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PROPAGATION OF RAYLEIGH-TYPE WAVES IN AEOLOTROPIC MATERIAL WITH CUBIC SYMMETRY

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INTRODUCTION

Miller and Musgrave studied the propagation of elastic waves in cubic materials. In this paper we have studied the possibility of propagation of Rayleigh-type waves in aelotropic material with cubic symmetry.Three different models have been considered: (1)a semi-infinite medium of the material,(ii) a layer of the material, of finite thickness, resting on a rigid base, and (iii) a semi-infinite isotropic medium overlaid by a layer of finite thickness of the material. For numerical calculation, the values of the elastic constants are taken to be those for pyrites [Love – p. 163].

SOLUTION OF THE PROBLEM

(A) With the origin at the free surface and the axes of symmetry as the axes of reference, Z-axis being directed into the medium, the equations of motion for cubic material in two dimensions are

$$c_{11}\frac{\partial^2 u}{\partial x^2} + c_{44}\frac{\partial^2 u}{\partial z^2} + (c_{12} + c_{44})\frac{\partial^2 \omega}{\partial x\,\partial z} = \varrho\frac{\partial^2 u}{\partial t^2} \tag{1}$$

and

$$c_{44}\frac{\partial^2 \omega}{\partial x^2} + c_{11}\frac{\partial^2 \omega}{\partial z^2} + (c_{12} + c_{44})\frac{\partial^2 u}{\partial x\,\partial z} = \varrho\frac{\partial^2 \omega}{\partial t^2}, \tag{2}$$

where u and w are the components of displacement in the x and z directions, respectively.

Substituting

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial^2 \psi}{\partial x\,\partial z}, \tag{3}$$

$$\omega = \frac{\partial \varphi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} + \alpha\psi$$

in equations (1) and (2), α being a constant to be chosen suitably, we get

$$(4)\qquad \frac{\partial}{\partial x}\left[c_{11}\frac{\partial^2\varphi}{\partial x^2} + (c_{12} + 2c_{44})\frac{\partial^2\varphi}{\partial z^2} - \varrho\frac{\partial^2\varphi}{\partial t^2} + \right.$$

$$\left. + \frac{\partial}{\partial z}\left\{c_{11}\frac{\partial^2\psi}{\partial x^2} + (c_{12} + 2c_{44})\frac{\partial^2\psi}{\partial z^2} + (c_{12} + c_{44})\,\alpha\psi - \varrho\frac{\partial^2\psi}{\partial t^2}\right\}\right] = 0$$

and

$$(5)\qquad \frac{\partial}{\partial z}\left[c_{11}\frac{\partial^2\varphi}{\partial z^2} + (c_{12} + 2c_{44})\frac{\partial^2\varphi}{\partial x^2} - \varrho\frac{\partial^2\varphi}{\partial t^2}\right] +$$

$$+ \frac{\partial^2}{\partial z^2}\left[c_{11}\frac{\partial^2\psi}{\partial z^2} + (c_{12} + 2c_{44})\frac{\partial^2\psi}{\partial x^2} - \varrho\frac{\partial^2\psi}{\partial t^2}\right] +$$

$$+ \alpha\left[c_{44}\frac{\partial^2\psi}{\partial x^2} + c_{11}\frac{\partial^2\psi}{\partial z^2} - \varrho\frac{\partial^2\psi}{\partial t^2}\right] = 0\,.$$

Now let us assume

$$(6)\qquad \varphi = A\cos kx e^{-qz} e^{ipt}$$

$$\psi = B\cos kx e^{-qz} e^{ipt}\,.$$

Then equations (4) and (5) are satisfied if

$$(7)\qquad A[(c_{12} + 2c_{44})\,q^2 + (\varrho p^2 - c_{11}k^2)] - Bq[(c_{12} + 2c_{44})\,q^2 +$$

$$+ \{(c_{12} + c_{44})\,\alpha + \varrho p^2 - c_{11}k^2\}] = 0$$

and

$$(8)\qquad Aq[c_{11}q^2 + \{\varrho p^2 - k^2(c_{12} + 2c_{44})\}] -$$

$$- B[q^2\{c_{11}q^2 + \varrho p^2 - k^2(c_{12} + 2c_{44})\} + \alpha\{c_{11}q^2 + \varrho p^2 - c_{44}k^2\}] = 0\,.$$

