\partial x} - \frac{\partial w}{\partial y\, \partial x^2}$$

and similarly on $J_j, j = 1, \dots, m$,

where ∂G denotes the boundary of G, $[Aw]_i$ is the difference between the values of Aw at the left and right hand sides of the i-th rib, that is $[Aw]_i = Aw/_{+0} - Aw/_{-0}$.

From the mathematical standpoint, there are two basic problems to be dealt with:

- to give an exact definition of the space in which the solution of the variational problem is to be found,
- to propose a numerical method approximating the trial problem.

An exact formulation of the trial problem is given in the following chapter for the above mentioned type of the energy functional. The nonconforming method is used for numerical solution of the problem.

2. MATHEMATICAL FORMULATION

Let G be a domain in the n-dimensional space R^n. Let $D(G)$ be a set of infinitely differentiable functions with compact support in G. For each integer m we denote by $W^{m,2}(G)$ the usual Sobolev space of all functions whose derivatives of order up to m (in generalized sense) are square integrable on G. For each $w \in W^{m,2}(G)$ we define the seminorm

$$|w|_{m,G} = \left(\int_G \sum_{|\alpha|=m} |D^\alpha w|^2 \, \mathrm{d}G\right)^{1/2}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i integer, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$,

$$D^\alpha w = \frac{\partial^{|\alpha|} w}{\partial x_1^{\alpha_1}, \dots, \partial x_n^{\alpha_n}}$$

It is well known that $\|w\|_{m,G} = |w|_{m,G} + |w|_{0,G}$ is a norm on $W^{m,2}(G)$. Let $W_0^{m,2}(G)$ be the completion of $D(G)$ with respect to the norm $|\cdot|_{m,G}$. Then $W_0^{m,2}(G) \subset W^{m,2}(G)$ and the seminorm $|\cdot|_{m,G}$ is a norm on $W_0^{m,2}(G)$. For further details concerning definitions and assertions see [6].

We are going to formulate the equilibrium condition of the problem on the following space:

$$V = \left\{ w;\ w \in W_0^{2,2}(G),\ w \in W_0^{2,2}(\Gamma),\ \frac{\partial w}{\partial x} \in W_0^{1,2}(\Gamma) \text{ for each } \Gamma \in I,\ w \in W_0^{2,2}(\gamma),\right.$$

$$\left. \frac{\partial w}{\partial y} \in W_0^{1,2}(\gamma) \text{ for each } \gamma \in J \right\}$$

with the norm $|||\cdot|||$ given by

$$|||w||| = |w|_{2,G} + \sum_{I_i \in I} \left(|w|_{2,I_i} + \left|\frac{\partial w}{\partial x}\right|_{1,I_i} \right) + \sum_{J_j \in J} \left((|w|_{2,J_j} + \left|\frac{\partial w}{\partial y}\right|_{1,J_j} \right).$$

Equilibrium condition: Let $f \in L_2(G)$. Find $u \in V$ such that

$$(2.1)\quad \begin{aligned} & a(u, \varphi) + \sum_{I_i \in I} \int_{I_i} \left(\frac{\partial^2 u}{\partial y^2} \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 \varphi}{\partial x\, \partial y} \right) \mathrm{d}y \; + \\ & + \sum_{J_j \in J} \int_{J_j} \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 \varphi}{\partial x\, \partial y} \right) \mathrm{d}x = 2 \int_G f\varphi \, \mathrm{d}x \, \mathrm{d}y \end{aligned}$$

for all $\varphi \in V$, where

$$a(u, \varphi) = \int_G \left(\Delta u\, \Delta \varphi + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 \varphi}{\partial x\, \partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 \varphi}{\partial y^2} \right) \mathrm{d}x \, \mathrm{d}y \, .$$

It is easy to verify that V is a Banach space and the bilinear form from the left hand side of (2.1) is V-elliptic (see e.g. [6]). Hence, by virtue of the well known Lax-Milgram theorem (see [6]), there exists a unique solution of the problem (2.1).

The approximation of the problem defined above is based on the following idea: We introduce a family $\{V_h\}_{h \in (0,1)}$ of finite-dimensional spaces V_h. The family is used to find an approximative solution $u_h \in V_h$ by applying the equilibrium condition (2.1) on the finite dimensional space V_h. But, in general $V_h \not\subset V$ and we cannot utilize the principle (2.1). We are forced to formulate a perturbed variational principle (2.5) on each V_h in the natural way.

We consider a division $G_h = \{G_{ih}\}_{i=1}^{k(h)}$ of G consisting of open rectangles G_{ih}, $i = 1, \ldots, k(h)$ for $h \in (0, 1)$ such that

$$\text{a) } \bar{G} = \bigcup_{i=1}^{k(h)} \bar{G}_{ih} ,$$

$$\text{b) } G_{ih} \cap G_{jh} = \emptyset \quad i, j = 1, \ldots, k(h) ; \quad i \neq j ,$$

$$\text{c) } G_{ih} \cap \Gamma = \emptyset , \quad G_{ih} \cap \gamma = \emptyset , \quad i = 1, \ldots, k(h) , \quad \Gamma \in I , \quad \gamma \in J ,$$

d) an edge of any rectangle $G_{ih} \in G_h$ is either an edge of another rectangle

$$G_{jh} \in G_h, j \neq i \text{ or a portion of } \partial G \text{ or } \Gamma \text{ or } \gamma, \Gamma \in I, \gamma \in J .$$

If a point Q is a vertex of a rectangle $G_{ih} \in G_h$ then we say that Q is a node of the division G_h.

We shall suppose that the sequence $\{G_h\}_{h \in (0,1)}$ of rectangulation is a regular family (see [3]), i.e. with each rectangle G_{ih} of a given division G_h we associate the following geometrical parametres:

$h(G_{ih})$ ... diameter of G_{ih} ,

$\varrho(G_{ih})$ ... supremum of diameters of spheres inscribed in G_{ih} .

