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SUBSONIC IRROTATIONAL FLOW OF COMPRESSIBLE FLUID IN AXIALLY-SYMMETRIC CHANNELS

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Increasing demands on the quantity of converted energy in stream machines (i.e. in turbomachines, compressors, pumps etc.) compel engineers to design machines with higher parameters. Along with other factors, the compressibility of fluids has to be taken into consideration, since the results obtained under the assumption of an incompressible fluid (see e.g. [1]) do not give a sufficiently exact image of reality.

In this paper, we present a theoretical analysis of the three-dimensional axially-symmetric irrotational adiabatic and isentropic steady channel flow of an ideal gas. We limit ourselves to the study of subsonic stream fields, which is very important e.g. in the investigation of a flow through some types of turbomachines.

Before concentrating on the problem under discussion, let us realize the complexity of questions connected with the flow of a compressible fluid. It is known that the irrotational compressible flow can be described by a nonlinear second-order partial differential equation of mixed type for the velocity potential. This equation is of elliptic or hyperbolic type in the subsonic or supersonic region respectively (see e.g. [2]–[4]). The situation is still more complicated than it could seem because the regions of ellipticity and hyperbolicity are not known in advance. The boundary value problems formulated for nonlinear equations of mixed type have hitherto resisted the efforts of mathematicians to solve them.

Therefore, we shall deal with the formulation and study of a simplified model problem, which exactly describes a subsonic stream field in which we are interested and has "acceptable" properties from the mathematical point of view.

1. SOME ASSUMPTIONS AND GEOMETRIC DESCRIPTION OF A CHANNEL

In this paper we shall consider stream fields in channels which are used in technical practice (e.g. in the construction of steam turbomachines). Contrary to [1], we shall immediately suppose that the channel is a bounded part of space filled up with fluid.

For the geometric description of the channel, as well as for the solution of the problem, we shall use cylindrical coordinates z, r, φ. Let z be the axis of symmetry of the channel. In view of axial symmetry, the form of channel is given by its projection into the upper half-plane (z, r), i.e. into the set $\{(z, r, 0); r \geqq 0\}$. Let us denote this projection by P. (P is a bounded domain.) The three-dimensional channel will be obtained by the rotation of the set P round the z-axis. Let L_1, L_2 denote the projections of the walls of the channel. The projection of the inlet and the exit of the channel will be denoted by Γ^1 and Γ^2, respectively. It means that the boundary of P is $\partial P = L_1 \cup L_2 \cup \Gamma^1 \cup \Gamma^2$. Let $\bar{P} = P \cup \partial P$. We shall call the region P simply a channel, Γ^1-inlet and Γ^2-exit.

Let us assume: a) Both L_1 and L_2 are parallel to z- or r-axis in a neighbourhood of the inlet and the exit. b) For every point $(z, r) \in \bar{P}$, it is $r > 0$. c) The arcs Γ^1 and Γ^2 are perpendicular to L_1 and L_2. d) Γ^1 and Γ^2 are "sufficiently" distant from the curved parts of the channel.

In virtue of a) and d) we may assume that the stream field is nearly parallel to the walls L_1, L_2 of the channel in a neighbourhood of the inlet and the exit. See e.g. Figs. 1 and 2 in [1].

Finally, we put

$$R_1 = \min \{r; \exists z \in E_1, (z, r) \in \bar{P}\},$$
$$R_2 = \max \{r; \exists z \in E_1, (z, r) \in \bar{P}\},$$

where E_1 denotes the set of all real numbers. It is evident that $0 < R_1 < R_2 < +\infty$.

2. EQUATIONS OF AXIALLY-SYMMETRIC FLOW

In this paragraph, we shall study the system of equations governing an irrotational axially-symmetric steady flow of an ideal gas. We shall deal only with an adiabatic and isentropic flow.