Eliminating A and B from (7) and (8) we see that in order that (6) may satisfy equations (4) and (5), q must satisfy

$$(9)\qquad c_{11}c_{44}q^4 + [(c_{44} + c_{11})\,\varrho p^2 + (c_{12}^2 - c_{11}^2 + 2c_{12}c_{44})\,k^2]\,q^2 +$$

$$+ (\varrho p^2 - c_{11}k^2)(\varrho p^2 - c_{44}k^2) = 0\,.$$

If $\pm q_1$ and $\pm q_2$ are the roots of equation (9) we may write

$$(10)\qquad \varphi = (A_1e^{q_1z} + A_2e^{-q_1z} + A_3e^{q_2z} + A_4e^{-q_2z})\cos kx e^{ipt}\,,$$

$$\psi = (B_1e^{q_1z} + B_2e^{-q_1z} + B_3e^{q_2z} + B_4e^{-q_2z})\cos kx e^{ipt}\,,$$

where

$$(11)\qquad \begin{aligned} q_1 B_1 &= \xi A_1 , \\ q_1 B_2 &= -\xi A_2 , \\ q_2 B_3 &= \eta A_3 , \\ q_2 B_4 &= -\eta A_4 , \end{aligned}$$

ξ and η being given by

$$(12)\qquad \xi = \frac{(c_{12} + 2c_{44})\, q_1^2 + \varrho p^2 - c_{11} k^2}{(c_{12} + 2c_{44})\, q_1^2 + \varrho p^2 - c_{11} k^2 + (c_{12} + c_{44})\, \alpha},$$

$$\eta = \frac{(c_{12} + 2c_{44})\, q_2^2 + \varrho p^2 - c_{11} k^2}{(c_{12} + 2c_{44})\, q_2^2 + \varrho p^2 - c_{11} k^2 + (c_{12} + c_{44})\, \alpha}.$$

From (3), (10) and (11) we get

$$(13)\qquad \begin{aligned} u = -k \sin k\varkappa e^{ipt}\big[&(1 + \xi)\, A_1 e^{q_1 z} + (1 + \xi)\, A_2 e^{-q_1 z} + \\ &+ (1 + \eta)\, A_3 e^{q_2 z} + (1 + \eta)\, A_4 e^{-q_2 z}\big] , \end{aligned}$$

$$(14)\qquad \begin{aligned} \omega = \cos k\varkappa e^{ipt}\Bigg[&\left\{q_1 + \left(\frac{q_1^2 + \alpha}{q_1}\right)\xi\right\} A_1 e^{q_1 z} - \left\{q_1 + \left(\frac{q_1^2 + \alpha}{q_1}\right)\xi\right\} A_2 e^{-q_1 z} + \\ &+ \left\{q_2 + \left(\frac{q_2^2 + \alpha}{q_2}\right)\eta\right\} A_3 e^{q_2 z} - \left\{q_2 + \left(\frac{q_2^2 + \alpha}{q_2}\right)\eta\right\} A_4 e^{-q_2 z}\Bigg]. \end{aligned}$$

(B) If, however, instead of (3), we substitute

$$(15)\qquad \begin{aligned} u &= \frac{\partial \varphi}{\partial \varkappa} + \frac{\partial^2 \psi}{\partial \varkappa\, \partial z}, \\ \omega &= \frac{\partial \varphi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} \end{aligned}$$

in equations (1) and (2) and assume the same form of solutions for φ and ψ as in (6), we find that the values of $\pm q_1$ and $\pm q_2$ are given by

$$(16)\qquad \begin{aligned} q_1^2 &= \frac{(c_{11} - \varrho c^2)\, k^2}{c_{12} + 2c_{44}}, \\ q_2^2 &= \frac{(c_{12} + 2c_{44} - \varrho c^2)\, k^2}{c_{11}}. \end{aligned}$$