Then

$$h \geqq \max \{h(G_{ih}); i = 1, \ldots, k(h)\}$$

and

$$0 < \bar{\gamma} \leqq \min \{\varrho(G_{ih})/h(G_{ih}); \; i = 1, \ldots, k(h)\}$$

for each $h \in (0,1)$, where $\bar{\gamma}$ is a fixed real number.

Let R be a "reference" rectangle*) in R^2. Then

$$A(R) = \{\varphi / \varphi = \sum_{\substack{i,j=0 \\ i+j \leq 3}} a_{ij} x^i y^j + a_{31} x^3 y + a_{13} x y^3 , \; [x, y] \in R\} .$$

Elements of the space $A(R)$ are the so called Ari-Adiniś polynomials.

We recall some basic properties of such polynomials. Let φ belong to $A(R)$. Then it holds:

(2.2) if $A_i = 1, \ldots, 4$ are the vertices of the rectangle R then the values $\{\varphi(A_i)$, $(\partial\varphi/\partial x)(A_i)$, $(\partial\varphi/\partial y)(A_i)$, $i = 1, \ldots, 4$ define the degrees of freedom;

(2.3) if the variable x is fixed then $(\partial\varphi/\partial x)(0, y) - (\partial\varphi/\partial x)(x, y)$ is a linear function with respect to y;

(2.4) if the variable y is fixed then $(\partial\varphi/\partial y)(x, 0) - (\partial\varphi/\partial y)(x, y)$ is a linear function with respect to x.

*) i.e. R is a fixed nondegenerate rectangle.

There exists an invertible affine mapping $F_{ih} : G_{ih} \to R$ for each $i = 1, \ldots, k(h)$, $h \in (0, 1)$. Let $h \in (0, 1)$. Then with regard to (2.2) the following definition is reasonable:

$V_h = \{\varphi; \varphi \circ F_{ih}^{-1} \in A(R)$ for $i = 1, \ldots, k(h)$; if Q is a nodal point of a division G_h then φ, $\partial\varphi/\partial x$, $\partial\varphi/\partial y$ are continuous at the point Q with respect to $\bar{G}$ and, moreover, if $Q \in \partial G$ then $\varphi(Q) = (\partial\varphi/\partial x)(Q) = (\partial\varphi/\partial y)(Q) = 0\}$.

Remark 1. The spaces V_h need not be subspaces of V because $\varphi \notin W_0^{2,2}(G)$, $\partial\varphi/\partial x \notin W_0^{1,2}(\Gamma)$ and $\partial\varphi/\partial y \notin W_0^{1,2}(\gamma)$ for $\varphi \in V_h$, $\Gamma \in I$, $\gamma \in J$. But it can be easily verified that

a) $\varphi \in W_0^{2,2}(\Gamma)$ and $\varphi \in W_0^{2,2}(\gamma)$ for $\Gamma \in I$, $\gamma \in J$,
b) $\varphi \in C(\bar{G})$, i.e. φ is continuous on G including the boundary ∂G,
c) $\varphi \equiv 0$ on ∂G.

Along the edges of a rectangle $G_{ih} \in G$ and R we define the linear interpolation operator $\mathscr{L}_{ih}$ and $\mathscr{L}$ respectively: We denote by A, B, C, D the vertices of G_{ih} or R. Assuming ψ defined on the boundary ∂G_{ih} and ∂R a.e., the value of the function $\mathscr{L}_{ih}\psi$ and $\mathscr{L}\psi$ at the point $X = tA + (1 - t)B$, $t \in (0, 1)$ of the edge AB is equal respectively to

$$t \lim_{\tau \to 1-} \psi(\tau A + (1 - \tau) B) + (1 - t) \lim_{\tau \to 0+} \psi(\tau A + (1 - \tau) B)$$

providing the limits exist. The functions $\mathscr{L}_{ih}\psi$ and $\mathscr{L}\psi$ are defined in the same way on the other edges BC, CD, AD.

Approximate problem: For $f \in L_2(G)$ and $h \in (0, 1)$ find $u_h \in V_h$ such that

$$(2.5) \qquad a_h(u_h, \varphi) + \sum_{\Gamma \in I} \left\{ \int_\Gamma \frac{\partial^2 u_h}{\partial y^2} \frac{\partial^2 \varphi}{\partial y^2} \, \mathrm{d}y + \right.$$

$$\left. + \frac{1}{2} \sum_{i=1}^{k(h)} \int_{\Gamma \cap \partial G_{ih}} \frac{\partial}{\partial y} \left(\mathscr{L}_{ih} \frac{\partial u_h}{\partial x} \right) \frac{\partial}{\partial y} \left(\mathscr{L}_{ih} \frac{\partial \varphi}{\partial x} \right) \mathrm{d}y \right\} +$$

$$\sum_{\gamma \in J} \left\{ \int_\gamma \frac{\partial^2 u_h}{\partial x^2} \frac{\partial^2 \varphi}{\partial x^2} \, \mathrm{d}x + \frac{1}{2} \sum_{i=1}^{k(h)} \int_{\gamma \cap \partial G_{ih}} \frac{\partial}{\partial x} \left(\mathscr{L}_{ih} \frac{\partial u_h}{\partial y} \right) \frac{\partial}{\partial x} \left(\mathscr{L}_{ih} \frac{\partial \varphi}{\partial y} \right) \mathrm{d}x \right\} = 2 \int_G f \varphi \, \mathrm{d}x \, \mathrm{d}y$$

for each $\varphi \in V_h$, where

$$a_h(u_h, \varphi) = \sum_{i=1}^{k(h)} \int_{G_{ih}} \left(\Delta u \, \Delta \varphi + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \, \partial y} \frac{\partial^2 \varphi}{\partial x \, \partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 \varphi}{\partial y^2} \right) \mathrm{d}x \, \mathrm{d}y \, .$$

