Aplikace matematiky

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Aplikace matematiky, Vol. 20 (1975), No. 4, 280–297

Persistent URL: http://dml.cz/dmlcz/103593

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INHOMOGENEOUS BOUNDARY VALUE PROBLEMS FOR THE VON KÁRMÁN EQUATIONS. II

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(Received August 13, 1974)

INTRODUCTION

In the present part of our paper, we discuss the boundary value problems of "free" and "partially free" plates, i.e. such that the boundary conditions do not eliminate the possibility of motions of a "non-flexible" plate.

Some investigations in this direction have been accomplished by Naumann [4], who considered an elastic plate, the edge of which is completely free of forces and supports.

From the mathematical point of view, the mechanical assumptions imply that the bilinear form, associated with the plate bending energy and the energy of elastic support and clamping, is not coercive on the whole energy space but merely on its appropriate subspace, namely, the orthogonal complement of all kinematically admissible rigid deflections.

Restating the boundary value problem in terms of a system of integral identities, we replace them by an abstract operator equation in the subspace mentioned above. If the data satisfy an orthogonality condition (total equilibrium), the solution of the operator equation represents a variational solution of our boundary value problem.[1])

In Section 1 we introduce a class of boundary value problems, recalling some of the notations and assumptions of Part I [1] and defining a new subspace of kinematically admissible deflections of the "non-flexible" plate, which involves the influence of a "prestressing" by the tension forces.

Section 2 contains the variational formulation of the problem and a discussion of solvability. Here we study the configurations of boundary conditions such that the solutions form a certain class of equivalence, i.e., an element of a quotient space.

[1]) Note that the approach differs from that of [4], where the abstract operator equation has been considered in the corresponding factor-space.

The main result of the paper is given in Section 3. It presents the necessary and sufficient conditions for the existence of the class of solutions, mentioned above. In Section 4 we prove the main theorem of existence and in Section 5 we introduce a sequence of perturbed boundary value problems of the coercive type the solutions of which converge to the solution of the given problem.

1. SETTING OF THE BOUNDARY VALUE PROBLEM

We preserve all the assumptions of Part I [1] concerning the domain Ω (cf. the beginning of Section 2 there).

The equilibrium of a thin elastic plate, which is subjected both to a perpendicular load q and to forces acting along the edges, is governed by the von Kármán equations

$$\Delta^2 w = [\Phi, w] + q \tag{1.1}$$

$$\Delta^2 \Phi = -[w, w] \tag{1.2}$$

(see Part I for the notation).

Let the boundary Γ of Ω consist of three mutually disjoint parts

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 ,$$

where each of Γ_i is either empty or has a positive length and does not contain isolated points. We consider the following boundary conditions

$$\begin{gathered} w = 0 , \quad M(w) + k_1 w_n = m_1 \quad \text{on} \quad \Gamma_1 , \\ M(w) + k_{21} w_n = m_2 , \quad T(w) + k_{22} w = t_2 \quad \text{on} \quad \Gamma_2 , \\ w_n = 0 , \quad T(w) + k_3 w = t_3 \quad \text{on} \quad \Gamma_3 , \end{gathered} \tag{1.3}$$

where

$$w_n = \frac{\partial w}{\partial n} ,$$

$$M(w) = \mu \Delta w + (1 - \mu)(w_{xx} n_x^2 + 2 n_x n_y w_{xy} + n_y^2 w_{yy}) ,$$

$$T(w) = -\frac{\partial}{\partial n} \Delta w + (1 - \mu) \frac{\partial}{\partial s} [w_{xx} n_x n_y - w_{xy}(n_x^2 - n_y^2) - w_{yy} n_x n_y] + \\ + X w_x + Y w_y .$$

Throughout the whole paper we assume that

$$\begin{aligned} & k_1 \in L^p(\Gamma_1) , \quad k_1 \geqq 0 \quad \text{a.e. on} \quad \Gamma_1 , \\ & k_{21} \in L^p(\Gamma_2) , \quad k_{22} \in L^1(\Gamma_2) , \quad k_{2j} \geqq 0 \quad \text{a.e. on} \quad \Gamma_2 , \quad (j = 1, 2) , \\ & k_3 \in L^1(\Gamma_3) , \quad k_3 \geqq 0 \quad \text{a.e. on} \quad \Gamma_3 , \end{aligned} \tag{1.4}$$

$$X, Y \in L^p(\Gamma_2 \cup \Gamma_3),$$
$$m_1 \in L^p(\Gamma_1), \quad m_2 \in L^p(\Gamma_2),$$
$$t_2 \in L^1(\Gamma_2), \quad t_3 \in L^1(\Gamma_3),$$
$$q \in [C(\bar{\Omega})]',$$

where $1 < p < \infty$ (cf. also Part I).

According to (1.3), the plate is supported and elastically clamped along Γ_1 (if $k_1 > 0$) or loaded only by a moment distribution m_1 (if $k_1 = 0$). In particular, it is simply supported along Γ_1 if $k_1 = m_1 = 0$.

On Γ_2 elastic supports (if $k_{22} > 0$) and elastic clamping (if $k_{21} > 0$) or transversal load and moment distribution (if $k_{22} = k_{21} = 0$) are prescribed. On Γ_3 the edge is elastically supported (if $k_3 > 0$) or loaded only by transversal loads (if $k_3 = 0$), being prevented from rotation.

If there are corners in the interior of $\Gamma_2 \cup \Gamma_3$ with coordinates $(x(s_i), y(s_i))$, $i = 1, 2, \dots, r$, then (1.3) will be completed by the conditions

$$H(w(s_i^+), n(s_i^+)) - H(w(s_i^-), n(s_i^-)) = h_i, \quad i = 1, 2, \dots, r,$$

where h_i are given constants and the operator H is defined in Section 2 of Part I.