The relations between the constants A_1, B_1 etc. are now given by

$$(17)\qquad \begin{aligned} q_1 B_1 &= A_1 , \quad q_1 B_2 = -A_2 , \\ q_2 B_3 &= A_3 , \quad q_2 B_4 = -A_4 \end{aligned}$$

and the displacements by

$$u = -2k \sin kx e^{ipt}[A_1 e^{q_1 z} + A_2 e^{-q_1 z} + A_3 e^{q_2 z} + A_4 e^{-q_2 z}], \tag{18}$$

$$\omega = 2 \cos kx e^{ipt}[q_1 A_1 e^{q_1 z} - q_1 A_2 e^{-q_1 z} + q_2 A_3 e^{q_2 z} - q_2 A_4 e^{-q_2 z}]. \tag{19}$$

Case (i). For a semi-infinite medium of cubic material, the suitable solutions for u and ω as obtained from (13) and (14) in case (A) are

$$u = -k \sin kx e^{ipt}[(1 + \xi) A_2 e^{-q_1 z} + (1 + \eta) A_4 e^{-q_2 z}], \tag{20}$$

$$\omega = - \cos kx e^{ipt}\left[\left\{q_1 + \left(\frac{q_1^2 + \alpha}{q_1}\right)\xi\right\} A_2 e^{-q_1 z} + \right. \left. + \left\{q_2 + \left(\frac{q_2^2 + \alpha}{q_2}\right)\eta\right\} A_4 e^{-q_2 z}\right]. \tag{21}$$

The stress-strain relations give

$$\widehat{zz} = c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial \omega}{\partial z}, \qquad \widehat{zx} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x}\right). \tag{22}$$

The boundary conditions require that the stress-components $\widehat{zz}$ and $\widehat{zx}$ vanish at $z = 0$. These conditions together with (20), (21) and (22) yield

$$A_2[c_{11}\{q_1^2 + (q_1^2 + \alpha)\xi\} - c_{12}k^2(1 + \xi)] + + A_4[c_{11}\{q_2^2 + (q_2^2 + \alpha)\eta\} - c_{12}k^2(1 + \eta)] = 0, \tag{23}$$

$$A_2[q_1(1 + \xi) + 1/q_1\{q_1^2 + (q_1^2 + \alpha)\xi\}] + + A_4[q_2(1 + \eta) + 1/q_2\{q_2^2 + (q_2^2 + \alpha)\eta\}] = 0. \tag{24}$$

Eliminating A_2 and A_4 from (23) and (24) we get the frequency equations as

$$[c_{11}\{q_1^2 + (q_1^2 + \alpha)\xi - c_{12}k^2(1 + \xi)][q_2(1 + \eta) + 1/q_2\{q_2^2 + (q_2^2 + \alpha)\eta\}] - - [c_{11}\{q_2^2 + (q_2^2 + \alpha)\eta\} - c_{12}k^2(1 + \eta)][q_1(1 + \xi) + 1/q_1\{q_1^2 + (q_1^2 + \alpha)\xi\}] = 0. \tag{25}$$

Again, if we take the values of q_1 and q_2 as obtained from (16) in case (B) the frequency equation becomes

$$\left(\frac{\varrho c^2}{c_{44}}\right)^2 - \left(\frac{c_{11} + c_{12}}{c_{44}} + 2\right)\left(\frac{\varrho c^2}{c_{44}}\right) + \left(2 + \frac{c_{12}}{c_{44}}\right)\left(\frac{c_{11}}{c_{44}} - \frac{c_{12}^2}{c_{11}c_{44}}\right) = 0. \tag{26}$$

Case (ii). For a layer of the cubic material of finite thickness h resting on a rigid base, the boundary conditions are

$$\widehat{zz} = 0, \quad \widehat{zx} = 0 \quad \text{at} \quad z = 0, \qquad u = 0, \quad \omega = 0 \quad \text{at} \quad z = h. \tag{27}$$