We equip the space V_h with the natural norm $|||\cdot|||_h$. For $\varphi \in V_h$,

$$|||\varphi|||_h = \left\{|\varphi|^2_{2,h} + \sum_{\Gamma\in I}\left(|\varphi|^2_{2,\Gamma} + \frac{1}{2}\sum_{i=1}^{k(h)}\left\|\frac{\partial}{\partial y}\left(\mathscr{L}_{ih}\frac{\partial\varphi}{\partial x}\right)\right\|^2_{L_2(\Gamma\cap\partial G_{ih})}\right) + \right.$$

$$(2.6) \qquad \left. + \sum_{\gamma\in J}\left(|\varphi|^2_{2,\gamma} + \frac{1}{2}\sum_{i=1}^{k(h)}\left\|\frac{\partial}{\partial x}\left(\mathscr{L}_{ih}\frac{\partial\varphi}{\partial y}\right)\right\|^2_{L_2(\gamma\cap\partial G_{ih})}\right)\right\}^{1/2},$$

$$|\varphi|_{2,h} = \left(\sum_{i=1}^{k(h)}|\varphi|^2_{2,G_{ih}}\right)^{1/2}$$

The bilinear form from the left hand side of (2.5) can be easily shown to be V_h-elliptic. Following the Lax-Milgram theorem we obtain the existence and uniqueness of the solution of problem (2.5).

The functional $|||\cdot|||_h$ defined by (2.6) on V_h can be extended onto $V \oplus V_h = \{\psi;\ \psi = \varphi_1 + \varphi_2, \varphi_1 \in V, \varphi_2 \in V_h\}$. It is sufficient to recall that $\partial\varphi_1/\partial x$ and $\partial\varphi_1/\partial y$ is continuous on Γ and γ, respectively, $\Gamma \in I$, $\gamma \in J$ and $\varphi_1 \in V$. Thus it is sensible to define $\mathscr{L}_{ih}(\partial\varphi_1/\partial x)$ and $\mathscr{L}_{ih}(\partial\varphi_1/\partial y)$ on $\Gamma \cap \partial G_{ih}$ and $\gamma \cap \partial G_{ih}$ respectively. Evidently $|||\cdot|||_h$ is a norm on $V \oplus V_h$.

3. ERROR ANALYSIS, PATCH TEST

For the sake of simplicity we assume: $J = \emptyset$ and the set I contains one rib only (denoted, say, Γ). All results of this chapter can be easily extended to more general cases.

In the following C denotes a generic constant, not necessarily the same in each two occurencies, and independent on the parameter h.

Theorem 3.1. *Let u and u_h be a solution of the problem* (2.1) and (2.5), *respectively. Then*

$$(3.1) \qquad |||u - u_h|||_h \leqq C\{\inf_{\varphi\in V_h}|||u - \varphi|||_h + \sup_{\varphi\in V_h}(E_h(u, \varphi)\,|||\varphi|||_h^{-1}\}$$

where

$$E_h(u, \varphi) = 2\int_G f\varphi\,\mathrm{d}x\,\mathrm{d}y - a_h(u, \varphi) - \int_\Gamma \frac{\partial^2 u}{\partial y^2}\frac{\partial^2\varphi}{\partial y^2}\,\mathrm{d}y -$$

$$- \frac{1}{2}\sum_{i=1}^{k(h)}\int_{\Gamma\cap\partial G_{ih}}\frac{\partial}{\partial y}\left(\mathscr{L}_{ih}\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial y}\left(\mathscr{L}_{ih}\frac{\partial\varphi}{\partial x}\right)\mathrm{d}y\,.$$

Proof. We shall, for simplicity, introduce the following notation:

$$A_h(w, \varphi) = a_h(w, \varphi) + \int_\Gamma \frac{\partial^2 w}{\partial y^2}\frac{\partial^2\varphi}{\partial y^2}\,\mathrm{d}y + \frac{1}{2}\sum_{i=1}^{k(h)}\int_{\Gamma\cap\partial G_{ih}}\frac{\partial}{\partial y}\left(\mathscr{L}_{ih}\frac{\partial w}{\partial x}\right)\frac{\partial}{\partial y}\left(\mathscr{L}_{ih}\frac{\partial\varphi}{\partial x}\right)\mathrm{d}y\,.$$

With regard to (2.1) and (2.5) we obtain

$$A_h(u_h, \varphi) = E_h(u, \varphi) + A_h(u, \varphi)$$

for $\varphi \in V_h$. Let $v \in V_h$. Subtracting $A_h(v, \varphi)$ from both sides of the above equality we get

$$A_h(u_h - v, \varphi) = E_h(u, \varphi) + A_h(u - v, \varphi)$$

Since the estimate

$$\sup_{\varphi \in V_h} \left(|A_h(u_h - v, \varphi)| \cdot |||\varphi|||_h^{-1}\right) \geqq C|||u_h - v|||_h$$

and

$$\sup_{\varphi \in V_h} \left(|A_h(u - v, \varphi)| \cdot |||\varphi|||_h^{-1}\right) \leqq C|||u - v|||_h$$

are evident, we can easily reach the inequality

$$|||u_h - v|||_h \leqq C\{\sup_{\varphi \in V_h} (E_h(u, \varphi) \cdot |||\varphi|||_h^{-1} + |||u - v|||_h\} \tag{3.2}$$

which holds for arbitrary $v \in V_h$ and a certain constant C. Using the triangle inequality $|||u - u_h|||_h \leqq |||u - v|||_h + |||u_h - v|||_h$ and the estimate (3.2), we obtain (3.1) immediately.

In the next three lemmas our attention will be drawn to the approximation of a function $\varphi \in V_h$ by a "conforming" function $\psi \in V$.

We denote the outward normal of ∂G_{ih} and ∂R by $\nu_{ih} = (\nu_{ih}^{(1)}, \nu_{ih}^{(2)})$ and $\nu = (\nu^{(1)}, \nu^{(2)})$ respectively.

