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# WEAK SOLUTION OF BOUNDARY VALUE PROBLEM FOR THE ORTHOTROPIC PLATE REINFORCED WITH STIFFENING RIBS

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The present paper is a continuation of the paper [4] dealing with the existence of solution of Boundary value problem for thin isotropic plate stiffened with ribs. It is intended to prove by methods of the abstract variational calculus the existence and the uniqueness of the weak solution of an orthotropic plate with stiffening ribs for the same class of boundary conditions in the subspace $V(\Omega) \subset W_2^2(\Omega)$.

Let $\Omega$ be a bounded region in the plane $x$, $y$ with the boundary $\partial\Omega$.

Let us study in $\Omega$ a system of partial differential equations

(1) $$L(w) = D_1 w_{xxxx} + 2D_3 w_{xxyy} + D_2 w_{yyyy} = p(x, y)^{1)}$$

(2) $$L_{\tau k,s} = \varrho_k^{-1} L_{nk} - m_k(s)\,, \quad L_{nk,s} = -\varrho_k^{-1} L_{\tau k} + V_{bk}\,,$$

where

$L_{\tau k}$, $L_{nk}$ are the internal forced of the rib $\gamma_k$:

(3) $$L_{\tau k} = C_k(\theta_{\tau k,s} - \varrho_k^{-1}\theta_{nk})\,, \quad L_{nk} = A_k(\theta_{nk,s} - \varrho_k^{-1}\theta_{\tau k})\,,$$

$\varrho_k$ is a variable radius of curvature of a sufficiently smooth curve $\gamma_k$ for $k = 1, 2, \ldots, l$,

$C_k$ is the torsional rigidity of the rib,

$A_k$ is the bending rigidity of the rib with respect to the axis $n$, where

$n$ is the normal to the tangent $\tau$ of the curve $\gamma_k$ on the section $s$,

$V_{bk}$ is the shearing force on the section $s$ of the stiffening rib $\gamma_k$ defined by the equations

$V_{bk} = - \int_0^s p_k(s)\, ds + \text{const.}$, and

$p_k(s) = p_{0_k}(s) - p_{k_k}(s)$ is the vertical load (shearing forces), which acts on the $k$-th rib and which according to the definition is equal to the difference of the shearing forces $p_{0_k}(s)$, $p_{k_k}(s)$ on the $k$-th rib $\gamma_k$ from the side of the regions $\Omega_0$, $\Omega_k$ respectively.

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[1] Hence forth we shall denote $\dfrac{\partial w}{\partial x} = w_x$; $L_{\tau k,s} = \dfrac{dL_{\tau k}}{ds}$.

Analogously, the $k$-th stiffening rib $\gamma_k$ will be acted upon by the resulting moment load (bending moments) $m_k(s) = m_{0_k}(s) - m_{k_k}(s)$, where $m_{0_k}(s)$, $m_{k_k}(s)$ are individual bending moments acting upon the $k$-th rib from the side of the regions $\Omega_0$ and $\Omega_k$, respectively.

$\theta_{\tau k}$, $\theta_{nk}$ are the deformation parameters of the $k$-th rib defined by

$$\theta_{\tau k} = \frac{\partial \Delta_k}{\partial n}; \quad \theta_{nk} = -\frac{\mathrm{d}\Delta_k}{\mathrm{d}s} \quad \text{and} \quad \Delta_k = w_0 = w_k$$

is the deflection ordinate on the $\gamma_k$ axis of the $k$-th rib, $w_0$, $w_k$ are the deflection ordinates of the plate in the regions $\Omega_0$ and $\Omega_k$ and for each $k = 1, 2, \ldots, l$ it holds

$$\text{(i)} \qquad \frac{\partial w_k}{\partial n} + \frac{\partial w_k}{\partial s} = \theta_{\tau k} - \theta_{nk} .$$