As for the boundary conditions upon Φ, we adopt (2.5) as well as (2.5′) from Part I (replacing only Γ_3 by $\Gamma_2 \cup \Gamma_3$), i.e.,

$$\Phi = \varphi_0, \quad \Phi_n = \varphi_1 \quad \text{on} \quad \Gamma, \tag{1.5}$$

$$\Phi_{yy}n_x - \Phi_{xy}n_y = X, \quad \Phi_{xx}n_y - \Phi_{xy}n_x = Y \quad \text{on} \quad \Gamma_2 \cup \Gamma_3. \tag{1.5'}$$ [1])

Assuming that φ_0, φ_1 satisfy the conditions (4.1) of Part I, there exists a unique function $F \subset W^{2,2}(\Omega)$ such that

$$F = \varphi_0, \quad F_n = \varphi_1 \quad \text{on } \Gamma,$$
$$(F, \psi)_{W_0^{2,2}} = 0 \quad \forall \psi \in W_0^{2,2}(\Omega)$$

(cf. Proposition 4.1 [1]). Moreover, it holds

$$\|F\|_{W^{2,2}} \leqq \text{const. } T_1(\varphi_0, \varphi_1).$$ [2])

F represents the Airy stress function of the associated linear plane stress problem.

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[1]) Note that (1.5) represents only another form of (1.5′) on $\Gamma_2 \cup \Gamma_3$ (cf. Part. I p. 257).

[2]) $T_1(\varphi_0, \varphi_1) = \sum_{j=1}^{l} [\|\varphi_0\|_{W^{3/2,2}(S_j)} + \|\varphi_1\|_{W^{1/2,2}(S_j)}] + \|\varphi_{01}\|_{W^{1/2,2}(\Gamma)} + \|\varphi_{10}\|_{W^{1/2,2}(\Gamma)}$

In accordance with the boundary conditions (1.3) we set

$$\mathscr{V} = \{u \in C^\infty(\bar{\Omega}) : u = 0 \text{ on } \Gamma_1,\ u_n = 0 \text{ on } \Gamma_3\}$$

and define

$$V = \text{closure of } \mathscr{V} \text{ in } W^{2,2}(\Omega)\,.$$

The functions of V may be interpreted as the kinematically admissible (virtual) deflections of the plate.

For $u, v \in W^{2,2}(\Omega)$ we introduce the bilinear forms

$$A(u, v) = \int_\Omega [u_{xx}v_{xx} + 2(1 - \mu)\, u_{xy}v_{xy} + u_{yy}v_{yy} + $$
$$+ \mu(u_{xx}v_{yy} + u_{yy}v_{xx})]\, dx\, dy\,,$$

$$a(u, v) = \int_{\Gamma_1} k_1 u_n v_n\, ds + \int_{\Gamma_2} (k_{21} u_n v_n + k_{22} uv)\, ds + \int_{\Gamma_3} k_3 uv\, ds\,,$$

and for φ, u, v $\in W^{2,2}(\Omega)$ the trilinear form

$$B(\varphi; u, v) = \int_\Omega [\varphi_{xy}(u_y v_x + u_x v_y) - \varphi_{yy} u_x v_x - \varphi_{xx} u_y v_y]\, dx\, dy$$

(cf. Part I). Note that B is symmetric with respect to u, v.

As in Part I we may consider the following condition upon the function F:

$$(+) \qquad B(F; u, u) \leqq 0 \quad \forall u \in V\,.$$

If u is the deflection of the plate, the term $-B(F; u, u)$ can be interpreted as a part of the bending energy, originating from the "prestressing" by tension forces, acting in the plane of the plate.

Let us define

$$P = \{u \in V : A(u, u) + a(u, u) = 0\}\,.$$

Clearly, $P \subseteq P_1$ (where P_1 denotes the space of linear polynomials); P is the set of all kinematically admissible deflections of a plate, the bending energy of which vanishes as well as the energy stored in the elastic supports.

If the condition $(+)$ is satisfied, we introduce the following subspace of P:

$$P_F = \{u \in V : A(u, u) - B(F; u, u) + a(u, u) = 0\} =$$
$$= \{u \in P : B(F; u, u) = 0\}\,.$$

In what follows we assume that precisely one of the following two cases takes place:

$$(1.6) \qquad \text{Condition } (+) \text{ is satisfied and } \quad P_F \neq \{\Theta\}\,;$$

(1.7) Condition $(+)$ does not hold and $P \neq \{\Theta\}$.

P_F is the set of all kinematically admissible deflections of a plate, the total bending energy of which (including the influence of "prestressing" by F) vanishes as well as the energy of elastic supports and clamping.

The assumptions (1.6), (1.7) are motivated by the fact that, in the contrary to Part I, here we intend to study boundary value problems for the system (1.1), (1.2) such that if $\{w, \Phi\}$ is a solution, then $\{w + p, \Phi\}$, where p is any element of P_F or P, respectively, is also a solution. Thus the solutions w will form an element of the quotient space V/P_F or V/P, respectively.

In order to satisfy (1.6) or (1.7) we have to negate all the conditions 1° to 5° of Section 2, Part I (where $\Gamma_1 = \emptyset$, Γ_2 is replaced by Γ_1 and Γ_3 by Γ_2), because each of those conditions yields $P = \{\Theta\}$. Thus we have to assume $\Gamma_2 \cup \Gamma_3 \neq \emptyset$.

The mechanical meaning of (1.6), (1.7) can be explained as follows: the kinematic boundary conditions (involved in the definition of V) restrict the total freedom of the "non-flexible" plate, lying on the rigid supports and with rigid clamping, to deflections contained in P_F or P, respectively. In other words, there exist non-zero virtual deflections of the "non-flexible" plate on the rigid supports and clamping.

Let us present some examples of the problems under consideration.

Example 1. Let $\Gamma = \Gamma_2$, $k_{21} = k_{22} = 0$. Then

$$a(u, u) = 0\,, \quad V = W^{2,2}(\Omega)\,, \quad P = P_1\,.$$

Moreover, let

$$F = F_1(x) + F_2(y)\,,$$

where F_1, F_2 are polynomials of the degree at most 3, such that $F_{1xx} \geqq 0$, $F_{2yy} \geqq 0$ on Ω and at least one of the integrals

$$\int_\Omega F_{1xx}\,dx\,dy\,, \quad \int_\Omega F_{2yy}\,dx\,dy$$

is positive. Then $(+)$ holds and

$$1 \leqq \dim P_F < \dim P_1 = 3\,.$$

In fact, we may write

$$p = ax + by + c \quad \forall p \in P_1$$

and the condition

$$-B(F, p, p) = a^2 \int_\Omega F_{2yy}\,dx\,dy + b^2 \int_\Omega F_{1xx}\,dx\,dy = 0$$

yields that at least one of a, b must vanish. Thus we have the case of (1.6).