The fundamental system consists of ([2]–[5])

a) equation of continuity

$$\operatorname{div}(\varrho \mathbf{V}') = 0, \tag{2.1}$$

b) Euler equations of motion

$$\frac{\mathrm{d}\mathbf{V}'}{\mathrm{d}t} = \mathbf{F} - \frac{1}{\varrho} \operatorname{grad} p. \tag{2.2}$$

$\mathbf{V}' = (v_z, v_r, v_\varphi)$ is the velocity of the fluid, p – pressure, ϱ – fluid density, $\mathbf{F}$ – vector of exterior volume force related to the unit of mass, t – time. Let us suppose that the quantities $\mathbf{V}'$, p, ϱ, $\mathbf{F}$ as well as the arcs L_1, L_2 are sufficiently smooth.

Further, we consider

c) equation of irrotational flow

$$\text{rot}\, \boldsymbol{V}' = 0\,, \tag{2.3}$$

d) equation of adiabatic and isentropic flow

$$p = C\varrho^{\varkappa}\,, \tag{2.4}$$

where $C > 0$ and $\varkappa > 1$ are constants characteristic for the given fluid ($\varkappa$ is the so called Poisson constant).

The assumption of axial symmetry implies that all quantities considered depend only on the variables z and r. It means that the three-dimensional problem of a channel flow can be reduced to the problem solved in the two-dimensional region P.

If we express Eq. (2.1) in the cylindrical coordinates, taking into account the assumption of axial symmetry, we get

$$\frac{\partial(r\varrho v_z)}{\partial z} + \frac{\partial(r\varrho v_r)}{\partial r} = 0\,. \tag{2.1'}$$

From (2.3) it follows

$$\frac{\partial(rv_\varphi)}{\partial z} = \frac{\partial(rv_\varphi)}{\partial r} = 0$$

and hence

$$rv_\varphi = \text{const}\,.$$

For the sake of simplicity we limit ourselves to the case

$$v_\varphi = 0\,. \tag{2.5}$$

The condition (2.3) of irrotational flow reduces then to one equation of the form

$$\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0\,. \tag{2.3'}$$

Now, let us pay our attention to Euler equations (2.2). The force $\boldsymbol{F}$ represents gravity here. Our considerations refer mainly to stream fields in machines with not too large space size, so that the influence of gravity on the flow is small. Therefore, we shall neglect the force $\boldsymbol{F}$ and put $\boldsymbol{F} = 0$.

The left-hand side of Eq. (2.2) can be transformed similarly as in [1]:

$$\frac{\mathrm{d}\boldsymbol{V}'}{\mathrm{d}t} = \frac{\partial \boldsymbol{V}'}{\partial t} + \tfrac{1}{2}\,\text{grad}\,(V'^2) - \boldsymbol{V}' \times (\text{rot}\, \boldsymbol{V}')$$

(here $V' = |\mathbf{V}'|$). If we denote $\mathbf{V} = (v_z, v_r)$, $V = |\mathbf{V}|$, then by (2.5), $V = V'$. Using Eq. (2.3) and the assumption of stationarity, Eq. (2.2) can be written in the form

$$\tfrac{1}{2}\operatorname{grad}(V^2) = -\frac{1}{\varrho}\operatorname{grad} p\,. \tag{2.2'}$$

For further considerations it will be useful to introduce the local speed of sound a:

$$a^2 = \frac{\mathrm{d}p}{\mathrm{d}\varrho} = C\varkappa\varrho^{\varkappa-1}\,.$$

Let ϱ_0, p_0, a_0 be respectively the density, the pressure and the speed of sound corresponding to the velocity $V_0 = 0$.

If we define

$$\mathscr{P}(\varrho) = \int_{\varrho_0}^{\varrho} \frac{\frac{\mathrm{d}p}{\mathrm{d}\varrho}(\tau)}{\tau}\,\mathrm{d}\tau\,,$$

we have

$$\operatorname{grad}\mathscr{P} = \frac{1}{\varrho}\operatorname{grad} p\,.$$

Substituting into (2.2′) we obtain

$$\operatorname{grad}(\mathscr{P} + \tfrac{1}{2}V^2) = 0\,,$$

so that

$$\mathscr{P}(\varrho) + \tfrac{1}{2}V^2 = \mathscr{P}(\varrho_0)\,. \tag{2.6}$$

This equation is an analogue of the so called Bernoulli equation, which is valid for incompressible fluid.