Lemma 3.1. *Let Γ_i, $i = 1, 2, 3, 4$ be the edges of the rectangle R. Further, let φ_0 and φ_1 be functions on ∂R which satisfy the following conditions:*

$$\text{(a)} \quad \varphi_0 \in W^{3/2,2}(\Gamma_i), \quad \varphi_1 \in W^{1/2,2}(\Gamma_i), \quad i = 1, 2, 3, 4,$$

$$\text{(b)} \quad \varphi_0 \in W^{1,2}(\partial R),$$

$$\text{(c)} \quad \varphi_{10} = \nu^{(1)}\varphi_1 + \nu^{(2)} \frac{\partial \varphi_0}{\partial \tau} \in W^{1/2,2}(\partial R)$$

where $\tau = (\nu^{(2)}, -\nu^{(1)})$,

$$\varphi_{01} = \nu^{(2)}\varphi_1 - \nu^{(1)} \frac{\partial \varphi_0}{\partial \tau} \in W^{1/2,2}(\partial R)$$

(for further details concerning the spaces $W^{3/2,2}$ and $W^{1/2,2}$ defined on a variety see [6]). Then there exists a function $\psi \in W^{2,2}(R)$ such that

$$\psi = \varphi_0, \quad \frac{\partial \psi}{\partial \nu} = \varphi_1 \quad \text{a.e. on } \partial R$$

and

$$\|\psi\|_{2,R} \leqq C\left\{\sum_{i=1}^{4} (\|\varphi_0\|_{3/2,\Gamma_i} + \|\varphi_1\|_{1/2,\Gamma_i}) + \|\varphi_{01}\|_{1/2,\partial R} + \|\varphi_{10}\|_{1/2,\partial R}\right) .$$

Proof. see [4].

Lemma 3.2. *There exists a linear operator $r : A(R) \to W^{2,2}(R)$ with the following properties: $\Phi \in A(R)$, $\Psi = r\Phi \Rightarrow$*

$$\text{(i)} \quad \Psi = \Phi\,, \quad \frac{\partial \Psi}{\partial \nu} = \mathscr{L}\left(\frac{\partial \Psi}{\partial \nu}\right) \quad \textit{a.e. on } \partial R\,,$$

$$\text{(ii)} \quad \|\Phi - \Psi\|_{2,R} \leqq C|\Phi|_{2,R}\,.$$

Proof. For arbitrary $\Phi \in A(R)$ we set $\varphi_0 = \Phi$, $\varphi_1 = \mathscr{L}(\partial\Phi/\partial\nu)$ on ∂R and verify that the assumptions of Lemma 3.1 are fulfilled:

(a) functions φ_0, φ_1 are polynomials on each edge Γ_i, $i = 1, 2, 3, 4$, i.e. infinitely smooth,

(b) in addition to above property, the functipn φ_0 is continuous at any vertex of the rectangle R (hence $\varphi_0 \in W^{1,2}(\partial R)$),

(c) both functions φ_{10} and φ_{01} are polynomials along any edge Γ_i and continuous at any vertex of the rectangle R, hence $\varphi_{10} \in W^{1,2}(\partial R)$, $\varphi_{01} \in W^{1,2}(\partial R)$.

Thus there exists a function $\Psi \in W^{2,2}(R)$ such that

$$\Psi = \Phi\,, \quad \frac{\partial \Psi}{\partial \nu} = \mathscr{L}\frac{\partial \Phi}{\partial \nu} \quad \text{a.e. on } \partial R\,.$$

Let $\{1, x, y, xy, x^2, y^2, x^3, x^2y, yx^2, y^3, x^3y, xy^3\}$ be a basis of the space $A(R)$. We denote by Φ_n, $n = 1, \ldots, 12$ the elements of that basis. Let us define $r\Phi_n = \Psi_n$ for $n = 1, \ldots, 12$ so that

$$\Psi_n \in W^{2,2}(R)\,,$$

$$\Psi_n \in \Phi_n\,, \quad \frac{\partial \Psi_n}{\partial \nu} = \mathscr{L}\left(\frac{\partial \Phi_n}{\partial \nu}\right) \quad \text{a.e. on } \partial R\,.$$

If $n = 1, \ldots, 6$ then we obviously set $\Psi_n = \Phi_n$. For $n = 7, \ldots, 12$ the existence of functions Ψ_n is guaranteed by the previous part of the proof. Now, we extend the operator r linearly to the whole space $A(R)$:

$$\text{Let } \Phi \in A(R), \text{ i.e. } \Phi = \sum_{i=1}^{12} \alpha_n \Phi_n. \text{ Then}$$

$$r\Phi = \sum_{n=1}^{12} \alpha_n \Psi_n$$

In virtue of the finite dimension of $A(R)$ we get

$$\|r\Phi\|_{2,R} \leqq C\|\Phi\|_{2,R} \tag{3.3}$$

for each $\Phi \in A(R)$. The operator r is linear, bounded and has the property (i).

Let us denote by P_1 all polynomials of the 1-st degree. Provided $p \in P_1$, it holds $p \in A(R)$ and $rp = p$. From (3.3) we have

$$\|\Phi - r\Phi\|_{2,R} \leqq C \inf_{p \in P_1} \|\Phi - p\|_{2,R} \tag{3.4}$$

for each $\Phi \in A(R)$. Applying the inequality (see [6])

$$\inf_{p \in P_1} \|\Phi - p\|_{2,R} \leqq C|\Phi|_{2,R}$$

to the estimate (3.4) we prove (ii).

Lemma 3.3. *For each $h \in (0, 1)$ there exists a linear operator $r_h : V_h \to V$ with the following properties: $\varphi \in V_h$, $\psi = r_h\varphi \Rightarrow$*

$$\psi = \varphi\,, \quad \frac{\partial \psi}{\partial v_{ih}} = \mathscr{L}_{ih}\left(\frac{\partial \varphi}{\partial v_{ih}}\right) \quad \textit{a.e. on } \partial G_{ih}\,, \quad i = 1, \ldots, k(h)\,; \tag{3.5}$$

