Joachim Naumann

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ON SOME UNILATERAL BOUNDARY VALUE PROBLEMS FOR THE VON KÁRMÁN EQUATIONS

PART I: THE COERCIVE CASE

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JOACHIM NAUMANN

Contents

1. INTRODUCTION

This paper deals with the existence of equilibrium states of a thin elastic plate under penpendicular loading, where the edge of the plate is subjected partly to boundary equations of the type studied by Hlaváček and Naumann [4], and partly to certain unilateral constraints.

The present Part I is devoted to systems of boundary conditions which imply that any motion of the plate is eliminated, if the elastic energy of the plate bending and of the elastic clampings vanishes (this means that the bilinear form representing the energy of the plate bending and of the elastic clampings is coercive on the associated energy space).

In [2], Duvaut and Lions have studied the existence and uniqueness of equilibrium states of an elastic plate (linear case) under various types of unilateral conditions. These boundary value problems are restated in terms of variational inequalities to which abstract results directly apply. Unilateral boundary value problems for the nonlinear system governing the equilibrium of a thin elastic plate (formulation in displacements) are studied by the same authors in [3]. Here only the coercive case (coerciveness with respect to the associated energy space) is considered. A method of Galerkin's type is used to solve the problems under consideration in their unified form.

In Section 2 we give the formulation of unilateral boundary value problems which we shall investigate. The following Section 3 is devoted to the preliminaries needed as well as to the variational formulation of our unilateral problems. We then show in Section 4 in which sense the equations are satisfied by a variational solution. Further, assuming appropriate regularity properties of the variational solution and using a result from [5], we are able to give an interpretation of some unilateral boundary conditions.

Section 5 presents the unification of all variational formulations introduced as well as our main existence result. The uniqueness of the variational solution can be proved in the case of sufficiently small deflections. The proof of the existence theorem which will be given in the following section, uses directly an abstract result. The last Section 7 concerns the passage to limit with respect to certain parameters. It turns out that one obtains in this way a variational solution of some types of boundary value problems discussed in [4].

2. SETTING OF THE UNILATERAL BOUNDARY VALUE PROBLEMS

Let Ω be a bounded domain in the x,y-plane (constituting the middle plane of the plate) with boundary Γ.[1]) Then the equilibrium states of a thin elastic plate subjected to a perpendicular loading are characterized by solutions of the following system of partial differential equations (the so-called von Kármán equations):

$$\Delta^2 f = -[w, w] \quad \text{in} \quad \Omega\,, \tag{2.1}$$

$$\Delta^2 w = [f, w] + q \quad \text{in} \quad \Omega\,. \tag{2.2}$$

Here the function $f = f(x, y)$ denotes the stress function, while $w = w(x, y)$ means the deflection of the plate. Δ^2 is the biharmonic operator with respect to the variables x and y, and

$$[u, v] \equiv u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}\,.$$

The perpendicular load is represented by the function q.

We impose upon f the boundary conditions

$$f = f_n = 0 \quad \text{on} \quad \Gamma\,. \tag{2.3}$$

The subscript n denotes the derivative along the outer unit normal $n = (n_x, n_y)$ with respet to Ω.

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[1]) Precise conditions upon Γ will be stated at the beginning of the next section.

The boundary conditions (2.3) imply, in a certain sense, that the edge of the plate is free of lateral tractions.[1])

Remark 2.1. We have restricted ourselves to the homogeneous boundary conditions (2.3) only for the sake of simplicity. The inhomogeneous boundary conditions $f = g_0$, $f_n = g_1$ can be treated by applying the method of reformulation presented in [4] (if corners are permitted on Γ, certain compatibility conditions upon g_0 and g_1 are necessary; cf. [4] for details).

Before we turn to the formulation of the unilateral boundary conditions, we introduce the boundary operators

$$M(u) = \mu \, \Delta u + (1 - \mu)\left(u_{xx}n_x^2 + 2u_{xy}n_xn_y + u_{yy}n_y^2\right),$$

$$T(u) = -\frac{\partial}{\partial n}\Delta u + (1 - \mu)\frac{\partial}{\partial s}\left[u_{xx}n_xn_y - u_{xy}(n_x^2 - n_y^2) - u_{yy}n_xn_y\right],$$

where $\mu = \text{const}\,(0 < \mu < \frac{1}{2})$ is the Poisson ratio of the plate material, $s = (-n_y, n_x)$. . $M(w)$ may be interpreted as the bending moment of the plate along the edge Γ, while $T(w)$ may be understood as the shearing force.