Example 2. Let $\Gamma_1 = \emptyset$, $k_3 = k_{22} = 0$ and the set

$$\Gamma_3 \cup \text{supp } k_{21}$$

belong to straight lines parallel to x-axis (being non-empty). Then we have

$$P = \{p \in P_1 : p = ax + c,\ a, c \in R^1\} .$$

Suppose that F is of the same type as in Example 1 and

$$\int_\Omega F_{2yy} \, dx \, dy > 0 .$$

We obtain

$$P_F = \{p \in P_1 : p = c, c \in R^1\} .$$

Example 3. Let Γ_1 be a segment on the y-axis, $\Gamma_3 = \emptyset$,

$$k_1 = k_{21} = 0$$

and let supp k_{22} belong to the y-axis.
Then we have

$$V = \{u \in \dot{W}^{2,2}(\Omega),\ u = 0 \text{ on } \Gamma_1\} ,$$
$$a(u, u) = 0 \Rightarrow u = 0 \quad \text{on} \quad \text{supp } k_{22} .$$

Let F do not satisfy $(+)$. Then

$$P = \{p : p = ax,\ a \in R^1\} ,$$

i.e., (1.7) is satisfied.

Example 4. Consider the problem of Example 3 with the only change that $F = \frac{1}{2}\sigma x^2$, $\sigma = \text{const.} > 0$. Then $(+)$ holds, and

$$P_F = P \neq \{\Theta\} ;$$

consequently, we have the case of (1.6).

Example 5. Consider again the configuration of Example 3, but with $F = \frac{1}{2}\sigma y^2$, $\sigma = \text{const.} > 0$.

Then $(+)$ holds, whereas $P_F = \{\Theta\}$.
Consequently neither (1.6) nor (1.7) takes place. This example could be rather joined to the class of boundary value problems of Part I (cf. Remark 4.1 below).

The system (1.1), (1.2) with the boundary conditions (1.3), (1.4), (1.5), (1.5′) under suppositions (1.6) or (1.7), respectively, will be referred to as the boundary-value problem II.

To give a variational formulation of boundary value problem II and to apply abstract methods in the proof. of the solvability, it is convenient to introduce two other scalar products on V.

A. Let (1.6) hold. Consider a system $\{f_{1i}\}$ $(i = 1, \ldots, K_1)$ of linear continuous functionals on V such that

$$\text{(2.1)} \quad u \in V, \quad A(u, u) - B(F, u, u) + a(u, u) + \sum_{i=1}^{K_1} (f_{1i}(u))^2 = 0 \Rightarrow u = \Theta,$$
$$\sum_{i=1}^{K_1} a_i f_{1i}(p) = 0 \quad \forall p \in P_F \Rightarrow \sum_{i=1}^{K_1} a_i^2 = 0.$$

Using the method of proof of Theorem 2.3 in [2] and observing the estimate

$$|B(F, u, v)| \leqq \text{const} \|F\|_{W^{2,2}} \|u\|_{W^{2,2}} \|v\|_{W^{2,2}}$$

which holds for all $u, v \in W^{2,2}$, we obtain the existence of two positive constants c_1, c_2 such that

$$\text{(2.2)} \quad c_1 \|u\|^2_{W^{2,2}} \leqq A(u, u) - B(F, u, u) + a(u, u) +$$
$$+ \sum_{i=1}^{K_1} (f_{1i}(u))^2 \leqq c_2 \|u\|^2_{W^{2,2}} \quad \forall u \in V.$$

Thus, if (1.6) holds and if the functionals $\{f_{1i}\}$ satisfy (2.1), the scalar product

$$(u, v)_F = A(u, v) - B(F, u, v) + a(u, v) + \sum_{i=1}^{K_1} f_{1i}(u) f_{1i}(v)$$

turns V into a Hilbert space.

B. Let the weaker condition (1.7) be satisfied. Consider a system $\{f_{2i}\}$ $(i = 1, \ldots$ $\ldots, K_2)$ of linear continuous functionals on V such that

$$\text{(2.3)} \quad u \in V, \quad A(u, u) + a(u, u) + \sum_{i=1}^{K_2} (f_{2i}(u))^2 = 0 \Rightarrow u = \Theta,$$
$$\sum_{i=1}^{K_2} a_i f_{2i}(p) = 0 \quad \forall p \in P \Rightarrow \sum_{i=1}^{K_2} a_i^2 = 0.$$

The same argument as above yields the existence of two positive constants $\bar{c}_1, \bar{c}_2$ such that

$$\text{(2.4)} \quad \bar{c}_1 \|u\|^2_{W^{2,2}} \leqq A(u, u) + a(u, u) + \sum_{i=1}^{K_2} (f_{2i}(u))^2$$
$$\leqq \bar{c}_2 \|u\|^2_{W^{2,2}} \quad \forall u \in V.$$

Hence, in the present case, V becomes a Hilbert space with respect to the scalar product

$$(u, v) = A(u, v) + a(u, v) + \sum_{i=1}^{K_2} f_{2i}(u) f_{2i}(v) .$$

Remark 2.1. The systems $\{f_{1i}\}$ and $\{f_{2i}\}$, respectively, may be chosen in many ways. Thus if P_F (or P) equals P_1, we can set [1])

$$f_i(u) = u(x_i, y_i), \quad (i = 1, 2, 3),$$

where the points $(x_i, y_i) \in \bar{\Omega}$ do not belong to one straight line.

If P_F(or P) $= \{p \in P_1 : p = ax + c\}$, we can take

$$f_i(u) = u(x_i, y_i), \quad (i = 1, 2),$$

where $(x_i, y_i) \in \bar{\Omega}$, $x_1 \neq x_2$.