$$|\varphi - \psi|_{2,h} \leqq C|\varphi|_{2,h}\,; \tag{3.6}$$

$$|\varphi - \psi|_{L_2(G)} \leqq Ch^2|\varphi|_{2,h}\,. \tag{3.7}$$

Proof. Let an arbitrary $\varphi \in V_h$ be fixed. We set

$$\psi = \left(r(\varphi \circ F_{ih}^{-1})\right) \circ F_{ih} \tag{3.8}$$

on G_{ih}, $i = 1, \ldots, k(h)$. According to Lemma 3.2 we obtain

$$\psi \in W^{2,2}(G_{ih})\,, \tag{3.9}$$

$$\psi = \varphi\,, \quad \frac{\partial \psi}{\partial v_{ih}} = \mathscr{L}_{ih}\frac{\partial \varphi}{\partial v_{ih}} \quad \text{on} \quad \partial G_{ih} \tag{3.10}$$

for $i = 1, \ldots, k(h)$. By virtue of the definition of V_h and Remark 1 we derive four assertions concerning the smoothness of ψ:

a) With the aid of (3.10) we can easily prove the following fact: Let us denote $I = \bar{G}_{ih} \cap \bar{G}_{jh}$, $i \neq j$. Let g_i^α and g_j^α be the traces of functions $\{D^\alpha \psi(x, y), [x, y] \in G_{ih}\}$ and $\{D^\alpha \psi(x, y), [x, y] \in G_{jh}\}$ on the boundary ∂G_{ih} and ∂G_{jh} respectively. Then $g_i^\alpha = g_j^\alpha$ a.e. on I for $|\alpha| \leqq 1$. Using this fact, (3.9) and the integration by parts we can easily derive

$$\int_G \psi D^\alpha \varphi \,\mathrm{d}x\,\mathrm{d}y = (-1)^{|\alpha|} \sum_{i=1}^{k(h)} \int_{G_{ih}} D^\alpha \psi \varphi \,\mathrm{d}x\,\mathrm{d}y =$$

$$= (-1)^{|\alpha|} \int_G \psi \varphi \,\mathrm{d}x\mathrm{d}y$$

for each $\varphi \in D(G)$, $|\alpha| \leqq 2$. Since the generalized derivative $D^\alpha\psi$ belongs to $L_2(G)$ for $|\alpha| \leqq 2$, it means $\psi \in W^{2,2}(G)$.

b) Because of $\varphi \equiv 0$ and $\mathscr{L}_{ih}(\partial\varphi/\partial\nu_{ih}) \equiv 0$ on $\partial G_{ih} \cap \partial G$ for $i = 1, \ldots, k(h)$, we obtain (see (3.10) and the assertion above) $\psi \in W_0^{2,2}(G)$.

c) $\psi \in W_0^{2,2}(\Gamma)$ because of $\varphi \in W_0^{2,2}(\Gamma)$.

d) According to (3.10) we obtain $\partial\psi/\partial x = \mathscr{L}_{ih}(\partial\varphi/\partial x)$ on $\partial G_{ih} \cap \Gamma$, $i = 1, \ldots, k(h)$. Hence the function $\partial\psi/\partial x$ is piecewise linear and continuous on Γ. With respect to the boundary condition on ∂G we have $\partial\psi/\partial x \in W_0^{1,2}(\Gamma)$.

Summarizing the properties a)–d), we can state $\psi \in V$. Moreover, it holds

$$|\varphi - \psi|_{2,G_{ih}} \leqq Ch^{-1}\|\varphi \circ F_{ih}^{-1} - r(\varphi \circ F_{ih}^{-1})\|_{2,R}$$

and

$$|\varphi - \psi|_{L_2(G_{ih})} \leqq Ch\|\varphi \circ F_{ih}^{-1} - r(\varphi \circ F_{ih}^{-1})\|_{L_2(R)}\,.$$

With the aid of Lemma 3.2 we estimate:

$$\|\varphi \circ F_{ih}^{-1} - r(\varphi \circ F_{ih}^{-1})\|_{2,R} \leqq C|\varphi \circ F_{ih}^{-1}|_{2,R}$$

Because of

$$|\varphi \circ F_{ih}^{-1}|_{2,R} \leqq Ch|\varphi|_{2,G_{ih}}$$

we have

$$|\varphi - \psi|_{2,h} = \sum_{i=1}^{k(h)} |\varphi - \psi|_{2,G_{ih}} \leqq C\sum_{i=1}^{k(h)} |\varphi|_{2,G_{ih}} = C|\varphi|_{2,h}$$

and

$$|\varphi - \psi|_{L_2(G)} \leqq Ch^2|\varphi|_{2,h}\,.$$

For each $h \in (0, 1)$ and $\varphi \in V_h$ we put $r_h\varphi$ equal to ψ that is defined by (3.8). It has been verified that the operator r_h posseses the required properties (3.5)–(3.7). This completes the proof.

Lemma 3.4. *If $u \in V$ is a solution of the problem* (2.1) *then*

$$E_h(u, \varphi) = \int_G 2f(\varphi - \psi)\,\mathrm{d}x\,\mathrm{d}y - a_h(u, \varphi - \psi) \tag{3.11}$$

for $\varphi \in V_h$, $\psi = r_h\varphi$ (for r_h see Lemma 3.3), *$h \in (0, 1)$.*

Proof. Using the formula (2.1) and the fact $\psi \in V$ (see Lemma 3.3) we derive the identity

$$2\int_G f\psi\,\mathrm{d}x\,\mathrm{d}y - a(u, \psi) = \int_\Gamma \frac{\partial^2 u}{\partial y^2}\frac{\partial^2\psi}{\partial y^2}\,\mathrm{d}y + \int_\Gamma \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial^2\psi}{\partial y\,\partial x}\,\mathrm{d}y\,.$$

First, because of $\psi - \varphi$ on $\Gamma \cap \partial G_{ih}$ (see (3.5)), we have

$$\int_\Gamma \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 \psi}{\partial y^2} \, \mathrm{d}y = \int_\Gamma \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 \varphi}{\partial y^2} \, \mathrm{d}y$$

Further, since $\partial\psi/\partial x = \mathscr{L}_{ih}(\partial\varphi/\partial x)$ on $\Gamma \cap \partial G_{ih}$ evidently, we get

$$\int_\Gamma \frac{\partial^2 u}{\partial y\, \partial x} \frac{\partial^2 \psi}{\partial y\, \partial x} \, \mathrm{d}y = \frac{1}{2} \sum_{i=1}^{k(h)} \int_{\Gamma \cap \partial G_{ih}} \frac{\partial}{\partial y}\left(\mathscr{L}_{ih} \frac{\partial u}{\partial x}\right) \frac{\partial}{\partial y}\left(\mathscr{L}_{ih} \frac{\partial \varphi}{\partial x}\right) \mathrm{d}y \, .$$

If we recall the definition of the form $E_h(u, \varphi)$ (see Theorem 3.1) then the assertion (3.11) will be a simple consequence of the three identities above.