2.1. Conditions with respect to a rotation on Γ. We consider the boundary condition $w = 0$ on Γ, i.e., the plate is supported along Γ. If under this condition the elastic energy of the plate bending vanishes then it follows $w \equiv 0$ (cf. [4]).

Further, let us introduce the boundary condition

$$T(w) + e_0 w = m_0 \quad \text{on} \quad \Gamma$$

where

$$e_0, m_0 \in L^1(\Gamma)^{2)}, \quad e_0 \geqq 0 \quad \text{a.e. on} \quad \Gamma.$$

This condition corresponds to a plate whose edge is elastically supported and loaded by the transversal force m_0 (cf. [4]). The inequality $e_0 \geqq 0$ is based on the fact that the deformation energy of the elastic supports cannot be negative, i.e., $\int_\Gamma e_0 w^2 \, ds \geqq 0$.

In the present Part I, the above boundary condition will be considered under the following additional assumption:

[1]) In order to make this point clearer, let us consider an example of boundary conditions upon f which lead to (2.3), namely

$$f_{yy}n_x - f_{xy}n_y = 0\,, \quad f_{xx}n_y - f_{xy}n_x = 0 \quad \text{on } \Gamma\,,$$

i.e., the lateral tractions vanish along Γ. An easy calculation yields

$$f = A + Bx + Cy\,, \quad f_n = Bn_x + Cn_y \quad \text{on } \Gamma$$

where A, B, C are arbitrary real constants (cf. [4]). Putting $A = B = C = 0$ one gets (2.3).

[2]) We refer to the book [8] for the definition of the spaces $L^p(\Gamma)$.

(*) $\qquad \int_\Gamma e_0 p_1^2 \, ds = 0$ implies $p_1 \equiv 0$ for all polynomials of the degree

$$\leqq 1 \text{ in } x \text{ and } y\,.$$

The latter condition guarantees that if both the energy of the plate bending and that of the elastic supports vanish, then $w \equiv 0$ (cf. [4]).

1° *Unilateral rotation on Γ.* Let us suppose that if the plate rotates from its support, then there are no bending moments, i.e., $w_n > 0$ implies $M(w) = 0$ on Γ. On the other hand, if the bending moments are positive, then the plate is forced onto the support, i.e., $M(w) > 0$ implies $w_n = 0$.

Thus, we have the conditions

$$(2.4) \qquad \begin{cases} w = 0 \quad \text{on} \quad \Gamma\,, \\ w_n = 0\,, \quad M(w) \geqq 0\,, \quad w_n M(w) = 0 \quad \text{on} \quad \Gamma\,,^{1)} \end{cases}$$

(cf. [2], [3]), on

$$(2.5) \qquad \begin{cases} w_n \geqq 0\,, \quad M(w) \geqq 0\,, \quad w_n M(w) = 0 \quad \text{on} \quad \Gamma\,, \\ T(w) + e_0 w = m_0 \quad \text{on} \quad \Gamma\,. \end{cases}$$

In the presence of corners on Γ, (2.5) has to be completed by the conditions

$$(+) \qquad H(w^-) = H(w^+) = 0$$

at the corners, where

$$H(w) = (1 - \mu)\left[w_{xx} n_x n_y - w_{xy}(n_x^2 - n_y^2) - w_{yy} n_x n_y\right]$$

(see [4] for details)[2] The condition $(+)$ may be interpreted as the vanishing of the jump of the twisting moment at the corner under consideration.

Remark 2.2. We introduce the subsets

$$\Gamma' = \{(x, y) \in \Gamma : w_n = 0\}\,, \quad \Gamma'' = \{(x, y) \in \Gamma : w_n > 0\}\,.$$

Thus $\Gamma = \Gamma' \cup \Gamma''$, and one gets from (2.4)

$$w = w_n = 0 \quad \text{on} \quad \Gamma' \quad \text{(clamped part)}\,,$$

$$w = M(w) = 0 \quad \text{on} \quad \Gamma'' \quad \text{(simply supported part)}\,.$$

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[1]) Note that the direction of the corresponding inequalities converse to that in [2], [3] is due to our different notation.

[2]) It is readily seen that our existence theorem (see Section 5) still holds, with the same proof, for inhomogeneous conditions $(+)$.

However, the subsets Γ' and Γ'' are unknown in (2.4).