These boundary conditions together with (18), (19) and (22) give

$$(28)\qquad \begin{aligned} l_1A_1 + l_1A_2 + l_2A_3 + l_2A_4 &= 0\,, \\ q_1A_1 - q_1A_2 + q_2A_3 - q_2A_4 &= 0\,, \\ A_1e^{q_1h} + A_2e^{-q_1h} + A_3e^{q_2h} + A_4e^{-q_2h} &= 0\,, \\ A_1q_1e^{q_1h} - A_2q_1e^{-q_1h} + A_3q_2e^{q_2h} - A_4q_2e^{-q_2h} &= 0\,, \end{aligned}$$

where

$$(29)\qquad \begin{aligned} l_1 &= k^2c_{12} - q_1^2c_{11}\,, \\ l_2 &= k^2c_{12} - q_2^2c_{11} \end{aligned}$$

and q_1 and q_2 are given by (16).

Eliminating A_1, A_2, A_3, A_4 from equations (28), we get the frequency equation

$$(30)\qquad \begin{vmatrix} l_1 & l_1 & l_2 & l_2 \\ q_1 & -q_1 & q_2 & -q_2 \\ e^{q_1h} & e^{-q_1h} & e^{q_2h} & e^{-q_2h} \\ q_1e^{q_1h} & -q_1e^{-q_1h} & q_2e^{q_2h} & -q_2e^{-q_2h} \end{vmatrix} = 0\,.$$

Case (iii). Let us now consider an isotropic homogeneous semi-infinite medium overlaid by a layer of the cubic material of finite thickness h.

The equations of motion for the lower isotropic medium in two dimensions are

$$(31)\qquad (\lambda_1 + u_1)\frac{\partial \Delta}{\partial x} + \mu_1\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial z^2}\right) = \varrho_1\frac{\partial^2 u_1}{\partial t^2}\,,$$

$$(32)\qquad (\lambda_1 + \mu_1)\frac{\partial \Delta}{\partial z} + \mu_1\left(\frac{\partial^2 \omega_1}{\partial x^2} + \frac{\partial^2 \omega_1}{\partial z^2}\right) = \varrho_1\frac{\partial^2 \omega_1}{\partial t^2}\,,$$

where u_1 and ω_1 are the displacement components in the lower medium, ϱ_1 and μ_1 are respectively the density and the rigidity, and

$$\Delta = \frac{\partial u_1}{\partial x} + \frac{\partial \omega_1}{\partial z}\,.$$

Substituting

$$(33)\qquad \begin{aligned} u_1 &= \frac{\partial \varphi_1}{\partial x} + \frac{\partial \psi_1}{\partial z}\,, \\ \omega_1 &= \frac{\partial \varphi_1}{\partial z} - \frac{\partial \psi_1}{\partial x}\,, \end{aligned}$$

in (31) and (32), we find that these equations are satisfied if

$$(34)\qquad \frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial z^2} = \frac{1}{\alpha_1^2}\frac{\partial^2 \varphi_1}{\partial t^2}$$

and

(35) $$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial z^2} = \frac{1}{\beta_1^2}\frac{\partial^2\psi_1}{\partial t^2},$$

where

(36) $$\alpha_1 = \left(\frac{\lambda_1 + 2\mu_1}{\varrho_1}\right)^{1/2}, \quad \beta_1 = \left(\frac{\mu_1}{\varrho_1}\right)^{1/2}.$$

The solutions of equations (34) and (35) suitable for the problem are

(37) $$\varphi_1 = L_1 e^{-n_1 z} \cos kx e^{ipt}, \quad \psi_1 = M_1 e^{-n_2 z} \sin kx e^{ipt},$$

where

(38) $$n_1 = k\sqrt{(1 - c^2/\alpha_1^2)}, \quad n_2 = k\sqrt{(1 - c^2/\beta_1^2)}.$$