The following lemma was first proved by Ciarlet [2].

Lemma 3.5. *Patch test. Let $\tilde{u}_h$ be a function on G which is a polynomial of the 2-nd degree on each of G_{ih}, $i = 1, \ldots, k(h)$. Then*

$$a_h(\tilde{u}_h, \varphi - \psi) = 0 \tag{3.12}$$

for $\varphi \in V_h$, $\psi = r_h\varphi$, $h \in (0, 1)$ where r_h has been defined in Lemma 3.3.

Proof. We have $\Delta^2 \tilde{u}_h \equiv 0$ on each G_{ih}, $i = 1, \ldots, k(h)$. Using the well known Green formula (see [6]), we obtain

$$\begin{aligned} \int_{G_{ih}} \bigg(\Delta\tilde{u}_h\, \Delta(\varphi - \psi) + \frac{\partial^2 \tilde{u}_h}{\partial x^2} \frac{\partial^2(\varphi - \psi)}{\partial x^2} + 2\, \frac{\partial^2 \tilde{u}_h}{\partial x\, \partial y} \frac{\partial x\, \partial y}{\partial^2(\varphi - \psi)} + \\ + \frac{\partial^2 \tilde{u}_h}{\partial y_2} \frac{\partial^2(\varphi - \psi)}{\partial y^2} \bigg) \mathrm{d}x\, \mathrm{d}y = \int_{\partial G_{ih}} M\tilde{u}_h \left(\frac{\partial \varphi}{\partial v_{ih}} - \frac{\partial v_{ih}}{\partial \psi} \right) \mathrm{d}\sigma \end{aligned} \tag{3.13}$$

where

$$M\tilde{u}_h = \Delta\tilde{u}_h + \left(\frac{\partial^2 \tilde{u}_h}{\partial x^2} (v_{ih}^{(1)})^2 + 2\, \frac{\partial^2 \tilde{u}_h}{\partial x\, \partial y} v_{ih}^{(1)} v_{ih}^{(2)} + \frac{\partial^2 \tilde{u}_h}{\partial y^2} (v_{ih}^{(2)})^2 \right).$$

Let us denote the vertices of the rectangle G_{ih} by $A = [a_1, a_2]$, $B = [b_1, a_2]$, $C = [b_1, c_2]$, $D = [a_1, c_2]$, $a_1 < b_1$, $a_2 < c_2$. Then $M\tilde{u}_h/_{AB} = M\tilde{u}_h/_{CD} =$ const. and $M\tilde{u}_h/_{AD} = M\tilde{u}_h/_{BC} =$ const.*)

If we verify that

$$\int_{AB} \left(\frac{\partial \varphi}{\partial v_{ih}} - \frac{\partial \psi}{\partial v_{ih}} \right) \mathrm{d}\sigma + \int_{CD} \left(\frac{\partial \varphi}{\partial v_{ih}} - \frac{\partial \psi}{\partial v_{ih}} \right) \mathrm{d}\sigma = 0 \tag{3.14}$$

and

$$\int_{AD} \left(\frac{\partial \varphi}{\partial v_{ih}} - \frac{\partial \psi}{\partial v_{ih}} \right) \mathrm{d}\sigma + \int_{BC} \left(\frac{\partial \varphi}{\partial v_{ih}} - \frac{\partial \psi}{\partial v_{ih}} \right) \mathrm{d}\sigma = 0 \tag{3.15}$$

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*) $M\tilde{u}_h =$ const. on $AB \cup CD$ and $AD \cup BC$, respectively.

then the right hand side of (3.13) has to be zero. Summing (3.13) over $i = 1, \ldots, k(h)$, we reach the assertion (3.12).

The left hand side of (3.14) satisfies

$$\int_{AB} \left(\frac{\partial \varphi}{\partial v_{ih}} - \frac{\partial \psi}{\partial v_{ih}} \right) \mathrm{d}\sigma + \int_{CD} \left(\frac{\partial \varphi}{\partial v_{ih}} - \frac{\partial \psi}{\partial v_{ih}} \right) \mathrm{d}\sigma =$$

$$= - \int_{a_1}^{b_1} \left(\frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial y} \right)_{y = a_2} \mathrm{d}x + \int_{a_1}^{b_1} \left(\frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial y} \right)_{y = b_2} \mathrm{d}x =$$

$$= \int_{a_1}^{b_1} \left(\left(\frac{\partial \varphi}{\partial y}(x, c_2) - \frac{\partial \varphi}{\partial y}(x, a_2) \right) - \left(\frac{\partial \psi}{\partial y}(x, c_2) - \frac{\partial \psi}{\partial y}(x, a_2) \right) \right) \mathrm{d}x \, .$$

According to the property (2.4) of Ari-Adini's polynomials, the function $X(x) = (\partial\varphi/\partial y)(x, c_2) - (\partial\varphi/\partial y)(x, a_2)$ is linear on the interval $\langle a_1, b_1 \rangle$. Using (3.5), we easily derive that $(\partial\psi/\partial y)(x, c_2) - (\partial\psi/\partial y)(x, a_2)$ is a linear interpolation of the function X. Hence the integrand of the last term vanishes and the identity (3.14) holds. The conclusion of lemma follows by observing that we can similarly prove the identity (3.15).