2° *Rotation with friction.* Let k be a positive constant (connected with the plate material). We assume that if the absolute value of the bending moment $M(w)$ is less than k, then there is no rotation of the points of Γ, i.e., $|M(w)| < k$ implies $w_n = 0$ on Γ. But if $|M(w)|$ attains the value k, rotation of the plate on Γ may take place provided there is a small perturbation.

Describing the situation considereed we obtain the following conditions upon w:

$$\tag{2.6} \begin{cases} w = 0 \quad \text{on} \quad \Gamma\,, \\ |M(w)| \leqq k \quad \text{on} \quad \Gamma : \\ |M(w)| < k \quad \text{implies} \quad w_n = 0\,, \\ M(w) = k \quad \text{implies} \quad w_n \leqq 0\,, \\ M(w) = -k \quad \text{implies} \quad w_n \geqq 0\,, \end{cases}$$

(cf. [2], [3]), and

$$\tag{2.7} \begin{cases} |M(w)| \leqq k \quad \text{on} \quad \Gamma : \\ |M(w)| < k \quad \text{implies} \quad w_n = 0\,, \\ M(w) = k \quad \text{implies} \quad w_n \leqq 0\,, \\ M(w) = -k \quad \text{implies} \quad w_n \geqq 0\,, \\ T(w) + e_0 w = m_0 \quad \text{on} \quad \Gamma\,, \end{cases}$$

where in the latter system conditions of the type $(+)$ have to be included if the boundary is permitted to have corners.

It is readily verified that the system of inequalities in (2.6) (and (2.7) may be replaced equivalently by

$$\tag{2.6'} |M(w)| \leqq k\,, \quad w_n M(w) + k|w_n| = 0 \quad \text{on} \quad \Gamma\,.$$

Remark 2.3. Turning back to (2.6), we define

$$\Gamma^{(1)} = \{(x, y) \in \Gamma : w_n = 0\}\,,$$
$$\Gamma^{(2)} = \{(x, y) \in \Gamma : w_n > 0\}\,, \quad \Gamma^{(3)} = \{(x, y) \in \Gamma : w_n < 0\}\,.$$

Then $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)} \cup \Gamma^{(3)}$, and

$$w = w_n = 0 \quad \text{on} \quad \Gamma^{(1)}\,,$$
$$w = 0\,, \quad M(w) = -k \quad \text{on} \quad \Gamma^{(2)}\,,$$
$$w = 0\,, \quad M(w) = k \quad \text{on} \quad \Gamma^{(3)}\,,$$

i.e., the plate is clamped on $\Gamma^{(1)}$, and supported and loaded by a moment distribution on $\Gamma^{(2)} \cup \Gamma^{(3)}$. As above, the decomposition of Γ is unknown.

2.2 Conditions with respect to a displacement on a part of Γ. In this section we also subject w to boundary conditions such that $w \equiv 0$ if the energy of the plate bending vanishes (cf. [4]).

To this end, let Γ be decomposed into two mutually disjoint (measurable) subsets Γ_0 and Γ_1, i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$. We introduce the conditions

$$w = w_n = 0 \quad \text{on} \quad \Gamma_0\,, \quad M(w) + e_1 w_n = m_1 \quad \text{on} \quad \Gamma_1\,,$$

where

$$e_1, m_1 \in L^p(\Gamma_1)\,(1 < p < \infty)\,, \quad e_1 \geqq 0 \quad \text{a. e. on} \quad \Gamma_1\,.$$

These conditions mean that the plate is clamped along Γ_0, while along Γ_1 it is elastically clamped and loaded by a moment distribution m_1. The inequality $e_1 \geqq 0$ is based on the fact that the deformation energy of the elastic clampings is non-negative. Moreover, we suppose that the following condition is fulfilled:

$$(**) \qquad \operatorname{mes}(\Gamma_0) > 0\,.$$

If (**) holds then the vanishing of the energy of the plate bending implies $w \equiv 0$.

Without further reference, in the presence of corners on Γ_1, the boundary condition (+) is assumed to hold at the corners.

We complete the boundary condition on Γ_1 by various unilateral conditions upon w and $T(w)$.

1° *Unilateral displacement on* Γ_1. Let the plate partially lie on a rigid support. In more detail, if the plate can move from the support then there are no shearing forces, i.e., $w > 0$ implies $T(w) = 0$. On the other hand, if the shearing forces become positive then the plate lies on the support, i.e., $T(w) >$ implies $w = 0$.