Since

$$\mathscr{P}(\varrho) = C\frac{\varkappa}{\varkappa - 1}(\varrho^{\varkappa-1} - \varrho_0^{\varkappa-1})\,,$$

$$\mathscr{P}(\varrho_0) = 0\,,$$

$$a_0^2 = C\varkappa\varrho_0^{\varkappa-1}\,,$$

we obtain from (2.6) after simple operations an important relation between the fluid density and the velocity:

$$\frac{\varrho}{\varrho_0} = \left(1 - \frac{\varkappa - 1}{2}\left(\frac{V}{a_0}\right)^2\right)^{1/(\varkappa-1)} \tag{2.7}$$

Let us introduce the dimensionless quantities

$$\mathbf{V}^* = \frac{\mathbf{V}}{a_0}\,,\quad \varrho^* = \frac{\varrho}{\varrho_0}\,,\quad V^* = |\mathbf{V}^*|$$

and write Eq. (2.1′) in the form

$$\frac{\partial(r\varrho^* v_z^*)}{\partial z} + \frac{\partial(r\varrho^* v_r^*)}{\partial r} = 0 .$$

Since P is a simply connected domain, the last equation and the assumption of smoothness of v_z^*, v_r^*, ϱ^* imply the existence of the so called stream function ψ such that

$$\frac{\partial\psi}{\partial r} = r\varrho^* v_z^* , \quad \frac{\partial\psi}{\partial z} = -r\varrho^* v_r^* . \tag{2.8}$$

The smoothness of ψ depends on the smoothness of v_z^*, v_r^*, ϱ^*. Usually, if we need to be more precise about it, we suppose that ψ is continuous in $\bar{P}$, has continuous derivatives of the first order in $P \cup \Gamma^1 \cup \Gamma^2$ and continuous derivatives of the second order in P (see Paragraph 4).

If we substitute (2.8) into (2.3′), we get the fundamental equation

$$\frac{\partial}{\partial z}\left(\frac{1}{r\varrho^*}\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r\varrho^*}\frac{\partial\psi}{\partial r}\right) = 0 \tag{2.9}$$

for the stream function of an irrotational compressible flow. Let us remember that Eq. (2.9) was derived under the assumptions that the flow was adiabatic, isentropic, axially-symmetric and the exterior volume force was neglected. The assumption $v_\varphi = 0$ has not yet been essential. It will be used in the following paragraphs.

Eq. (2.9) is a quasilinear partial differential equation of the second order. Nonlinearity of this equation is given by the relation (2.7) between the density ϱ and the velocity V. This dependence will be studied in detail in the following paragraph.

Remark 1. It is easy to show that the stream function is constant along an arbitrary stream line. The difference of two values ψ_1, ψ_2 of the stream function is equal to the mass-flow per sec between the stream surfaces, determined by the values ψ_1, ψ_2, divided by $2\pi a_0 \varrho_0$. The stream function is determined by the given stream field up to an additive constant. In practice we choose, as a rule, the so called zero-stream line S_0 (usually a part of the boundary) and put $\psi \mid S_0 = 0$.

3. DEPENDENCE OF THE FLUID DENSITY ON THE GRADIENT OF A STREAM FUNCTION

The well-known formula (2.7) can be written in the form

$$(\varrho^*)^2 = \left(1 - \frac{\varkappa - 1}{2}(V^*)^2\right)^{2/(\varkappa-1)} .$$

From this relation, we shall derive an equation which determines the fluid density as a function of the variables r and $\zeta = (\nabla\psi)^2 = (\partial\psi/\partial z)^2 + (\partial\psi/\partial r)^2$, where ψ is a stream function. We denote

$$\nabla\psi = \left(\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial r}\right).$$

In view of (2.8),

$$(V^*)^2 = \frac{1}{(\varrho^*)^2}\left(\frac{\nabla\psi}{r}\right)^2. \tag{3.1}$$