The curve $\Gamma = \bigcup_{k=1}^{l} \gamma_k$ divides the region $\Omega$ into $l + 1$ subregions. Then $\bar{\Omega} = \bar{\Omega}_0 + \bar{\Omega}_1 + \bar{\Omega}_2 + \ldots + \bar{\Omega}_l$. For the sake of simplicity let us suppose that regions $\Omega_k$ $(k = 1, 2, \ldots, l)$ are simply connected while $\Omega_0$ is a multiple-connected region bounded by the set of curves $\Gamma$ and by $\partial\Omega = \bigcup_{j=1}^{m+1} \partial\Omega_j^*$. The curve $\partial\Omega_{m+1}^*$ encircles all the other curves $\partial\Omega_j^*$ and the curves $\Gamma$ and $\partial\Omega$ do not touch or intersect each other. The given vertical load of the plate applied to the region is defined by a function $p = p(x, y)$.

The following notation is introduced for the constants:

$D = D_{11}$, $D_2 = D_{22}$, $D_3 = D_{12} + 2D_{66}$, where $D_{ij}$ are the rigiditiets of an orthotropic thin plate.

The physical reality, i.e., the non-negativity of work of the internal forces of an orthotropic plate with stiffening ribs implies that the matrix of coefficients

$$(1^\circ) \qquad D = \begin{bmatrix} D_1; & D_3 \\ D_3; & D_2 \end{bmatrix}$$

is positive definite. Taking into account the Silvester Theorem we obtain from the positive definiteness of the quadratic form of the five unknown quantities $w_{xx}$, $w_{xy}$, $w_{yy}$, $L_{\tau k}$, $L_{nk}$

$$(2^\circ) \qquad D_1 > 0\,; \quad D_2 > 0\,; \quad D_{66} > 0\,; \quad D_1 D_2 - D_{12}^2 > 0 \quad \text{(see [7])}.$$

Equations (1), (2) will be examined in the bounded region $\Omega$ which is a multiple-connected region with a Lipschitzian boundary $\partial\Omega$. It is assumed that $\partial\Omega$ consists of two disjoint parts $\partial\Omega_1$ and $\partial\Omega_2$ such that

$$\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 ,$$

where mes $(\partial\Omega_1) > 0$. The boundary conditions are considered in the form

$4_1$) $$w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_1 ,$$

$4_2$) $$w = 0 ,$$

$$\overline{m}_n = -w_{xx}(D_1 n_x^2 + D_{12} n_y^2) - w_{yy}(D_{12} n_x^2 + D_2 n_y^2) -$$
$$- 4 w_{xy} D_{66} n_x n_y = m_2 \quad \text{on} \quad \partial\Omega_2 ,$$

where $n = (n_x, n_y)$ is the external normal with regard to $\Omega$, $\overline{m}_n$ is a statical quantity, the bending moment on the surface of the boundary $\partial\Omega_2$ with the normal $n$.

$4_3$) Within the region $\Omega$ the following geometric conditions are formulated for the function $w(x, y)$:

$$w^+ = w^- ,$$

$$\frac{\partial w^+}{\partial n} = \frac{\partial w^-}{\partial n} \quad \text{on} \quad \gamma_k \quad \text{for} \quad k = 1, 2, \ldots, l .$$

The superscript plus or minus indicates that the quantity in question refers to the region $\Omega_0$ or $\Omega_k$, respectively. The mechanical meaning of these conditions is as follows: in $\partial\Omega_1$ the plate is clamped, in $\partial\Omega_2$ it is simply supported and the deflection surface is smooth over the region $\Omega$.

**Terminology.** Let $\Omega$ be a bounded region in the plane $x$, $y$ whose boundary is Lipschitzian (see [6]).

$L_p(\Omega)$ denotes the space of all measurable functions which are integrable with the power $p$ on $\Omega$ (with regard to the Lebesque measure d$x$ d$y$).