Thus, one obtains the system of conditions

$$(2.8) \qquad \begin{cases} w = w_n = 0 \quad \text{on} \quad \Gamma_0\,, \\ \left.\begin{array}{l} w \geqq 0\,, \quad T(w) \geqq 0\,, \quad w\,T(w) = 0 \\ M(w) + e_1 w_n = m_1 \end{array}\right\} \text{on } \Gamma_1\,. \end{cases}$$

2° *Displacement with friction on* Γ_1. Let a positive constant g be given (depending on the friction properties of the plate material). We then assume that no motion of the plate (along Γ_1) takes place if the absolute value of $T(w)$ is less than g, i.e., $|T(w)| < g$ implies $w = 0$. But if $|T(w)|$ attains g, a motion of the plate may occur provided there is a small perturbation on Γ_1.

Describing such a boundary configuration, we state the conditions

$$(2.9) \qquad \begin{cases} w = w_n = 0 \quad \text{on} \quad \Gamma_0\,, \\ |T(w)| \leqq g \quad \text{on} \quad \Gamma_1: \\ |T(w)| < g \quad \text{implies} \quad w = 0\,, \\ T(w) = g \quad \text{implies} \quad w \leqq 0\,, \\ T(w) = -g \quad \text{implies} \quad w \geqq 0\,, \\ M(w) + e_1 w_n = m_1 \quad \text{on} \quad \Gamma_1\,. \end{cases}$$

The system of inequalities in (2.9) is equivalent to

$$(2.9') \qquad |T(w)| \leqq g\,, \quad w\,T(w) + g|w| = 0 \quad \text{on} \quad \Gamma_1\,.$$

3° *Plate elastically constrained on* Γ_1. We suppose that the plate can move freely on Γ_1 if the absolute value of the deflection on Γ_1 remains bounded by a constant, say 1, i.e., $T(w) = 0$ (no shearing forces) if $|w| \leqq 1$. Beyond the bound 1 there are elastic constraints, i.e., $T(w) = -\sigma w(1 - 1/|w|)$ if $|w| > 1$ (here σ is a given positive constant).

We get the system of conditions

$$(2.10) \qquad \begin{aligned} &w = w_n = 0 \quad \text{on} \quad \Gamma_0\,, \\ &\left.\begin{aligned} &T(w) = 0 \quad \text{if} \quad |w| \leqq 1\,, \\ &T(w) = -\sigma w(1 - 1/|w|) \quad \text{if} \quad |w| > 1\,, \\ &M(w) + e_1 w_n = m_1\,. \end{aligned}\right\} \text{on } \Gamma_1\,. \end{aligned}$$

The second group of the conditions in (2.10) corresponds to (13) (second line) and (19) in [3] ($e_1 = m_1 = 0$).

Remark 2.4. The conditions $w = w_n = 0$ on Γ_0 represent only one case of boundary conditions to which the edge of the plate can be subjected along Γ_0. Other conditions are

$$w = 0\,, \quad M(w) + g_1 w_n = r_1 \quad \text{on} \quad \Gamma_0$$

$$(\Gamma_0 \text{ is not a segment of a straight line})\,,$$

or

$$M(w) + k_1 w_n = t_1\,, \quad T(w) + k_0 w = t_0 \quad \text{on} \quad \Gamma_0\,,$$

$$\int_{\Gamma_0} k_0 p_1^2 \, ds = 0 \quad \text{implies} \quad p_1 \equiv 0 \quad \text{for all polymomials of the degree}$$

$$\leqq 1 \text{ in } x \text{ and } y$$

(cf. [4]). Each of these conditions implies $w \equiv 0$ if the energy of the plate bending vanishes.

As will be seen in the following section, the variational formulation of (2.1)–(2.3), (2.8) (or 2.9), (2.10)) can be extended in a straightforward manner to the above boundary conditions.

Remark 2.5. If we are given a decomposition of Γ into disjoint parts, various mixed unilateral boundary conditions of the types stated above can be investigated. Moreover, one can combine unilateral boundary conditions with inhomogeneous boundary conditions (equations) in [4]. For example, if the decomposition $\Gamma = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ is given one may introduce the conditions

$$w = w_n = 0 \quad \text{on} \quad \Gamma_1 ,$$

$$\left.\begin{array}{l} w \geqq 0, \ \ T(w) \geqq 0 , \quad w\, T(w) = 0 \\ M(w) + k_1 w_n = t_1 \end{array}\right\} \text{on } \Gamma_2 ,$$

$$M(w) + e_1 w_n = m_1 , \quad T(w) + e_0 w = m_0 \quad \text{on } \Gamma_3 .$$

The variational formulation of all these more general boundary value problems is included in our unified setting introduced in Section 5.