If we put

$$\begin{aligned} \alpha &= \frac{2}{\varkappa - 1} > 0\,, \\ \omega &= (\varrho^*)^2\,, \\ x &= \frac{1}{\alpha}\left(\frac{\nabla\psi}{r}\right)^2 = \frac{1}{\alpha}\frac{\zeta}{r^2} \end{aligned} \tag{3.2}$$

(x is a function of r and ζ), we get

$$\omega = \left(1 - \frac{x}{\omega}\right)^\alpha. \tag{3.3}$$

This equation, with $\alpha > 0$ fixed, $x \geqq 0$ and $\omega > 0$, will serve for the calculation of the density ϱ in dependence on the variables r and $\zeta = (\nabla\psi)^2$.

Let us study the solvability of Eq. (3.3). For this purpose, we shall write it in an equivalent form

$$x = \omega - \omega^{1+1/\alpha}. \tag{3.4}$$

From the conditions $x \geqq 0$, $\omega > 0$, we get $\omega \leqq 1$, so that

$$0 < \omega \leqq 1.$$

For $\omega \in (0, 1\rangle$ let

$$f(\omega) = \omega - \omega^{1+1/\alpha}. \tag{3.5}$$

Then

$$f'(\omega) = 1 - \left(1 + \frac{1}{\alpha}\right)\omega^{1/\alpha}.$$

There exists exactly one value

$$\omega_{kr} = \left(\frac{\alpha}{1+\alpha}\right)^\alpha$$

such that

$$f'(\omega_{kr}) = 0 .$$

It holds: $f'(\omega) < 0$ for $\omega \in (\omega_{kr}, 1\rangle$, $f'(\omega) > 0$ for $\omega \in (0, \omega_{kr})$,

$$f(\omega_{kr}) = \frac{1}{1 + \alpha}\left(\frac{\alpha}{1 + \alpha}\right)^{\alpha} .$$

It follows from the preceding that the function f is increasing on the interval $(0, \omega_{kr}\rangle$, decreasing on the interval $\langle\omega_{kr}, 1\rangle$, $f(\omega) \leqq f(\omega_{kr})$ for every $\omega \in (0,1\rangle$ and $f(\omega) = f(\omega_{kr})$ if and only if $\omega = \omega_{kr}$. In addition, $f(1) = 0 = \lim_{\omega \to 0+} f(\omega)$.

Let us denote $x_{kr} = f(\omega_{kr})$. Since the function f is continuous, we can assert that if $x = x_{kr}$, then Eq. (3.3) has exactly one solution ω_{kr}. For $0 < x < x_{kr}$ we get exactly two solutions ω_1, ω_2 satisfying the inequalities

$$0 < \omega_1 < \omega_{kr} < \omega_2 < 1 .$$

If $x = 0$, we have the solution $\omega = 1$. Hence Eq. (3.3) is solvable if and only if

$$0 \leqq x \leqq x_{kr} = \frac{1}{1 + \alpha}\left(\frac{\alpha}{1 + \alpha}\right)^{\alpha} . \tag{3.6}$$

Before we proceed to physical interpretation of the solutions ω_1, ω_2, let us point out that the Mach number M is defined as the quotient of the fluid velocity and the local speed of sound (at a given point):

$$M = \frac{V}{a} .$$

We say that the stream field is subsonic or supersonic in a region $P_1 \subset P$, if $M < 1$ in P_1 or $M > 1$ in P_1, respectively. It is evident that $M < 1$ ($M > 1$) is equivalent to $V < a$ ($V > a$).

In the following, we shall prove that the solutions ω_1 and ω_2 correspond to the supersonic or subsonic stream field, respectively.