If P_F(or P) $= \{p \in P_1 : p = ax\}$, we can set

$$f_1(u) = u(x_1, y_1), \quad (x_1, y_1) \in \bar{\Omega}, \quad x_1 \neq 0$$

and for $P_F(P) = R^1$, we take $f_1(u) = u(x_1, y_1)$, $(x_1, y_1) \in \bar{\Omega}$.
We introduce the orthogonal decompositions

$$V = R_F \oplus P_F \quad \text{with respect to} \quad (\cdot, \cdot)_F ,$$
$$V = R \oplus P \quad \text{with respect to} \quad (\cdot, \cdot) .$$

As in [2] one concludes that

$$R_F = \{u \in V : \sum_{i=1}^{K_1} (f_{1i}(u))^2 = 0\} ,$$
$$R = \{u \in V : \sum_{i=1}^{K_2} (f_{2i}(u))^2 = 0\} .$$

Observing (2.2) and (2.4), respectively, we infer: R_F and R is a Hilbert space with respect to the scalar product

$$((u, v))_F = A(u, v) - B(F, u, v) + a(u, v)$$

and

$$((u, v)) = A(u, v) + a(u, v) ,$$

respectively.

The associated norm will be denoted by $|||\cdot|||_F$ and $|||\cdot|||$, respectively.

---

[1]) Note that, by Sobolev's Imbedding Theorem, $u(x, y)$ makes sense pointwise for each $u \in W^{2,2}(\Omega)$.

Remark 2.2. Let V/P_F denote the vector space of all classes $\tilde{u}$ such that $u, v \in \tilde{u}$ if and only if $u - v \in P_F$. V/P_F will be provided with the norm

$$\|\tilde{u}\|_{V/P_F} = \inf_{u \in \tilde{u}} \|u\|_F$$

where $\|u\|_F = (u, u)_F^{1/2}$. If $\hat{u}$ denotes the (uniquely determined) representative of $\tilde{u}$ which belongs to R_F, we have $\|\tilde{u}\|_{V/P_F} = \|\hat{u}\|_F = |||u|||_F$ ($u \in \tilde{u}$ arbitrary). Consequently, V/P_F is a Hilbert space with respect to the scalar product

$$(\tilde{u}, \tilde{v})_{V/P_F} = (\hat{u}, \hat{v})_F = ((u, v))_F \quad (u \in \tilde{u},\ v \in \tilde{v} \text{ arbitrary}),$$

i.e., R_F and V/P_F are isomorphic Hilbert spaces.

An analogous remark applies to the space V/P.

Definition 2.1. *The pair* $\{w, f\} \in V \times W_0^{2,2}$ *is called an excess variational solution of boundary value problem* II *if*

1° *the identity*

$$A(w, \varphi) + a(w, \varphi) = B(F, w, \varphi) + B(f, w, \varphi) + L(\varphi) \tag{2.5}$$

holds for all $\varphi \in V$, where

$$L(\varphi) = \int_{\Gamma_1} m_1 \varphi_n \, ds + \int_{\Gamma_2} (m_2 \varphi_n + t_2 \varphi) \, ds + \int_{\Gamma_3} t_3 \varphi \, ds + \\ + \sum_{i=1}^{r} h_i \, \varphi(x(s_i), y(s_i)) + \langle q, \varphi \rangle ;$$

2° *the identity*

$$(f, \psi)_{W_0^{2,2}} = -B(w, w, \psi) \tag{2.6}$$

holds for all $\psi \in W_0^{2,2}(\Omega)$.

Let $\{w, f\}$ be an excess variational solution of boundary value problem II. Inserting $\varphi = p \in P$ in (2.5) and observing that (cf. Part I, Section 5)

$$B(\psi, u, p) = B(p, u, \psi) = 0 \quad \forall \psi \in W_0^{2,2}(\Omega), \quad \forall u \in V, \quad \forall p \in P_1, \tag{2.7}$$

we obtain the necessary solvability condition

$$B(F, w, p) + L(p) = 0 \quad \forall p \in P. \tag{2.8}$$

A. Let (1.6) be satisfied. Then we have, using Schwarz's inequality,

$$B(F, u, p) = 0 \quad \forall u \in V, \quad \forall p \in P_F.$$

Consequently, (2.8) implies

$$L(p) = 0 \quad \forall p \in P_F. \tag{2.9}$$

Further, observing that $A(u, p) + a(u, p) = 0$ for any $u \in V$ and any $p \in P_F$ (in fact for any $p \in P$) and using (2.7) we obtain: *If* (1.6) *is satisfied and if* $\{w, f\}$ *is an excess variational solution of boundary value problem* II *then each pair* $\{w + p, f\}$, *where* $p \in P_F$, *is also an excess variational solution* (i.e., excess variational solutions may be identified with elements of V/P_F).

Condition (2.9) expresses a kind of equilibrium of external forces and moments.

B. Next let (1.7) be satisfied. We now *suppose* that besides $\{w, f\}$ each pair $\{w + p, f\}$, where $p \in P$, is also an excess variational solution of boundary value problem II. Then it holds

$$B(F, u, p) = 0 \quad \forall u \in V, \quad p \in P, \tag{2.10}$$

and (2.8) reduces to

$$L(p) = 0 \quad \forall p \in P. \tag{2.9'}$$

To clarify the nature of the condition (2.10), let us consider Example 3, with

$$F = F(x), \quad F_{xx} = C_1 x + C_2 < 0 \quad \text{on} \quad \Omega.$$

Then (+) does not hold, $P = \{p : p = ax, a \in \mathbf{R}^1\}$ and (2.10) holds, because

$$B(F, u, p) = -\int_\Omega F_{xx} u_y p_y \, dx \, dy = 0 \quad \forall u \in V, \quad p \in P.$$

Considering the same example with $F = -\tau xy$, $\tau = \text{const.}$, then (+) again fails to hold and

$$B(F, u, p) = -a\tau \int_\Omega u_y \, dx \, dy \quad \forall u \in V, \quad p \in P.$$

Consequently, (2.10) is not satisfied.

3. THE MAIN RESULT

Theorem 1. *Suppose the data* $q, m_1, m_2, t_2, t_3, X, Y$ *satisfy the conditions* (1.4) *and* φ_0, φ_1 *the conditions* (4.1) *of Part I, respectively.*

(i) *Let* (1.6) *hold. Then there exists at least one pair* $\{u_1, f_1\} \in R_F \times W_0^{2,2}(\Omega)$ *such that*

$$A(u_1, \varphi) + a(u_1. \varphi) = B(F, u_1, \varphi) + B(f_1, u_1, \varphi) + L(\varphi) \tag{3.1}$$

holds for all $\varphi \in R_F$,

$$(f_1, \psi)_{W_0^{2,2}(\Omega)} = -B(u_1, u_1, \psi) \tag{3.2}$$

holds for all $\psi \in W_0^{2,2}(\Omega)$.