3. NOTATION. VARIATIONAL FORMULATION

Let the boundary Γ of the domain Ω be Lipschitzian (i.e., Γ is permitted to have corners).[1]

We denote by $L^p(\Omega)$ $(1 \leqq p < \infty)$ the space of all real functions which are integrable with power p on Ω (with respect to the Lebesgue measure $\mathrm{d}x\, \mathrm{d}y$).

Using the notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\, \partial y^{\alpha_2}} , \quad |\alpha| = \alpha_1 + \alpha_2 ,$$

we define for any integer $m \geqq 1$

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \quad \text{for} \quad |\alpha| \leqq m\}$$

(the derivatives are taken in the sense of distributions). $W^{m,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{m,p} = \left\{\int_\Omega |u|^p \,\mathrm{d}x\, \mathrm{d}y + \sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p \,\mathrm{d}x\, \mathrm{d}y\right\}^{1/p} .$$

The scalar product

$$(u, v)_{m,2} = \int_\Omega uv \,\mathrm{d}x\, \mathrm{d}y + \sum_{|\alpha|=m} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x\, \mathrm{d}y$$

turns $W^{m,2}(\Omega)$ a Hilbert space.

Let $\mathscr{D}(\bar{\Omega})$ denote the space of all infinitely continuously differentiable functions in Ω which together with all their derivatives can be continuously extended onto $\bar{\Omega}$.

Putting

$$\mathscr{V}_1 = \{u \in \mathscr{D}(\bar{\Omega}) : u = 0 \text{ on } \Gamma\} ,$$

$$\mathscr{V}_2 = \{u \in \mathscr{D}(\bar{\Omega}) : u = u_n = 0 \text{ on } \Gamma_0\} ,$$

where $\Gamma = \Gamma_0 \cap \Gamma_1$ is the decomposition according to Section 2.2, we introduce the spaces

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[1]) Using the terminology of [8] we write: $\Omega \in \mathfrak{N}^{(0),1}$. This is sufficient for the Sobolev Imbedding Theorem and the Trace Theorem to hold.

$$V_1 = \text{closure of } \mathscr{V}_1 \text{ in } W^{2,2}(\Omega),$$
$$V_2 = \text{closure of } \mathscr{V}_2 \text{ in } W^{2,2}(\Omega).$$

Observing the Sobolev Imbedding Theorem and the Trace Theorem, one obtains

$$u = 0 \quad \text{pointwise on } \Gamma, \quad \forall u \in V_1,$$
$$\left.\begin{array}{ll} u = 0 & \text{pointwise on } \Gamma_0, \\ u_n = 0 & \text{a.e. on } \Gamma_0 \text{ (in the trace sense)} \end{array}\right\} \forall u \in V_2.$$

For treating the boundary value problems stated in Section 2, it is convenient to introduce another product on V_1, $W^{2,2}(\Omega)$ and V_2. To this end, we define for $u, v \in W^{2,2}(\Omega)$ the bilinear form

$$A(u, v) \equiv \int_\Omega [u_{xx}v_{xx} + 2(1 - \mu)\, u_{xy}v_{xy} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx})]\, dx\, dy$$

(recall that μ denotes the Poisson ratio of the plate material).

From [4] we obtain:

There exist constants c_i and c_i' $(i = 1, 2, 3)$ such that

(3.1) $$c_1\|u\|_{2,2}^2 \leqq A(u, u) \leqq c_1'\|u\|_{2,2}^2 \quad \forall u \in V_1;$$

(3.2) $$c_2\|u\|_{2,2}^2 \leqq A(u, u) + \int_\Gamma e_0 u^2\, ds \leqq c_2'\|u\|_{2,2}^2 \quad \forall u \in W^{2,2}(\Omega);$$

(3.3) $$c_3\|u\|_{2,2}^2 \leqq A(u. u) \leqq c_3'\|u\|_{2,2}^2 \quad \forall u \in V_2.$$

From (3.1) or (3.3) we conclude respectively that V_1 and V_2 are Hilbert spaces with respect to the scalar product $A(u, v)$. By (3.2), the scalar product $A(u, v) + \int_\Gamma e_0 uv\, ds$ turns $W^{2,2}(\Omega)$ a Hilbert space. Henceforth, V_1 as well as $W^{2,2}(\Omega)$, V_2 will be understood to be furnished with the scalar product mentioned.