Let us adopt further the notation

$$|\alpha| = \sum_{i=1}^{2} \alpha_i ; \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \, \partial y^{\alpha_2}} .$$

Let us define the Sobolev space $W_2^2(\Omega)$ by

$$W_2^2(\Omega) = \{u \mid u \in L_2(\Omega) ; \quad D^\alpha u \in L_2(\Omega) \quad \text{for} \quad |\alpha| \leqq 2\} ,$$

(the derivatives are taken in the sense of distributions).

$W_2^2(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W_2{}^2(\Omega)} = \left\{\int\!\!\int_\Omega |u|^2 \, \mathrm{d}x \, \mathrm{d}y + \sum_{|\alpha|=2} \int_\Omega |D^\alpha u|^2 \, \mathrm{d}x \, \mathrm{d}y\right\}^{1/2} .$$

The space $W_2^2(\Omega)$ with a scalar product

$$(u, v)_{W_2^2(\Omega)} = \int_\Omega uv \, dx \, dy + \sum_{|\alpha|=2} \int_\Omega D^\alpha u D^\alpha v \, dx \, dy$$

is a Hilbert space.

$\dot{W}_2^2(\Omega)$ denotes a subspace $W_2^2(\Omega)$ which is obtained as the closure of the space $\mathscr{D}(\Omega)$ in the norm $\|\cdot\|_{W_2^2(\Omega)}$ $(\mathscr{D}(\Omega) = \{\varphi \mid \varphi$ are infinitely differentiable functions with a compact carrier in $\Omega\})$. The scalar product defined in $\dot{W}_2^2(\Omega)$ by

$$(u, v)_{\dot{W}_2^2(\Omega)} = \sum_{|\alpha|=2} \int_\Omega D^\alpha u D^\alpha v \, dx \, dy$$

generates in $\dot{W}_2^2(\Omega)$ the norm $\|\cdot\|_{\dot{W}_2^2(\Omega)}$ which is equivalent to the norm $\|\cdot\|_{W_2^2(\Omega)}$ in $\dot{W}_2^2(\Omega)$.

For the purpose of studying the boundary value problem (1), (2), $(4_1, 4_2, 4_3)$ let us define a space $V(\Omega)$ in the following way:
Let

$$\mathscr{V}(\Omega) = \left\{u \mid u \in \varepsilon(\bar{\Omega}) ; \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_1 , \quad u = 0 \quad \text{on} \quad \partial\Omega_2\right\} ;$$

then $V(\Omega)$ is the closure of $\mathscr{V}(\Omega)$ in the norm

(ii) $$\|u\|_{H(\Omega)} = \left(\|u\|^2_{\dot{W}_2^2(\Omega)} + \sum_{k=1}^{l} \int_{\gamma_k} [L^2_{\tau k}(u) + L^2_{nk}(u)] \, ds\right)^{1/2} = $$
$$= (\|u\|^2_{\dot{W}_2^2(\Omega)} + \|u\|^2_{K(\Gamma)})^{1/2} ,$$

where

$$\|u\|^2_{K(\Gamma)} = \sum_{k=1}^{l} (\|L_{\tau k}(u)\|^2_{L_2(\gamma_k)} + \|L_{nk}(u)\|^2_{L_2(\gamma_k)})$$

$(\varepsilon(\bar{\Omega})$ is a set of functions infinitely differentiable in $\Omega$, whose all derivatives may be continuously extended to the boundary).

Let us further denote by $C_0(\bar{\Omega})$ the space of all real functions which are continuous in $\bar{\Omega}$ and equal to zero on $\partial\Omega$, $C_c(\bar{\Omega})$ being the space of real functions which are continuous in $\Omega$ and equal to zero outside a certain compact subset of $\Omega$. From the Sobolev theorems on imbeddings (see [6]) we obtain that $V(\Omega) \subset C_0(\bar{\Omega})$ both algebraically and topologically.