Let us express the absolute value V_{kr} of the velocity, corresponding to the value ω_{kr}. By (3.1) and (3.2), it is

$$V^2 = a_0^2 \alpha \frac{x}{\omega} . \tag{3.7}$$

If we put $x = x_{kr}$, $\omega = \omega_{kr}$, we get

$$V_{kr}^2 = a_0^2 \frac{\alpha}{1 + \alpha} .$$

The speed of sound a can be expressed by the formula

(3.8) $$a^2 = C\varkappa\varrho^{\varkappa-1} = C\varkappa(\varrho^*)^{\varkappa-1}\,\varrho_0^{\varkappa-1} = a_0^2\omega^{1/\alpha}\,.$$

Hence, for $\omega = \omega_{\text{kr}}$ we have

$$a_{\text{kr}}^2 = a_0^2\omega_{\text{kr}}^{1/\alpha} = a_0^2\frac{\alpha}{1+\alpha} = V_{\text{kr}}^2\,.$$

It means that if $\omega = \omega_{\text{kr}}$, then the velocity V_{kr} is equal to the local speed of sound a_{kr}.

If we use (3.4), then the relation (3.7) can be written in the form

(3.7′) $$V^2 = a_0^2\alpha(1 - \omega^{1/\alpha})\,.$$

It follows from (3.7′) and (3.8) that if $0 < \omega < \omega_{\text{kr}}$, then $V > V_{\text{kr}} = a_{\text{kr}} > a$ and if $\omega_{\text{kr}} < \omega \leqq 1$, then $V < V_{\text{kr}} = a_{\text{kr}} < a$. In the preceding, we have proved that to every $x \in (0, x_{\text{kr}})$ there exist two solutions ω_1, ω_2 of Eq. (3.3). Since the inequalities $0 < \omega_1 < \omega_{\text{kr}} < \omega_2 < 1$ are valid, then it is evident that ω_1 corresponds to the supersonic and ω_2 to the subsonic flow. If $x = 0$, then $\omega_2 = 1$, which corresponds to the value of the velocity $V = 0$.

Here, we see that our approach to the problem is not suitable for the study of transonic flow when in an unknown part P_1 of the channel P the Mach number $M < 1$, and $M \geqq 1$ in $P - P_1$. To put it precisely, we could not calculate the fluid density uniquely in this case. Since we limit ourselves to the study of subsonic flow, we shall use the root ω_2 of Eq. (3.3) for the calculation of the density. In the following, we shall omit the index 2 and the symbol ω will denote the solution ω_2.

It is evident that ω depends on $x \in \langle 0, x_{\text{kr}}\rangle$, i.e.

$$\omega = \tilde{\omega}(x)\,.$$

$\tilde{\omega}(x)$, as the inverse with respect to $f(\omega)$ on the interval $\langle\omega_{\text{kr}}, 1\rangle$, is decreasing on the interval $\langle 0, x_{\text{kr}}\rangle$. Further, $\tilde{\omega}(0) = 1$, $\tilde{\omega}(x_{\text{kr}}) = \omega_{\text{kr}}$ and for every $x \in \langle 0, x_{\text{kr}})$,

(3.9) $$\frac{\mathrm{d}\tilde{\omega}(x)}{\mathrm{d}x} = \frac{\alpha}{\alpha - (1+\alpha)\,(\tilde{\omega}(x))^{1/\alpha}} < 0\,.$$

In Eq. (2.9), the expression

(3.10) $$\tilde{\beta} = \frac{1}{r\varrho^*}$$

occurs as a coefficient at partial derivatives of a stream function. Let us study some important properties of $\tilde{\beta}$.

It follows from the preceding that we can consider $\tilde{\beta}$ a function of the variables r and $\zeta = (\nabla\psi)^2$:

$$(3.11) \qquad \tilde{\beta} = \tilde{\beta}(r, \zeta) = \frac{1}{r\sqrt{\left(\tilde{\omega}\left(\frac{\zeta}{\alpha r^2}\right)\right)}}$$

(see (3.2) and (3.10)).
Since

$$\frac{\zeta}{\alpha r^2} = x \in \langle 0, x_{\mathrm{kr}} \rangle ,$$

it is

$$\zeta \in \langle 0, \alpha r^2 x_{\mathrm{kr}} \rangle .$$

It means that the function $\tilde{\beta}(r, \zeta)$ is defined on the set

$$\mathscr{D}(\tilde{\beta}) = \{(r, \zeta);\ r \in \langle R_1, R_2 \rangle,\ \zeta \in \langle 0, \alpha r^2 x_{\mathrm{kr}} \rangle\} .$$