(ii) *Let* (1.7) *hold. For sufficiently small* $T_1(\varphi_0, \varphi_1)$, *there exists at least one pair* $\{u_2, f_2\} \in R \times W_0^{2,2}(\Omega)$ *such that*

$$A(u_2, \varphi) + a(u_2, \varphi) = B(F, u_2, \varphi) + B(f_2, u_2, \varphi) + L(\varphi) \tag{3.3}$$

holds for all $\varphi \in R$,

$$(f_2, \psi)_{W_0^{2,2}(\Omega)} = -B(u_2, u_2, \psi) \tag{3.4}$$

holds for all $\psi \in W_0^{2,2}(\Omega)$.

A. Let the assumptions of Theorem 1, (i) be fulfilled, and let $\{u_1, f_1\} \in R_F \times$ $\times$ $W_0^{2,2}(\Omega)$ be a pair satisfying (3.1), (3.2).

Let (2.9) hold. Observing that each $\varphi \in V$ has the (unique) decomposition $\varphi =$ $= \hat{\varphi} + p$ where $\hat{\varphi} \in R_F$, $p \in P_F$, it is readily seen that the pair $\{u_1, f_1\}$ is an excess variational solution of boundary value problem II. But then each pair $\{u_1 + p, f_1\}$, $(p \in P_F)$ possesses the same property (cf. Section 2).

Thus, *if the assumptions of Theorem* 1, (i) *are satisfied*, (2.9) *is necessary and sufficient for the existence of a class* $\{u_1 + p, f_1\}$ $(p \in P_F)$ *of excess variational solutions of boundary value problem* II.

B. Let the assumptions of Theorem 1, (ii) be fulfilled, and let $\{u_2, f_2\} \in R \times$ $\times$ $W_0^{2,2}(\Omega)$ be a pair satisfying (3.3), (3.4). It is then easy to see that the condition

$$B(F, u_2, p) + L(p) = 0 \quad \forall p \in P$$

is sufficient for $\{u_2, f_2\}$ to be an excess variational solution of boundary value problem II. If the sharper conditions (2.10), (2.9′) are satisfied, then each pair $\{u_2 + p, f_2\}$ . . $(p \in P)$ is also an excess variational solution.

Observing the corresponding remarks in Sections 2, we may conclude: *if the assumptions of Theorem* 1, (ii) *are satisfied, then* (2.10), (2.9′) *are necessary and sufficient for the existence of a class* $\{u_2 + p, f_2\}$ $(p \in P)$ *of excess variational solutions of boundary value problem* II.

4. PROOF OF THEOREM 1

Proof of (i). Following the method of proof of our main theorem in [1], we introduce an abstract operator formulation of boundary value problem II. Then the solution of the operator equation leads to a solution of the identities (3.1) and 3.2).

Let $\psi \in W_0^{2,2}(\Omega)$, $u \in R_F$ be arbitrary. Using Sobolev's Imbedding Theorem, Hölder's inequality and the first inequality in (2.2) one obtains, for any $\varphi \in R_F$,

$$B(\psi, u, \varphi) \leqq (\text{const } \|\psi\|_{W^{2,2}(\Omega)} \|u\|_{W^{1,4}(\Omega)}) \, |||\varphi|||_F \tag{4.1}$$

(cf. (5.3) of Part I). By the Riesz representation theorem for linear functionals, there exists a (uniquely determined) element $C_1(\psi, u) \in R_F$ such that

$$(4.2) \qquad ((C_1(\psi, u), \varphi))_F = B(\psi, u, \varphi) \quad \forall \varphi \in R_F .$$

By virtue of the Trace Theorem, the Sobolev Imbedding Theorem and the first inequality (2.2) we get the estimate

$$(4.3) \qquad |L(\varphi)| \leqq \text{const}\, K(m_1; m_2; t_2; t_3; h_1, \ldots, h_r; q)\, |||\varphi|||_F ,$$

which holds for all $\varphi \in R_F$, where

$$K(m_1; m_2; t_2; t_3; h_1, \ldots, h_r; q) =$$
$$= \|m_1\|_{L^p(\Gamma_1)} + \|m_2\|_{L^p(\Gamma_2)} + \|t_2\|_{L^1(\Gamma_2)} + \|t_3\|_{L^1(\Gamma_3)} +$$
$$+ \sum_{i=1}^{r} |h_i| + \|q\|_{[C(\bar{\Omega})]'}$$

(cf. also (5.7) of Part I). Hence there exists a (unique) element $q_1 \in R_F$ such that

$$(4.4) \qquad ((q_1, \varphi))_F = L(\varphi) \quad \forall \varphi \in R_F .$$

Further, let $u, v \in R_F$ be arbitrary. Arguing as above, we obtain the estimate

$$(4.5) \qquad |B(u, v, \psi)| \leqq (\text{const}\, \|u\|_{W^{2,2}(\Omega)}\, \|v\|_{W^{1,4}(\Omega)})\, \|\psi\|_{W_0^{2,2}(\Omega)}$$

for all $\psi \in W_0^{2,2}(\Omega)$. Then there exists a (uniquely determined) element $C_2(u, v) \in W_0^{2,2}(\Omega)$ such that

$$(4.6) \qquad (C_2(u, v), \psi)_{W_0^{2,2}(\Omega)} = -B(u, v, \psi) \quad \forall \psi \in W_0^{2,2}(\Omega) .$$

We define the mapping $C : R_F \to R_F$ as follows

$$C(u) = -C_1(C_2(u, u), u)$$

and consider the equation

$$(*) \qquad u + C(u) = q_1 \quad \text{on} \quad R_F .$$

Let $u_1 \in R_F$ be a solution of (*). Putting $f_1 = C_2(u_1, u_1)$ we have $\{u_1, f_1\} \in R_F \times W_0^{2,2}(\Omega)$; the equation (*) is equivalent with the following system:

$$((u_1, \varphi))_F = ((C_1(f_1, u_1), \varphi))_F + ((q_1, \varphi))_F \quad \forall \varphi \in R_F ,$$
$$(f_1, \psi)_{W_0^{2,2}(\Omega)} = -B(u_1, u_1, \psi) \quad \forall \psi \in W_0^{2,2}(\Omega) .$$

By (4.2) and (4.4), the last system is identical with (3.1), (3.2). The converse is also true: If the pair $\{u_1, f_1\} \in R_F \times W_0^{2,2}(\Omega)$ satisfies (3.1), (3.2) then u_1 is a solution of (*).