Because $C_c(\Omega)$ is dense in $C_0(\bar{\Omega})$, each of the elements in $(C_0(\Omega))^*$ may be identified by means of a transposition with the element in $(C_c(\Omega))^*$. Hence in our further considerations the Dirac measure $\delta = \delta_{(x_0, y_0)}((x_0, y_0) \in \Omega)$ represents singular vertical load in (1). If $p \in (C_0(\Omega))^*$ and $\varphi \in C_0(\bar{\Omega})$ then the value of $p$ for the function $\varphi$ will be denoted by $\langle p, \varphi \rangle$. Let us note that $L_1(\Omega) \subset (C_0(\bar{\Omega}))^*$ implies

$$\langle p, \varphi \rangle = \int_\Omega p(x, y)\, \varphi(x, y)\, \mathrm{d}x\, \mathrm{d}y\, .$$

It follows from the Sobolev theorems on imbeddings and from the theorems on traces that for each $u \in V(\Omega)$, $u = 0$ pointwise in $\partial\Omega_1 \cup \partial\Omega_2$ (see [5]; [6]) and $\partial u / \partial n$ in the sense of traces in $\partial\Omega_1$.

For the study of our boundary value problem it is necessary to introduce a new scalar product in $V(\Omega)$ by means of a bilinear form of the form

(5) $$a(u, v) = \int_\Omega [D_{11} u_{xx} v_{xx} + 2D_{12} u_{xx} v_{yy} + D_{22} u_{yy} v_{yy} + \\ + 4D_{66} u_{xy} v_{xy}]\, \mathrm{d}x\, \mathrm{d}y + \sum_{k=1}^{l} \int_{\gamma_k} [L_{\tau k}(u)\, L_{\tau k}(v) + L_{nk}(u)\, L_{nk}(v)]\, \mathrm{d}s\, .$$

In the case of a rectangular plate the bilinear form (5) may be taken in a simpler form

$$a(u, v) = \int_\Omega [D_1 u_{xx} v_{xx} + 2D_3 u_{xy} v_{xy} + D_2 u_{yy} v_{yy}]\, \mathrm{d}x\, \mathrm{d}y + \\ + \sum_{k=1}^{l} \int_{\gamma_k} [L_{\tau k}(u)\, L_{\tau k}(v) + L_{nk}(u)\, L_{nk}(v)]\, \mathrm{d}s\, .$$

**Lemma 1.** *There exist positive constants $c_1, c_2 > 0$ for which*

(6) $$c_1 \|u\|^2_{H(\Omega)} \leqq a(u, u) \leqq c_2 \|u\|^2_{H(\Omega)}$$

*holds for all $u \in V(\Omega)$*

Proof. The first inequality in (6) follows from 1°, 2° and from [2; Theorem 2.1] while the other inequality is evident.

**Lemma 2.** *The bilinear form $a(u, v)$ is symmetric in the space $V(\Omega)$.*

Proof. It is easy to prove by integration by parts – the Green Theorem, the integral identity

(7) $$a(u, v) = (L(u), v)_{L_2(\Omega)} - (\overline{m}_n(u), v_n)_{L_2(\partial\Omega)} + \\ + \sum_{k=1}^{l} \int_{\gamma_k} [L_{\tau k}(u)\, L_{\tau k}(v) + L_{nk}(u)\, L_{nk}(v)]\, \mathrm{d}s$$

for each $u, v \in \mathscr{V}(\Omega)$.

The identity (7) shows that the bilinear form $a(u, v)$ is symmetric in $\mathscr{V}(\Omega)$ if the linear operator $L(w)$ is symmetric in the space $\mathscr{V}(\Omega)$. The symmetry of the operator $L(w)$ is, however, proved by a usual application of the Green Theorem under the assumption that the boundary $\partial\Omega$ is smooth. After a simple rearrangement we obtain

$$(8) \qquad (L(u), v)_{L_2(\Omega)} - (\overline{m}_n(u), v_n)_{L_2(\partial\Omega)} = (u, L(v))_{L_2(\Omega)} - \\ - (u_n, \overline{m}_n(v))_{L_2(\partial\Omega)}$$

which substituted into (7) yields

$$a(u, v) = (L(v), u)_{L_2(\Omega)} - (\overline{m}_n(v), u_n)_{L_2(\partial\Omega)} + \\ + \sum_{k=1}^{l} \int_{\gamma_k} [L_{\tau k}(v)\, L_{\tau k}(u) + L_{nk}(v)\, L_{nk}(u)]\, \mathrm{d}s\,.$$