The stress-components are given by

(39) $$(\widehat{zz})_1 = \lambda_1 \frac{\partial u_1}{\partial x} + (\lambda_1 + 2\mu_1)\frac{\partial \omega_1}{\partial z}, \quad (\widehat{zx})_1 = \mu_1\left(\frac{\partial u_1}{\partial z} + \frac{\partial \omega_1}{\partial x}\right).$$

The boundary conditions in this case are

(40) $$\widehat{zz} = 0, \quad \widehat{zx} = 0 \quad \text{at} \quad z = 0$$
$$u = u_1, \quad \omega = \omega_1$$
$$\widehat{zz} = (\widehat{zz})_1, \; \widehat{zx} = (\widehat{zx})_1 \quad \text{at } z = h.$$

These conditions lead to

$$l_1A_1 + l_1A_2 + l_2A_3 + l_2A_4 = 0,$$
$$q_1A_1 - q_1A_2 + q_2A_3 - q_2A_4 = 0,$$
$$2e^{q_1h}A_1 + 2e^{-q_1h}A_2 + 2e^{q_2h}A_3 + 2e^{-q_2h}A_4 - L_1e^{-n_1h} - \frac{M_1}{k}n_2e^{-n_2h} = 0,$$
$$2q_1e^{q_1h}A_1 - 2q_1e^{-q_1h}A_2 + 2q_2e^{q_2h}A_3 - 2q_2e^{-q_2h}A_4 + L_1n_1e^{-n_1h} + M_1ke^{-n_2h} = 0,$$
$$-2l_1e^{q_1h}A_1 - 2l_1e^{-q_1h}A_2 - 2l_2e^{q_2h}A_3 - 2l_2e^{-q_2h}A_4 - m_1e^{-n_1h}L_1 - 2\mu_1\alpha_1n_2e^{-n_2h}M_1 = 0,$$
$$-2kc_{44}q_1e^{q_1h}A_1 + 2kc_{44}q_1e^{-q_1h}A_2 - 2kc_{44}q_2e^{q_2h}A_3 + 2kc_{44}q_2e^{-q_2h}A_4 - 2\mu_1kn_1e^{-n_1h}L_1 - \mu_1(n_2^2 + k^2)e^{-n_2h}M_1 = 0,$$

where

(41) $$m_1 = \lambda(n_1^2 - k^2) + 2\mu_1 n_1^2$$

and q_1, q_2 are given by (16).

Eliminating A_1, A_2, A_3, A_4, L_1 and M_1 from these relations we get

(42) $$\begin{vmatrix} l_1 & l_1 & l_2 & l_2 & 0 & 0 \\ q_1 & -q_1 & q_2 & -q_2 & 0 & 0 \\ 2e^{q_1 h} & 2e^{-q_1 h} & 2e^{q_2 h} & 2e^{-q_2 h} & -e^{-n_1 h} & -n_2/k\, e^{-n_2 h} \\ 2q_1 e^{q_1 h} & -2q_1 e^{-q_1 h} & 2q_2 e^{q_2 h} & -2q_2 e^{-q_2 h} & n_1 e^{-n_1 h} & ke^{-n_2 h} \\ -2l_1 e^{q_1 h} & -2l_1 e^{-q_1 h} & -2l_2 e^{q_2 h} & -2l_2 e^{-q_2 h} & -m_1 e^{-n_1 h} & -2\mu_1 \alpha_1 n_2 e^{-n_2 h} \\ -2kc_{44} q_1 e^{q_1 h} & 2kc_{44} q_1 e^{-q_1 h} & -2kc_{44} q_2 e^{q_2 h} & 2kc_{44} q_2 e^{-q_2 h} & -2\mu_1 k_1 n_1 e^{-n_1 h} & -\mu_1(n_2^2 + k^2)\, e^{-n_2 h} \end{vmatrix} = 0$$

NUMERICAL RESULTS

For numerical calculation of the roots of the frequency equations, we take the values of the eleastic constants to be those for pyrites, i.e.

$$c_{11} = \quad 3680 \times 10^6 \text{ grammes wt. per sq. cm.}$$
$$c_{44} = \quad 1075 \times 10^6 \text{ grammes wt. per sq. cm.}$$
$$c_{12} = -\ 483 \times 10^6 \text{ grammes wt. per sq. cm.}$$