Let us calculate the derivative $\partial\tilde{\beta}/\partial\zeta$. From (3.11), we get

$$(3.12) \qquad \frac{\partial\tilde{\beta}(r, \zeta)}{\partial\zeta} = -\frac{1}{2\alpha r^3}\left(\tilde{\omega}\left(\frac{\zeta}{\alpha r^2}\right)\right)^{-3/2} \frac{\mathrm{d}\tilde{\omega}\left(\frac{\zeta}{\alpha r^2}\right)}{\mathrm{d}x} > 0 ,$$

if $r \in \langle R_1, R_2 \rangle$ and $\zeta \in \langle 0, \alpha r^2 x_{\mathrm{kr}})$. On the basis of (3.12), we can show easily that if the flow is subsonic, then Eq. (2.9) is elliptic. However, it is an unpleasant fact that $\tilde{\beta}(r, \zeta)$ is defined only for the values ζ from the bounded interval $\langle 0, \alpha r^2 x_{\mathrm{kr}} \rangle$ (if r is given). It means that we ought to seek a subsonic solution of Eq. (2.9) in a class of functions which have a square of their gradient uniformly bounded. This class is not, of course, a linear set, which makes the problem still more complicated. We can avoid this difficulty, if we define $\tilde{\beta}$ suitably for all values $\zeta \geqq 0$, i.e. if we extend the function $\tilde{\beta}$ to the set $\langle R_1, R_2 \rangle \times \langle 0, +\infty)$.

For this purpose, we shall limit ourselves to stream fields for which the Mach number M satisfies the inequality

$$(3.13) \qquad (0 \leqq)\, M \leqq \overline{M} < 1$$

at every point of the channel. $\overline{M}$ is a given constant. This restriction is not essential if the stream field in question is strictly subsonic, because the number $\overline{M}$ can be chosen as close to one as desired.

In virtue of the definition of M, (3.7′) and (3.8), it holds

$$M = \left(\alpha\left(\frac{1}{\omega^{1/\alpha}} - 1\right)\right)^{1/2} .$$

Thus M is a decreasing function of $\omega \in \langle\omega_{kr}, 1\rangle$: $M = M(\omega)$. Therefore, the requirement $(0 \leqq)\, M \leqq \overline{M} < 1$ is equivalent to

$$\omega_{kr} < \overline{\omega} \leqq \omega(\leqq 1)\,, \tag{3.14}$$

where $\overline{M} = M(\overline{\omega})$, $M = M(\omega)$. If we express the values ω, $\overline{\omega}$ as $\omega = \tilde{\omega}(x)$, $\overline{\omega} = \tilde{\omega}(\bar{x})$ for suitable $x, \bar{x} \in \langle 0, x_{kr}\rangle$, then we see that (3.13) is equivalent to

$$(0 \leqq)\, x \leqq \bar{x} < x_{kr}\,, \tag{3.15}$$

which follows immediately from the behaviour of the function $\tilde{\omega}(x)$. It means, further, that $M \leqq \overline{M}$ if and only if

$$(\nabla\psi)^2 = \zeta \in \langle 0, \alpha r^2 \bar{x}\rangle\,.$$

Now, let us extend the function $\tilde{\beta}$ to the domain $\langle R_1, R_2\rangle \times \langle 0, +\infty)$. It is evident (see (3.11)) that it will do to extend the function $\tilde{\omega}(x)$ from the interval $\langle 0, \bar{x}\rangle$ to $\langle 0, +\infty)$. It is possible to prove that there exists a function $\omega(x)$ defined on $\langle 0, +\infty)$ with the following properties:

a) if $x \in \langle 0, \bar{x}\rangle$, then $\omega(x) = \tilde{\omega}(x)$,

b) $\omega(x) = \omega_{kr}$ for $x \in \langle x_{kr}, +\infty)$,

c) $d\omega/dx$ is continuous on $\langle 0, +\infty)$,

d) $\dfrac{d\omega(x)}{dx} < 0$ for $x \in \langle 0, x_{kr})$.