In view of the identity (8) we obtain

$$(9) \qquad a(v, u) = (L(u), v)_{L_2(\Omega)} - (\overline{m}_n(u), v_n)_{L_2(\partial\Omega)} + \\ + \sum_{k=1}^{l} \int_{\gamma_k} [L_{\tau k}(u)\, L_{\tau k}(v) + L_{nk}(u)\, L_{nk}(v)]\, \mathrm{d}s = a(u, v)\,.$$

In this way, the symmetry of the bilinear form $a(u, v)$ in the space $\mathscr{V}(\Omega)$ is proved.

Lemma 1,2 implies that $V(\Omega)$ with a scalar product defined by

$$(10) \qquad (u, v)_{V(\Omega)} = a(u, v)$$

is a Hilbert space. Hence the inclusion $V(\Omega) \subset W_2^2(\Omega)$ is evident, the bedding being continuous.

The weak solution is introduced by the following

**Definition.** A function $w(x, y)$ will be called *a weak solution of the boundary value problem* (1), (2), ($4_1$, $4_2$, $4_3$) if
a) $w \in V(\Omega)$,
b) the integral identity (the equation of virtual work)

$$(11) \qquad a(w, \varphi) = \langle p, \varphi \rangle + \int_{\partial\Omega_2} m_2 \frac{\partial\varphi}{\partial n}\, \mathrm{d}s$$

holds for all $\varphi \in V(\Omega)$.

The integral identity (11) can be obtained formally multiplying, (1) by the test function $\varphi \in V(\Omega)$, integrating by parts and making use of the boundary conditions ($4_1$, $4_2$). Then we multiply second and the first equation from (2) by the test function $C_k\varphi_k \in V(\Omega)$ and $A_k(\partial\varphi_k/\partial n)$ respectively. After integration by parts over a closed curve $\gamma_k$ for $k = 1, 2, \ldots, l$ with regard to (i), the Clebsch's and Kirchhoff's relationsships [8], we obtain finally the integral identity (11).

**Lemma 3.** *A functional* $g(\varphi) = \langle p, \varphi \rangle + \int_{\partial\Omega_2} m_2 (\partial\varphi/\partial n)\, \mathrm{d}s$ *represents a bounded linear functional in* $V(\Omega)$.

Proof. It is obvious that the functional $g(\varphi)$ is linear in the space $V(\Omega)$. In virtue of the continuity of operators of traces in $W_2^2(\Omega)$ and of the Sobolev Theorem on imbeddings, the following estimate for $g(\varphi)$ is obtained

$$\left| \int_{\partial\Omega_2} m_2 \frac{\partial\varphi}{\partial n}\, \mathrm{d}s + \langle p, \varphi \rangle \right| \leqq \text{const} \left( \|m_2\|_{L_2(\partial\Omega)} + \|p\|_{(C_0(\Omega))^*} \right) \|\varphi\|_{V(\Omega)}$$

from which the boundedness of the functional $g(\varphi)$ in $V(\Omega)$ follows. So we have according to the Riesz-Fréchet representation theorem

$$g(\varphi) = (p^*, \varphi)_{V(\Omega)} \tag{12}$$

where $p^* \in V(\Omega)$ and $\|g\|_{V^*(\Omega)} = \|p^*\|_{V(\Omega)}$. Hence $g(\varphi)$ is a continuous functional in $V(\Omega)$ such that $\|g\|_{V^*(\Omega)} \leqq \text{const}\left( \|m_2\|_{L_2(\partial\Omega_2)} + \|p\|_{(C_0(\Omega))^*} \right)$. Nevertheless, the integral identity (11) yields the following identity in virtue of Eq. (10):

$$a(w, \varphi) = (w, \varphi)_{V(\Omega)} = g(\varphi) \quad \text{for all} \quad \varphi \in V(\Omega)\,. \tag{13}$$

Hence the formula (12) together with the equality (13) imply $w = p^*$. This means, however, that the following theorem is a consequence of Lemma 3.