We shall deduce the existence of a solution of (*) from the following abstract result (cf. [3, Chapt. 2, Theorem 2.7]):

Let X be a reflexive Banach space with the dual X^*, the dual pairing between X and X^* being denoted by $(\cdot, \cdot)$,

Let $A : X \to X^*$ be a mapping such that

(i) *A maps bounded sets into bounded sets,*

(ii) *for any sequence $\{u_j\} \subset X$ such that $u_j \to u$ weakly in X and* $\limsup (Au_j, u_j - u) \leqq 0$, *it holds*

$$(Au, u - v) \leqq \liminf (Au_j, u_j - v) \quad \forall v \in X\,,$$

(iii) $\dfrac{(Av, v)}{\|v\|} \to \infty \quad \text{as} \quad \|v\| \to \infty\,.$

Then for each $f \in X^*$ the equation $Au = f$ possesses at least one solution.

This theorem may be applied to the solution of (*), if the following properties of C are verified:

1° $((C(u), u))_F \geqq 0 \quad \forall u \in R_F\,.$

2° If $\{u_j) \subset R_F$ is a sequence such that $u_j \to u$ weakly in R_F as $j \to \infty$, there exists a subsequence $\{u_{jn}\} \subset \{u_j\}$ such that

$$C(u_{j_n}) \to C(u) \quad \text{strongly in} \quad R_F \quad \text{as} \quad n \to \infty\,.$$

Proof of 1°. Observing the defining relations (4.2) and (4.6) we easily get, for any $u \in R_F$,

$$\begin{aligned} ((C(u), u))_F &= ((-C_1(C_2(u, u), u), u))_F = -B(C_2(u, u), u, u) = \\ &= -B(u, u, C_2(u, u)) = \|C_2(u, u)\|^2_{W_0^{2,2}(\Omega)} \geqq 0\,. \end{aligned}$$

Proof of 2°. Let $\{u_j\} \subset R_F$ be any sequence such that $u_j \to u$ weakly in R_F as $j \to \infty$. By (4.1), (4.5)

$$\begin{aligned} |||C(u_j) - C(u)|||_F &\leqq |||C_1(C_2(u_j, u_j), u_j - u)|||_F + \\ &+ |||C_1(C_2(u, u) - C_2(u_j, u_j), u)|||_F \leqq \text{const}\, \|u_j - u\|_{W^{1,4}(\Omega)} + \\ &+ \text{const}\, \|C_2(u_j, u_j) - C_2(u, u)\|_{W^{2,2}(\Omega)}\,. \end{aligned}$$

Observing that

$$B(u, v, \psi) = B(v, u, \psi) \quad \forall u, v \in W^{2,2}(\Omega)\,, \quad \forall \psi \in W_0^{2,2}(\Omega)\,,$$

we infer from (4.5) the estimate

$$\|C_2(u_j, u_j) - C_2(u, u)\|_{W^{2,2}(\Omega)} = \text{const}\, \|u_j - u\|_{W^{1,4}(\Omega)}\,.$$

Since the imbedding $W^{2,2}(\Omega) \subset W^{1,4}(\Omega)$ is compact, the assertion is readily obtained.

The proof of part (i) is complete.

Remark 4.1. In Example 5 we arrive at a mechanical "paradox", because the plate behaves as a "fixed" one. An arbitrarily small tension $\sigma_x = F_{yy} = \sigma$ can guarantee the (moment) equilibrium with arbitrary given moments m_1, m_2 and loads t_2, h_i, q. In fact, applying the same approach of proof as previously, (with $P_F = \{\Theta\}$, $R_F = V$) we obtain the existence of an excess variational solution even without any equilibrium condition of the type (2.9).

Proof of (ii). Since we use again the above method of proof, we restrict ourselves to the main points.

Let $u \in R$ be arbitrary. Applying the first inequality in (2.4) we obtain, for any $\varphi \in R$,

$$(4.7) \qquad |B(F, u, \varphi)| \leqq (\text{const } \|F\|_{W^{2,2}(\Omega)} \|u\|_{W^{1,4}(\Omega)}) |||\varphi||| \leqq \leqq (\text{const } T_1(\varphi_0, \varphi_1) \|u\|_{W^{1,4}(\Omega)}) |||\varphi||| .$$

Thus there exists a (uniquely determined) element $Lu \in R$ such that

$$(4.8) \qquad ((Lu, \varphi)) = B(F, u, \varphi) \quad \forall \varphi \in R .$$

The estimate (4.7) implies

$$(4.9) \qquad |||Lu||| \leqq \alpha_0 T_1(\varphi_0, \varphi_1) |||u||| \quad \forall u \in R ,$$

where $\alpha_0 = \text{const} > 0$.

It is easy to see that the estimates (4.1) and 4.3), with only different constants, hold for all $\varphi \in R(|||\varphi|||_F$ replaced by $|||\varphi|||)$. Consequently, for arbitrary $\psi \in W_0^{2,2}(\Omega)$, $u \in R$ there exists a (unique) $\bar{C}_1(\psi, u) \in R$ such that

$$((\bar{C}_1(\psi, u), \varphi)) = B(\psi, u, \varphi) \quad \forall \varphi \in R ,$$

and there exists a (unique) $q_2 \in R$ such that $((q_2, \varphi)) = L(\varphi)$ holds for all $\varphi \in R$.

Since an estimate of the type (4.5) is true for arbitrary $u, v \in R$ we get the existence (and uniqueness) of an element $\bar{C}_2(u, v) \in W_0^{2,2}(\Omega)$ such that

$$(\bar{C}_2(u, v), \psi)_{W_0^{2,2}(\Omega)} = -B(u, v, \psi) \quad \forall \psi \in W_0^{2,2}(\Omega) .$$

Defining the mapping $\bar{C} : R \to R$ by means of

$$\bar{C}(u) = -\bar{C}_1(\bar{C}_2(u, u), u) ,$$

we consider the equation

$$(**) \qquad u - Lu + \bar{C}(u) = q_2 \quad \text{on} \quad R .$$

It is easy to verify that if u_2 is a solution of (**), the pair $\{u_2, f_2\}$, in which $f_2 = \bar{C}_2(u_2, u_2)$, satisfies (3.3), (3.4). Conversely, if $\{u_2, f_2\} \in R \times W_0^{2,2}(\Omega)$ satisfies (3.3), (3.4) the element u_2 is a solution of (**).