By the help of the function $\omega(x)$ we can define the desired extension of $\tilde{\beta}$. Let $\mathscr{D}(\beta) = \langle R_1, R_2\rangle \times \langle 0, +\infty)$, and for $(r, \zeta) \in \mathscr{D}(\beta)$ put

$$\beta(r, \zeta) = \frac{1}{r\sqrt{\left(\omega\left(\frac{\zeta}{\alpha r^2}\right)\right)}}\,. \tag{3.16}$$

Let us denote

$$\bar{\zeta}_r = \alpha r^2 \bar{x}\,, \quad \hat{\zeta} = \alpha R_2^2 x_{kr}\,. \tag{3.17}$$

It is easy to show that it holds:

There exist constants $C_1 > 0$, $C_2 > 0$ such that

$$0 < \frac{1}{R_2} \leqq \beta(r, \zeta) \leqq C_1, \tag{3.18}$$

$$0 \leqq \frac{\partial\beta(r, \zeta)}{\partial\zeta} \leqq C_2 \quad \text{for every} \quad (r, \zeta) \in \mathscr{D}(\beta)\,;$$

$$\beta(r, \zeta) = \tilde{\beta}(r, \zeta) \quad \text{if} \quad r \in \langle R_1, R_2\rangle \quad \text{and} \quad \zeta \in \langle 0, \bar{\zeta}_r\rangle\,,$$

$$\frac{\partial\beta(r,\zeta)}{\partial\zeta} = 0 \quad \text{for every} \quad (r,\zeta) \in \langle R_1, R_2\rangle \times \langle\hat{\zeta}, +\infty),$$

$$\beta \in C^1(\mathscr{D}(\beta))$$

(i.e. β has continuous derivatives of the first order on its domain).

On the basis of properties (3.18) we can find out that if we write β instead of $1/(r\varrho^*)$ in (2.9), we get an equation of elliptic type. It is evident that the stream field obtained from the solution ψ of this equation, determines a real irrotational compressible flow, if (3.15) (where $x = (\nabla\psi)^2/(\alpha r^2)$) is valid at every point of the channel. Moreover, if $\zeta = (\nabla\psi)^2 \geqq \hat{\zeta}$ in a part P_1 of the channel, then the function ψ determines an irrotational incompressible flow in P_1.

4. BOUNDARY VALUE CONDITIONS AND CLASSICAL FORMULATION OF THE PROBLEM

The boundary value conditions to which a stream function is subjected on the arcs L_1, L_2 are given by the fact that L_1, L_2, as tight walls, are stream lines. With respect to Remark 1, we can write

(4.1)
$$\psi \mid L_1 = 0, \qquad \psi \mid L_2 = Q.$$

The constant Q is equal to the total mass-flow through the channel per sec, divided by $2\pi a_0 \varrho_0$.

The conditions at the inlet and the exit of the channel follow as a consequence of assumptions a) and d) in Paragraph 1: $v_r \mid \Gamma^i = 0$ for axial inlet ($i = 1$) or exit ($i = 2$) and $v_z \mid \Gamma^i = 0$ for radial inlet or exit. By (2.8), we have

$$\frac{\partial\psi}{\partial z}\bigg| \Gamma^i = 0 \quad \text{for axial inlet or exit},$$

$$\frac{\partial\psi}{\partial r}\bigg| \Gamma^i = 0 \quad \text{for radial inlet or exit.}$$

These two conditions can be written in the form

(4.2)
$$\frac{\partial\psi}{\partial n}\bigg| \Gamma^i = 0, \quad i = 1, 2,$$

where $\partial/\partial n$ denotes the derivative in the direction of the outer normal to Γ^i with respect to the domain P.

Finally, let us formulate the "classical" problem of irrotational, compressible flow.

We seek a function ψ, which

1) is continuous in $\bar{P}$,
2) has continuous partial derivatives of the first order in $P \cup \Gamma^1 \cup \Gamma^2$,
3) has continuous derivatives of the second order in P,
4) satisfies Eq. (2.9) in P, where we put β instead of $1/r\varrho^*$,
5) satisfies boundary value conditions (4.1) and (4.2) on ∂P.