**Theorem 1.** *There exists a weak solution of the boundary value problem* (1), (2), $(4_1, 4_2, 4_3)$ *in space* $V(\Omega)$.

Furthermore, it is directly evident that $a(w, w) = 0$ implies $w = 0$. Hence the uniqueness of the weak solution of the boundary value problem follows from the $V(\Omega)$-ellipticity of the bilinear form $a(w, \varphi)$ in $V(\Omega)$.

General theorems on the regularity of the weak solution [1] show that if $p \in (C_0(\bar{\Omega}))^*$, the solution $w$ of the boundary value problem (1), (2), $(4_1, 4_2, 4_3)$ is only from the space $V(\Omega)$. A smoother solution exists only if $p \in C^\infty(\Omega)$ and if the curves $\gamma_k$ are smooth.

**Classical Galerkin's method. Definition.** Let $\{\varphi_i\}_{i=1}^\infty$ be a set of base functions where $\varphi_i \in M_n(\Omega) \subset V(\Omega)$, $i \leqq n$ and $\{\varphi_i\}_1^\infty$ is a complete system in $V(\Omega)$. $M_n(\Omega)$ is a closed subspace of $V(\Omega)$. Galerkin's approximation of the exact solution $w$ in $M_n(\Omega)$ is a function $w_n \in M_n(\Omega)$ such that

$$w_n = \sum_{i=1}^{n} \alpha_{ni}\varphi_i \tag{14}$$

$$a(w_n, \varphi_i) = \langle p, \varphi_i \rangle + \int_{\partial\Omega_2} m_2 \frac{\partial\varphi_i}{\partial n}\, \mathrm{d}s\,; \quad \varphi_i \in M_n(\Omega)\,.$$

The unknown parameters $\alpha_{ni}$ are found from the integral identities (13) for $i = 1, 2, \ldots, n$.

(15) $$\sum_{i=1}^{n} \alpha_{ni} a(\varphi_i, \varphi_k) = \sum_{i=1}^{n} \alpha_{ni} (\varphi_i, \varphi_k)_{V(\Omega)} = \langle p, \varphi_k \rangle + \int_{\partial\Omega_2} m_2 \frac{\partial \varphi_k}{\partial n} \, ds \, ,$$

for $k = 1, 2, \ldots, n$.

The determinant of the system (15) is the Gram determinant which for a linearly independent system $\{\varphi_i\}_1^\infty$ is different from zero. Hence the system (15) has just one solution (for $m_2 = 0$, $p = 0$ it has the zero solution).

**Theorem 2.** *A sequence of Galerkin's approximations converges strongly to a weak solution $w$ of the boundary value problem* (1), (2), ($4_1$, $4_2$, $4_3$) *in the space $V(\Omega)$ and the estimate*

(16) $$\|w - w_n\|_{V(\Omega)} \leqq \inf_{\varphi \in M_n(\Omega)} \|w - \varphi\|_{V(\Omega)}$$

*holds.*

Proof. A sequence of functions $\{w_n\}$ in $V(\Omega)$ is uniformly bounded in virtue of the inequality

$$\sup_{\substack{\varphi \in M_n(\Omega) \\ \|\varphi\|_{M_n(\Omega)} = 1}} |\langle p^*, \varphi \rangle| \leqq \sup_{\substack{\varphi \in V(\Omega) \\ \|\varphi\|_{V(\Omega)} = 1}} |\langle p^*, \varphi \rangle| \, .$$