Roots of equation (25). (Approximate Solution.) Let us choose α to be so small that $q_1^2 + \alpha \simeq q_1^2$, $q_2^2 + \alpha \simeq q_2^2$,

$$(c_{12} + 2c_{44})\, q_1^2 + \varrho p^2 - c_{11}k^2 + (c_{12} + c_{44})\,\alpha \simeq (c_{12} + 2c_{44})\, q_1^2 + \varrho p^2 - c_{11}k^2$$

and

$$(c_{12} + 2c_{44})\, q_2^2 + \varrho p^2 - c_{11}k^2 + (c_{12} + c_{44})\,\alpha \simeq (c_{12} + 2c_{44})\, q_2^2 + \varrho p^2 - c_{11}k^2\,,$$

so that $\xi = 1$ and $\eta = 1$.

Equation (25) with the values of q_1 and q_2 given by equation (9) in (A) then reduces to

$$\left(\frac{c^2}{\beta^2}\right)^2 - \left(\frac{c_{11}}{c_{44}} + 1\right)\left(\frac{c^2}{\beta^2}\right) + \left(\frac{c_{11}}{c_{44}} - \frac{c_{12}^2}{c_{11}c_{44}}\right) = 0$$

where

$$\frac{c^2}{\beta^2} = \frac{\varrho c^2}{c_{44}}.$$

Hence

$$\frac{c}{\beta} = \left[\frac{1}{2}\left(\frac{c_{11}}{c_{44}} + 1 \pm \left\{\left(\frac{c_{11}}{c_{44}} - 1\right)^2 + 4\frac{c_{12}^2}{c_{11}c_{44}}\right\}^{1/2}\right)\right]^{1/2}.$$

With the values of the elastic constants given above, we get

$$c/\beta = 0.987,\ 1.85\,.$$

The former of these two values of c/β clearly corresponds to Rayleigh waves.

We, therefore, conclude that Rayleigh waves exist in a semi-infinite medium of aeolotropic material with cubic symmetry.

Roots of equation (26)

Here

$$\frac{c}{\beta} = \left[\frac{1}{2}\left(\frac{c_{11}+c_{12}}{c_{44}} + 2 \pm \left\{\left(\frac{c_{11}+c_{12}}{c_{44}} + 2\right)^2 - 4\left(2 + \frac{c_{12}}{c_{44}}\right)\left(\frac{c_{11}}{c_{44}} - \frac{c_{12}^2}{c_{11}c_{44}}\right)\right\}^{1/2}\right)\right]^{1/2}.$$

Hence $c/\beta = 1.23,\ 1.86$.

Roots of equation (30)

Four of the roots of the equation are readily obtained as

$$\frac{c}{\beta} = \left(2 + \frac{c_{12}}{c_{44}}\right)^{1/2},\quad 2^{1/2},\quad \left(\frac{c_{11}^2 - c_{12}^2 - 2c_{12}c_{44}}{c_{11}c_{44}}\right)^{1/2},\quad \left(\frac{c_{11}}{c_{44}}\right)^{1/2}.$$

Hence four of the roots are

$$c/\beta = 1.24,\ 1.41,\ 1.90 \text{ and } 2.03\,.$$

Roots of equation (42)

Two of the roots are easily found to be

$$\frac{c}{\beta} = \left(2 + \frac{c_{12}}{c_{44}}\right)^{1/2},\quad \left(\frac{c_{11}}{c_{44}}\right)^{1/2}.$$

Hence $c/\beta = 1.24,\ 2.03$.

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Souhrn

ŠÍŘENÍ VLN RAYLEIGHOVA TYPU V AELOTROPICKÉM MATERIÁLU S KUBICKOU SYMETRIÍ

SUBHAS DUTTA a PRIYATOSH ROY

V článku se zkoumá možnost šíření vln Rayleighova typu v aelotropickém materiálu kubického systému pro různé modely.

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