Let us denote this problem by (A). It is known that it is considerably difficult to solve nonlinear partial differential equations. In general, it is impossible to find the solution of problem (A) in an explicit form. The only way is to use a suitable approximate method. For instance, we can mention the report [6], where the finite-difference method was used. Numerical experiments carried out by a computer, show that it is possible to obtain by this method results valuable from the technical point of view.

CONCLUSION

Mathematical investigation of the inner aerodynamics of stream machines meets, in its generality, with considerable difficulties which we are not able to solve in all details at present. A general flow through these machines is three-dimensional, rotational and non-steady, connected with loss of energy and heat-transfer in regions of complicated geometric form. The fluid is compressible, viscous and non-perfect, consisting often of two media (e.g. steam and water). In order to be able to get at least some results applicable in technology, the problem must be simplified, and only fundamental elements of stream machines under idealized assumptions can be studied.

Therefore, we formulate here on the basis of a detailed analysis of a subsonic flow a model problem that exactly describes a subsonic flow and has relatively good mathematical properties. In this way, we can get the following information about the investigated stream field: a) we can tell whether the flow discussed is subsonic or not, b) in case that the solution ψ satisfies certain conditions stated above, our model problem gives a real subsonic flow, c) if the flow is not subsonic in the whole channel, we can determine approximately the regions where the flow is supersonic.

In this paper, we have not studied the solvability of problem (A). We deal with this in [7].

References

[1] *M. Feistauer:* Some cases of numerical solution of differential equations describing the vortex-flow through three-dimensional axially-symmetric channels, Aplikace matematiky, *16* (1971), 265—288.

[2] *L. Bers:* Mathematical aspects of subsonic and transonic gas dynamics, New York, John Wiley and Sons, Inc., 1958.
[3] *R. Mises:* Mathematical theory of compressible fluid flow, Academic Press, Inc. Publishers, New York, 1958.
[4] *Shih-I Pai:* Introduction to the theory of compressible flow, D. Van Nostrand Comp., Inc. Princeton, New Jersey.
[5] *Л. Г. Лойцянский:* Механика жидкости и газа, Физматгиз, Москва, 1973.
[6] *M. Feistauer* and *J. Polášek:* Solution of irrotational axially-symmetric compressible flow by finite difference method (in Czech), Technical research report, Škoda Plzeň, 1974.
[7] *M. Feistauer and J. Římánek:* Solution of subsonic axially-symmetric stream fields, Aplikace matematiky, *20* (1975), 266—279.

Souhrn

PODZVUKOVÉ NEVÍŘIVÉ PROUDĚNÍ STLAČITELNÉ TEKUTINY V OSOVĚ SYMETRICKÝCH KANÁLECH

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Studium vnitřní aerodynamiky proudových strojů ve své obecnosti představuje po matematické stránce obtíže, které za současného stavu vědy nejsou do všech podrobností řešitelné. Aby bylo možné získat výsledky použitelné pro praxi, bylo nutno celou problematiku zjednodušit a studovat pouze základní elementy proudových strojů za zidealizovaných předpokladů.

V tomto článku podáváme podrobný teoretický rozbor proudění ideální stlačitelné tekutiny v třírozměrných kanálech za předpokladu, že se jedná o proudění stacionární, nevířivé, osově symetrické, adiabatické, isoentropní a podzvukové. Na základě této analýzy byla formulována modelová úloha, která popisuje vyšetřované proudové pole a má po matematické stránce „rozumné“ vlastnosti. Jejím řešením můžeme získat pro praxi cenné informace: a) můžeme říci, zda je vyšetřované proudové pole v celém kanálu podzvukové či ne, b) v případě podzvukového proudění dává řešení naší modelové úlohy skutečné proudové pole stlačitelné tekutiny, c) v případě, že proudové pole není podzvukové v celém kanálu, můžeme alespoň přibližně stanovit oblasti, v nichž je pole nadzvukové.

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