This, however, means that

$$\|w_n\|_{V(\Omega)} = \|p^*\|_{[M_n(\Omega)]^*} \leqq \|p^*\|_{[V(\Omega)]^*} \quad \text{for all} \quad n \, .$$

Further, let us have a sequence $\{w^n\}$ such that $w^n \in M_n(\Omega)$ and $\|w - w^n\|_{V(\Omega)} \to 0$ for $n \to \infty$. The existence of such a sequence follows from the completeness of the system $\{\varphi_i\}_1^\infty$. The subspace $M_n(\Omega)$ consists of elements $\sum_{i=1}^{n} d_i \varphi_i(x, y)$ with arbitrary coefficients, hence, with regard to Eq. (14), the identity (17) holds for each $\varphi \in M_n(\Omega)$,

(17) $$a(w_n, \varphi) = \langle p, \varphi \rangle + \int_{\partial\Omega_2} m_2 \frac{\partial \varphi}{\partial n} \, ds \, .$$

Now if we subtract the identity (17) from (11) we obtain the formula

(18) $$a(w - w_n, \varphi) = 0 \quad \text{for all} \quad \varphi \in M_n(\Omega) \, .$$

Let us put

$$\varphi = w^n - w_n \in M_n(\Omega) \, .$$

Hence we have with respect to (18)

$$(19) \qquad a(w - w_n, w - w_n) = a(w - w_n, w - w^n).$$

The uniform boundedness of the norms $\|w_n\|_{V(\Omega)}$ in the Hilbert space $V(\Omega)$ and $\overline{\bigcup_{n=1}^{\infty} M_n(\Omega)} = V(\Omega)$ implies that $\{w_n\}$ converges to $w$ weakly in $V(\Omega)$. On the other hand, $\{w^n\}$ converges to $w$ strongly in $V(\Omega)$, which shows that the right hand term in (19) for $n \to \infty$ converges to zero. The first part of the theorem is thereby proved.

The bilinear form $a(u, v)$ is $V(\Omega)$-elliptic in the space $V(\Omega)$ and thus (see (10))

$$(20) \qquad \|w - w_n\|^2_{V(\Omega)} = a(w - w_n, w - w_n) = \\ = a(w - w_n, w - \varphi) + a(w - w_n, \varphi - w_n)$$

where the function $\varphi \in M_n(\Omega)$ is arbitrary.

The second summand in (20) is zero because $\varphi - w_n \in M_n(\Omega)$ and, hence the formula (18) is valid. Thus we have

$$\|w - w_n\|^2_{V(\Omega)} \leqq \|w - w_n\|_{V(\Omega)} \|w - \varphi\|_{V(\Omega)}.$$

Dividing the inequality by $\|w - w_n\|_{V(\Omega)}$ and passing to the infimum in the right hand term we obtain the inequality (16). Further, let us associate each $h \in (0, 1\rangle$ with a finite-dimensional subspace $V_h(\Omega)$ closed in $V(\Omega)$; hence $V_h(\Omega) \subset V(\Omega)$.

Let us denote by $w_h \in V_h(\Omega)$ Galerkin's approximation $w$ in $V_h(\Omega)$, i.e. the function defined by a relation of the form (17) for each $\varphi \in V_h(\Omega)$.

The problem now arises when $\|w - w_h\|_{V(\Omega)} \to 0$ for $h \to 0$. Let us suppose that a projective operator $P_h : S(\Omega) \to V_h(\Omega)$ is defined on a subset $S(\Omega) \subset V(\Omega)$ which is dense in $V(\Omega)$. The projective operator satisfies

$$(21) \qquad \|u - P_h u\|_{V(\Omega)} \to 0, \quad h \to 0, \quad \text{for each} \quad u \in S(\Omega).$$