Theorem 3.2. *Let u and u_h be a solution of the problem* (2.1) *and* (2.5), *respectively. Let $\mathfrak{M}_h = \{w; w \in L_2(G), w$ is a polynomial of the 2-nd degree on each $G_{ih}, i = 1, \ldots \ldots, k(h)\}$, $h \in (0, 1)$. Then*

$$|||u - u_h|||_h \leqq C\{ \inf_{\varphi \in V_h} |||u - \varphi|||_h + \inf_{\tilde{u}_h \in \mathfrak{M}_h} |u - \tilde{u}_h|_{2,h} + h^2 \} \, . \tag{3.16}$$

Proof. In accordance with (3.1), (3.11) and (3.12) we obtain

$$|||u - u_h|||_h \leq C\{ \inf_{\varphi \in V_h} |||u - \varphi|||_h + \sup_{\varphi \in V_h} [(2 \int_G f(\varphi - r_h\varphi) \, \mathrm{d}G -$$

$$- \inf_{\tilde{u}_h \mathfrak{M}_h} a_h(u - \tilde{u}_h, \varphi - r_h\varphi))] \cdot |\varphi|_{2,h}^{-1}\} \, . \tag{3.17}$$

The estimate (3.16) follows from (3.17) by using (3.6), (3.7) and the inequality

$$|a_h(u - \tilde{u}_h, \varphi - r_h\varphi)| \leqq C|u - \tilde{u}_h|_{2,h} \, |\varphi - r_h\varphi|_{2,h} \, .$$

Remark 2. The estimate (3.16) is valid also in the general case of the problem (2.1), i.e. without any restriction on the sets I and J.

The problem of convergence turns into the problem of the best approximation to the solution u on the spaces V_h and $\mathfrak{M}_h$.

4. CONVERGENCE

Let us consider the model problem from the previous chapter. We introduce a class of regular solutions of the problem (2.1):

Let G^1 and G^2 be open rectangles, $\bar{G} = \bar{G}^1 \cup \bar{G}^2$, $G^1 \cap G^2 = \emptyset$, $G^i \cap \Gamma = \emptyset$, $i = 1, 2$. We define a set

$$W = \{w;\, w \in V,\, w \in W^{3,2}(G^i),\ i = 1, 2,\, w \in W^{3,2}(\Gamma)\}\,.$$

If u is a solution of the problem (2.1) and $u \in W$ then we shall say that u is a regular solution of (2.1).

Lemma 4.1. *Let w be an element of the space W. Then we can estimate*:

$$\inf_{\varphi \in V_h} |||w - \varphi|||_h \leqq Ch(|w|_{3,G^1} + |w|_{3,G^2} + |w|_{3,\Gamma})\,, \tag{4.1}$$

$$\inf_{\varphi \in \mathfrak{M}_h} |w - \varphi|_{2,h} \leqq Ch(|w|_{3,G^1} + |w|_{3,G^2})\,. \tag{4.2}$$

Proof. We shall only sketch the proof, because it is based on the well known technique (see [1]). We define interpolation operators P_h^1 and P_h^2 on V_h and $\mathfrak{M}_h$, respectively. If $w \in W$ then $P_h^1 w \in V_h$ interpolates the function w at all degrees of freedom on the space V_h (i.e. $D^\alpha w = D^\alpha(P_h^1 w)$ at each nodal point of the division G_h, $|\alpha| \leqq 1$) and further $P_h^2 w \in \mathfrak{M}_h$ is defined in the following way: Let us denote the vertices of a rectangle G_{ih} by A, B, C, D — see the proof of Lemma 3.5. Then $P_h^2 w/_{G_{ih}}$ is a polynomial of the 2-nd degree satisfying the condition $P_h^2 w = w$ at A, B, C, D and $D^\alpha(P_h^2 w) = D^\alpha w$ at A for $|\alpha| \leqq 1$. It is easy to verify that

$$|w - P_h^1 w|_{2,G_{ih}} \leqq Ch|w|_{3,G_{ih}}\,,$$

$$|w - P_h^2 w|_{2,G_{ih}} \leqq Ch|w|_{3,G_{ih}}\,,$$

$$|w - P_h^1 w|_{2,\Gamma \cap \partial G_{ih}} \leqq Ch|w|_{3,\Gamma \cap \partial G_{ih}}\,,$$

$$\left\| \frac{\partial}{\partial y} \mathscr{L}_{ih} \frac{\partial}{\partial x} (w - P_h^1 w) \right\|_{L_2(\Gamma \cap \partial G_{ih})} = 0\,,$$

so that

$$|||w - P_h^1 w|||_h = \left(\sum_{i=1}^{k(h)} |w - P_h^1 w|_{2,G_{ih}}^2 + |w - P_h^1 w|_{2,\Gamma}^2 + \right.$$

$$\left. + \frac{1}{2} \sum_{i=1}^{k(h)} \left\| \frac{\partial}{\partial y} \mathscr{L}_{ih} \frac{\partial}{\partial x} (w - P_h^1 w) \right\|_{L_2(\Gamma \cap \partial G_{ih})}^2 \right)^{1/2} \leqq$$

$$\leqq Ch(|w|_{3,G^1} + |w|_{3,G^2} + |w|_{3,\Gamma})$$

and

$$|w - P_h^2 w|_{2,h} = \left(\sum_{i=1}^{k(h)} |w - P_h^2 w|_{2,G_{ih}}^2 \right)^{1/2} \leqq Ch(|w|_{3,G^1} + |w|_{3,G^2})\,.$$

This completes the proof.

Theorem 4.1. *Let u and u_h be solutions of the problem* (2.1) *and* (2.5), *respectively. Moreover, let u be regular. Then*

$$|||u - u_h|||_h = O(h) . \tag{4.3}$$

Proof. This is an easy consequence of (3.16), (4.1) and (4.2).

The main goal of this chapter is to prove the convergence $u \to u_h$ without an assumption of smoothness of the solution u. We must derive an assertion concerning density of the space W in the space V (weak solution).