Next, the inequality $((\bar{C}(u), u)) \geqq 0$ holds for all $u \in R$. Choosing $T_1(\varphi_0, \varphi_1)$ such that

$$\alpha_0 T_1(\varphi_0, \varphi_1) < 1$$

(cf. (4.9)) we obtain

$$((u - Lu + \bar{C}(u), u)) \,.\, |||u|||^{-1} \to +\infty \quad \text{as} \quad |||u||| \to \infty \,.$$

Since $\bar{C}$ possesses the compactness property 2° (cf. p. 292) the existence of a solution of (**) follows by means of the general existence theorem for semi-monotone operators (cf. [1], [3]).

5. A LIMIT OF PERTURBED PROBLEMS OF COERCIVE TYPE

In this section we show that a variational solution of boundary value problem II may be obtained as a limit element of a sequence of variational solutions of boundary value problems of the coercive type, which were studied in Part I.

Let us replace the boundary conditions on Γ_2 and Γ_3 by

$$M(w) + k_{21} w_n = m_2 \,, \quad T(w) + (k_{22} + \varepsilon)\, w = t_2 \quad \text{on} \quad \Gamma_2 \,,$$
$$w_n = 0 \,, \quad T(w) + (k_3 + \varepsilon)\, w = t_3 \quad \text{on} \quad \Gamma_3$$

(cf. (1.3)), where $\varepsilon = \text{const} > 0$ is arbitrary, and note that the following coerciveness holds:

$$v \in V \,, \quad A(v, v) + a(v, v) + \varepsilon \int_{\Gamma_2 \cup \Gamma_3} v^2 \, ds = 0 \Rightarrow v = 0 \,. \tag{5.1}$$

The new boundary conditions of the "perturbed" problem express that artificial elastic supports with a constant modulus ε along $\Gamma_2 \cup \Gamma_3$ are added to the original ones (if any). (Recall that $\Gamma_2 \cup \Gamma_3$ has a positive length due to (1,6), (1,7)).

Theorem 2. *Suppose the data $q, m_1, m_2, t_2, t_3, X, Y$ satisfy the conditions* (1.4) *and φ_0, φ_1 the conditions* (4.1) *of Part* I, *respectively.*

(i) Let $\varepsilon > 0$ be arbitrary, but fixed. Let Condition (+) be satisfied, or let $T_1(\varphi_0, \varphi_1)$ be sufficiently small.

Then there exists at least one pair $\{u_\varepsilon, f_\varepsilon\} \in V \times W_0^{2,2}(\Omega)$ such that

$$A(u_\varepsilon, \varphi) + a(u_\varepsilon, \varphi) + \varepsilon \int_{\Gamma_2 \cup \Gamma_3} u_\varepsilon \varphi \, ds = \tag{5.2}$$
$$= B(F, u_\varepsilon, \varphi) + B(f_\varepsilon, u_\varepsilon, \varphi) + L(\varphi)$$

holds for all $\varphi \in V$,

(5.3) $$(f_\varepsilon, \psi)_{W_0^{2,2}} = -B(u_\varepsilon, u_\varepsilon, \psi)$$

holds for all $\psi \in W_0^{2,2}(\Omega)$.

(ii) Consider a sequence $\varepsilon \to 0$. Let (1.6) hold. Moreover, let (2.9) be fulfilled. Let $u_\varepsilon = \hat{u}_\varepsilon + p_\varepsilon$, with $\hat{u}_\varepsilon \in R_F$, $p_\varepsilon \in P_F$, be the unique (orthogonal) decomposition of u_ε. Then there exists a subsequence $\{u_\eta, f_\eta\} \subset \{u_\varepsilon, f_\varepsilon\}$ such that

$$\hat{u}_\eta \to u^{(1)} \quad \text{weakly in} \quad W^{2,2}(\Omega)\,,$$
$$f_\eta \to f^{(1)} \quad \text{strongly in } W_0^{2,2}(\Omega)\,, \quad \text{as} \quad \eta \to 0\,,$$

where the pair $\{u^{(1)}, f^{(1)}\}$ is an excess variational solution of boundary value problem II.

(iii) Let (1.7) hold and $T_1(\varphi_0, \varphi_1)$ be sufficiently small. Suppose that (2.10) and (2.9′) are fulfilled. Let $u_\varepsilon = \bar{u}_\varepsilon + p_\varepsilon$, with $\bar{u}_\varepsilon \in R$, $p_\varepsilon \in P$, be the unique (orthogonal) decomposition of u_ε.

Then there exists a subsequence $\{u_\tau, f_\tau\} \subset \{u_\varepsilon, f_\varepsilon\}$ such that

$$\bar{u}_\tau \to u^{(2)} \quad \text{weakly in} \quad W^{2,2}(\Omega)\,,$$
$$f_\tau \to f^{(2)} \quad \text{strongly in} \quad W_0^{2,2}(\Omega) \quad \text{as} \quad \tau \to 0\,,$$

where the pair $\{u^{(2)}, f^{(2)}\}$ is an excess variational solution of boundary value problem II.

Proof of Theorem 2. Proof of (i).

Using the approach of the proof of the Theorem in Part I, one obtains for each (fixed) $\varepsilon > 0$ the existence of at least one pair $\{u_\varepsilon, f_\varepsilon\} \in V \times W_0^{2,2}(\Omega)$ satisfying (5.2) for all $\varphi \in V$, and (5.3) for all $\psi \in W_0^{2,2}(\Omega)$.