The answer to the above question is given by the following

**Theorem 3.** *Let a continuous bilinear form $a(u, v)$ be $V(\Omega)$-elliptic in the space $V(\Omega)$ and let in addition the condition (21) be satisfied. Then*

$$\|w - w_h\|_{V(\Omega)} \to 0; \quad h \to 0.$$

Proof. The subset $S(\Omega)$ is dense in $V(\Omega)$. Hence there exists $w_0 \in S(\Omega)$ for which

$$\|w - w_0\|_{V(\Omega)} \leqq \frac{\varepsilon}{2} \quad \text{for all} \quad \varepsilon > 0.$$

It follows also from Eq. (21) that there exists $h_0 \in (0, 1\rangle$ such that

$$\|w_0 - P_h w_0\|_{V(\Omega)} \leqq \frac{\varepsilon}{2} \quad \text{for} \quad h \leqq h_0.$$

As in the proof of Theorem 2 we conclude

$$\|w - w_h\|_{V(\Omega)} \leqq \|w - P_h w_0\|_{V(\Omega)} \leqq$$
$$\leqq [\|w - w_0\|_{V(\Omega)} + \|w_0 - P_h w_0\|_{V(\Omega)}] \leqq \varepsilon$$

which proves the theorem.

So if we do not known anything about the regularity we have

$$\|w - w_h\|_{V(\Omega)} \to 0 \quad \text{for} \quad h \to 0 . \tag{22}$$

If, however, $w \in S(\Omega)$, then $\|w - w_h\|_{V(\Omega)} \leqq \|w - P_h w\|_{V(\Omega)}$. Using the classical Galerkin's method, let $\varphi_1, \varphi_2, \ldots, \varphi_m, \ldots$ be a base in $V(\Omega)$, i.e., for every $m \geqq 1$ $w_1, w_2 \ldots, w_m$ are linearly independent elements.

Denoting $V_m(\Omega) = \{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ we have $\overline{\bigcup_{m=1}^{\infty} V_m(\Omega)} = V(\Omega)$. Note that $V_m(\Omega) \subset V_{m+1}(\Omega)$ for every $m$. In this way, we can apply Theorem 3 with $h = 1/m$, $V_h(\Omega) = V_m(\Omega)$, $S(\Omega) = V(\Omega)$ and $P_h = M_m$, where $M_m$ is the orthogonal projection of $V(\Omega)$ into $V_m(\Omega)$. This procedure is, however, depreciated by one great disadvantage. In more complicated cases it is difficult to find the corresponding base of the space $V_m(\Omega)$. In addition, if $\{\varphi_i\}_{i=1}^{\infty}$ are not orthogonal, the matrix $A^m = \{a(\varphi_i, \varphi_j)\}$ will be generally full. Hence a numerical analysis of the boundary value problem (1), (2), $(4_1, 4_2, 4_3)$ will be preferably carried out by the generalized Ritz-Galerkin's method – the method of finite elements. However, this means among other to find a space of functions $S(\Omega)$, i.e. a subspace of approximations which is dense in $V(\Omega)$. Similarly, it is necessary to prove that the corresponding finite elements (e.g. the polynomials of the fifth order) converge finitely in the norm $\|u\|_{K(\Gamma)}$ which requires to prove the convergence of their second derivatives on the curves $\gamma_k$. All these are, however, very difficult problems.

## *References*


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Súhrn

## SLABÉ RIEŠENIE OKRAJOVEJ ÚLOHY PRE ORTOTROPICKÚ DOSKU ZOSÍLENÚ TUHOSTNÝMI REBRAMI

JÁN LOVÍŠEK

V tejto práci sa metodou abstraktného variačného počtu dokazuje existencia a unicita slabého riešenia okrajovej úlohy pre ortotropickú dosku zosílenú tuhostnými rebrami. Okrajová úloha sa rieši na priestore $V(\Omega) \subset W_2^2(\Omega)$, na ktorom je odpovedajúce bilineárna forma $V(\Omega)$-eliptická. Pre numerické riešenie sa zavádza klasická Galerkinova metoda. Galerkinovské aproximácie silno konvergujú na priestore $V(\Omega)$ ku slabému riešeniu okrajovej úlohy.

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