Lemma 4.2 *Let u be a solution of the problem* (2.1). *There exists a sequence $\{w_n\}_{n=1}^{\infty}$, $w_n \in W$ such that*

$$\lim_{n \to \infty} |||w_n - u||| = 0 . \tag{4.4}$$

Proof. Because $u \in W_0^{2,2}(\Gamma)$ and $\partial u/\partial x \in W_0^{1,2}(G)$, there exist sequences $\{\varphi_{0n}\}_{n=1}^{\infty}$, $\{\varphi_{1n}\}_{n=1}^{\infty}$ such that $\varphi_{0n} \in D(\Gamma)$, $\varphi_{1n} \in D(\Gamma)$, $n = 1, 2, \ldots$, $\lim_{n \to \infty} \|u - \varphi_{0n}\|_{2,\Gamma} = 0$, $\lim_{n \to \infty} \|(\partial u/\partial x)\, \varphi_{1n}\|_{1,\Gamma} = 0$. For each integer n we introduce an auxiliary problem: To find $w_n \in W^{2,2}(G^1) \cap W^{2,2}(G^2)$ such that

$$\Delta^2 w_n = f \tag{4.5}$$

on G^i, $i = 1, 2$ (in the distribution sense),

$$w_n \big|_{\partial G^i \cap \partial G} = 0 , \quad \frac{\partial w_n}{\partial \nu} \bigg|_{\partial G_i \cap \partial G} = 0$$

$$w_n \big|_{\partial G^i \cap \Gamma} = \varphi_{0n} , \quad \frac{\partial w_n}{\partial x} \bigg|_{\partial G^i \cap \Gamma} = \varphi_{1n}$$

The weak solution w_n of the problem (4.5) depends continuously on the boundary conditions (see [6]), i.e.

$$|w_n - u|_{2,G^i} \to 0 , \quad i = 1, 2 .$$

Since evidently $w_n \in W_0^{2,2}(G)$, we obtain $|w_n - u|_{2,G} \to 0$ and finally

$$|||w_n - u||| \to 0 \quad \text{for} \quad n \to \infty .$$

Taking into account the results of Kondratěv [5], we can state that $w_n \in W^{3,2}(G^i)$, $i = 1, 2$ for $f \in L_2(G)$. Hence w_n belongs to W.

Lemma 4.3. *The inequality*

$$|||w|||_h \leqq C|||w||| \tag{4.6}$$

holds for each h and $w \in V$.

Proof. It is sufficient to show that

$$(4.7)\qquad \int_{\Gamma\cap\partial G_{ih}} \left\{\frac{\partial}{\partial y}\mathscr{L}_{ih}\frac{\partial w}{\partial x}\right\}^2 \mathrm{d}y \leqq C\int_{\Gamma\cap\partial G_{ih}} \left\{\frac{\partial^2 w}{\partial x\,\partial y}\right\}^2 \mathrm{d}y$$

holds independently of h and $w \in V$. But this is a consequence of a Bramble-Hilbert lemma (see [1]), because

$$\int_{\Gamma\cap\partial G_{ih}} \left\{\frac{\partial}{\partial y}\left(\mathscr{L}_{ih}\left(\frac{\partial w}{\partial x} + c\right)\right)\right\}^2 \mathrm{d}y = \int_{\Gamma\cap\partial G_{ih}} \left\{\frac{\partial}{\partial y}\left(\mathscr{L}_{ih}\frac{\partial w}{\partial x}\right)\right\}^2 \mathrm{d}y$$

where c is an arbitrary constant.

Theorem 4.2. *Let u and u_h be solutions of the problem* (2.1) *and* (2.5), *respectively. Then u_h converges to u in the following sense*:

$$(4.8)\qquad |||u_h - u|||_h \to 0 \quad \text{for} \quad h \to 0\,.$$

Proof. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence whose existence is guaranteed by Lemma 4.2. Using the estimate (3.16) and the triangle inequality, we get

$$|||u_h - u|||_h \leqq C\{\inf_{\varphi\in V_h} |||u - \varphi|||_h + \inf_{\varphi\in\mathfrak{M}_h} |u - \varphi|_{2,h} + h^2\} \leqq$$

$$(4.9)\qquad \leqq C\{|||u - w_n|||_h + \inf_{\varphi\in V_h} |||w_n - \varphi|||_h + |u - w_n|_{2,h} + \inf_{\varphi\in\mathfrak{M}_h} |w_n - \varphi|_{2,h} + h^2\}\,.$$

In accordance with (4.4) and (4.6) we can state that

$$|u - w_n|_{2,h} \leqq |||u - w_n|||_h \leqq C|||u - w_n||| \to 0\,,\quad n \to \infty\,,\quad h \in (0, 1)$$

and (4.1) and (4.2) imply

$$\inf_{\varphi\in V_h} |||w_n - \varphi|||_h \to 0$$

$$\inf_{\varphi\in\mathfrak{M}_h} |w_n - \varphi|_{2,h} \to 0$$

for $h \to 0$ and each fixed integer n.
The last three estimates together with (4.9) yield (4.8).

CONCLUSION

The main aim of this paper has been to verify the application of the nonconforming finite element method to a certain problem of a plate with ribs. This procedure is very advantageous especially when using smaller computer or minicomputers.

In this paper the case of one rib has been discussed. Some more general problem will be studied in another work of the authors. When extending the results of the last chapter to the case $I = \{I_i\}_{i=1}^{n}$, $n > 1$, $J = \emptyset$ the same technique as that of Lemma 4.2 can be used.

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NEKONFORMNÍ METODA KONEČNÝCH PRVKŮ V PROBLÉMU VETKNUTÉ DESKY SE ŽEBRY

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Tato práce se zabývá otázkami konvergence jedné nekonformní metody konečných prvků v problému, který se týká řešení deformace desky se žebry. Výchozí problém je formulován pro jistý typ funkcionálu simulujícího energii soustavy desky s konečným počtem žeber. Konvergence navrhované numerické metody je dokázána pro případ jednoho žebra za předpokladu, že není nic známo o regularitě řešení výchozího problému. Rozšíření na případ konečného počtu nezkřížených žeber je, ve smyslu poznámky v závěru práce, snadné.

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