Proof of (ii). 1) *A-priori estimates.* Set $\varphi = u_\varepsilon$ in (5.2). Observing (2.2) and (2.9) one gets

$$c_1 \|\hat{u}_\varepsilon\|^2_{W^{2,2}} = A(u_\varepsilon, u_\varepsilon) - B(F, u_\varepsilon, u_\varepsilon) + a(u_\varepsilon, u_\varepsilon) \leqq$$
$$\leqq A(u_\varepsilon, u_\varepsilon) - B(F, u_\varepsilon, u_\varepsilon) + a(u_\varepsilon, u_\varepsilon) + \varepsilon \int_{\Gamma_2 \cup \Gamma_3} u_\varepsilon^2 \, ds =$$
$$= B(f_\varepsilon, \hat{u}_\varepsilon, \hat{u}_\varepsilon) + L(\hat{u}_\varepsilon)\,.$$

Since $B(f_\varepsilon, \hat{u}_\varepsilon, \hat{u}_\varepsilon) \leqq 0$ (cf. Section 4, Proof of 1°) it follows (using also (4.3) and(2.2))

(5.4) $$\|\hat{u}_\varepsilon\|_{W^{2,2}} \leqq \text{const} \quad \text{for all} \quad \varepsilon > 0\,.$$

Again putting $\varphi = u_\varepsilon$ in (5.2), we obtain by virtue of (5.4)

$$\varepsilon \int_{\Gamma_2 \cup \Gamma_3} u_\varepsilon^2 \, ds \leqq \text{const} \quad \text{for all} \quad \varepsilon > 0 . \tag{5.5}$$

2) *Limit procedure.* By (5.4), there exists a subsequence $\{u_\eta\} \subset \{u_\varepsilon\}$ and $u^{(1)} \in$ such that

$$\hat{u}_\eta \to u^{(1)} \quad \text{weakly in} \quad W^{2,2}(\Omega) \quad \text{as} \quad \eta \to 0 .$$

Since $f_\varepsilon = C_2(u_\varepsilon, u_\varepsilon)$ (see (4.6)) and $C_2(u_\varepsilon, u_\varepsilon) = C_2(\hat{u}_\varepsilon, \hat{u}_\varepsilon)$, we have (by going to a subsequence if necessary)

$$C_2(u_\eta, u_\eta) \to C_2(u^{(1)}, u^{(1)}) \text{ strongly in } W_0^{2,2}(\Omega) \text{ as } \eta \to 0$$

(cf. Section 4, Proof of 2°).

Putting $f^{(1)} = C_2(u^{(1)}, u^{(1)})$, the pair $\{u^{(1)}, f^{(1)}\}$ is in $V \times W_0^{2,2}(\Omega)$ and satisfies (2.6).

From (5.5) we conclude the estimate

$$\varepsilon \left| \int_{\Gamma_2 \cup \Gamma_3} u_\varepsilon \varphi \, ds \right| \leqq \varepsilon^{1/2} \, \text{const} \left\{ \int_{\Gamma_2 \cup \Gamma_3} \varphi^2 \, ds \right\}^{1/2}$$

which holds for any $\varepsilon > 0$ and any $\varphi \in W^{2,2}(\Omega)$. Finally, we have, for arbitrary $\varphi \in W^{2,2}(\Omega)$,

$$A(u_\varepsilon, \varphi) - B(F, u_\varepsilon, \varphi) + a(u_\varepsilon, \varphi) =$$
$$= A(\hat{u}_\varepsilon, \varphi) - B(F, \hat{u}_\varepsilon, \varphi) + a(\hat{u}_\varepsilon, \varphi) ,$$

and

$$B(f_\varepsilon, u_\varepsilon, \varphi) = B(f_\varepsilon, \hat{u}_\varepsilon, \varphi) .$$

Letting $\varepsilon = \eta \to 0$ in (5.2), it is readily seen that $\{u^{(1)}, f^{(1)}\}$ satisfies (2.5) for all $\varphi \in V$.

The proof of (ii) is complete.

The proof of (iii) follows the same line of thoughts as that of (ii).

Remark 5.1. Let (1.7) hold, and let $u_\varepsilon = \bar{u}_\varepsilon + p_\varepsilon$, with $\bar{u}_\varepsilon \in R$, $p_\varepsilon \in P$, be the unique (orthogonal) decomposition of u_ε (u_ε according to (i)). To obtain the estimate

$$\|\bar{u}_\varepsilon\|_{W^{2,2}} \leqq \text{const} \quad \text{for all} \quad \varepsilon > 0 ,$$

it is easy to see that the condition

$$2B(F, \bar{u}_\varepsilon, p_\varepsilon) + B(F, p_\varepsilon, p_\varepsilon) + L(p_\varepsilon) = 0$$

is sufficient (provided that $T_1(\varphi_0, \varphi_1)$ is sufficiently small). But this condition is a consequence of (2.10), (2.9′).

Remark 5.2 An analogue of Theorem 2 can be proved, if

$$\varepsilon \int_{\Gamma_2 \cup \Gamma_3} uv \, ds \quad \text{is replaced by} \quad \varepsilon \int_{\Omega^*} uv \, dx \, dy \,, \quad \text{where} \quad \Omega^* \subset \Omega \,,$$

$$\text{mes}\,(\Omega^*) > 0 \,.$$

References

[1] *Hlaváček, I.* and *Naumann, J.:* Inhomogeneous boundary value problems for the von Kármán equations. I. Aplikace matematiky 19 (1974), 253—269.

[2] *Hlaváček, I.* and *Nečas, J.:* On inequalities of Korn's type. I. Arch. Ratl. Mech. Anal., 36 (1970), 305—311.

[3] *Lions, J. L.:* Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Gauthier-Villars, Paris 1969.

[4] *Naumann, J.:* An existence theorem for the von Kármán equations under the conditions of free boundary. Apl. mat., 19 (1974), 17—27.

Souhrn

NEHOMOGENNÍ OKRAJOVÉ ÚLOHY PRO KÁRMÁNOVY ROVNICE II.

IVAN HLAVÁČEK and JOACHIM NAUMANN

Dokazuje se existence řešení jisté třídy kombinovaných okrajových úloh, přičemž okrajové podmínky připouštějí možnost pohybu desky jako tuhého celku. Část okraje může být podepřená a pružně vetknutá, část pružně podepřená i vetknutá nebo volná, zatížená momenty a posouvajícími silami. Na celém okraji desky, který může mít rohy, působí též zatížení v rovině desky.

Variační formulace problému je převedena na operátorovou rovnici, pro kterou platí abstraktní existenční věta. Řešení existuje a tvoří zbytkovou třídu, právě když jsou splněny podmínky celkové rovnováhy vnějších sil a momentů.

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