We denote by $\mathscr{D}(\Omega)$ the space of all infinitely continuously differentiable functions having their support in Ω. Let $W_0^{m,2}(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ in $W^{m,2}(\Omega)$. $W_0^{m,2}(\Omega)$ is a Hilbert space with respect to the scalar product

$$(u, v)_{m,2;0} = \sum_{|\alpha|=m} \int_\Omega D^\alpha u\, D^\alpha v\, dx\, dy$$

(cf. [8]). The norms $\|\cdot\|_{m,2}$ and $\|\cdot\|_{m,2;0} = (\cdot,\cdot)_{m,2;0}^{1/2}$ are equivalent on $W_0^{m,2}(\Omega)$. In particular, the equivalence of $\|\cdot\|_{2,2;0}$ to $\{A(\cdot,\cdot)\}^{1/2}$ on the space $W_0^{2,2}(\Omega)$ is readily seen.

Thus, we have

$$W_0^{2,2}(\Omega) \subset V_i \subset W^{2,2}(\Omega), \quad (i = 1, 2)$$

where each injection is continuous.

For our discussion we also need the spaces $W^{s,2}(\Gamma)$ ($s > 0$, real). However, we omitt their detailed definition, and refer to [8] (cf. also [7]).

Let $C(\bar{\Omega})$ be the space of all continuous functions in Ω which have a continuous extension onto $\bar{\Omega}$. Then by $[C(\bar{\Omega})]'$ we denote the dual space of $C(\bar{\Omega})$ and by $\langle m, \varphi \rangle$ the dual pairing between $m \in [C(\bar{\Omega})]'$ and $\varphi \in C(\bar{\Omega})$. Unless otherwise stated, throughout the paper we assume $q \in [C(\bar{\Omega})]'$ (let us refer to [4] for some particular cases of q).

We introduce finally the sets

$$K_1 = \{u \in W^{2,2}(\Omega) : u_n \geqq 0 \text{ a.e. on } \Gamma\},$$
$$K_2 = \{u \in W^{2,2}(\Omega) : u \geqq 0 \text{ on } \Gamma_1\}$$

($\Gamma = \Gamma_0 \cup \Gamma_1$ denotes the decomposition according to Section 2.2). It is readily verified that K_i ($i = 1, 2$) is a closed convex cone in $W^{2,2}(\Omega)$.

Definition 3.1.

1° *The pair* $\{f, w\}$ *is called a variational solution to* (2.1)–(2.3), (2.4) *iff* $f \in W^{2,2}(\Omega)$ $w \in V_1 \cap K_1$, *and if*

$$(3.4_0) \qquad (f, \psi)_{2,2;0} = -\int_\Omega [w, w]\, \psi \,\mathrm{d}x\,\mathrm{d}y \quad \forall \psi \in W_0^{2,2}(\Omega);$$

$$(3.4) \quad A(w, v - w) \geqq \int_\Omega [f, w](v - w)\,\mathrm{d}x\,\mathrm{d}y + \langle q, v - w \rangle \quad \forall v \in V_1 \cap K_1.$$

2° *The pair* $\{f, w\}$ *is called a variational solution to* (2.1)–(2.3), (2.5) *if* $f \in W_0^{2,2}(\Omega)$, $w \in K_1$, $\{f, w\}$ *satisfies* (3.4_0), *and if*

$$(3.5) \qquad A(w, v - w) + \int_\Gamma e_0 w(v - w)\,\mathrm{d}s \geqq$$
$$\geqq \int_\Omega [f, w](v - w)\,\mathrm{d}x\,\mathrm{d}y + \int_\Gamma m_0(v - w)\,\mathrm{d}s + \langle q, v - w \rangle \quad \forall v \in K_1.$$

3° *The pair* $\{f, w\} \in W_0^{2,2}(\Omega) \times V_1$ *is called a variational solution to* (2.1)–(2.3), (2.6) *if* $\{f, w\}$ *satisfies* (3.4_0), *and if*

$$(3.6) \qquad A(w, v - w) + k\int_\Gamma |v_n|\,\mathrm{d}s - k\int_\Gamma |w_n|\,\mathrm{d}s \geqq$$
$$\geqq \int_\Omega [f, w](v - w)\,\mathrm{d}x\,\mathrm{d}y + \langle q, v - w \rangle \quad \forall v \in V_1.$$

4° *The pair* $\{f, w\} \in W_0^{2,2}(\Omega) \times W^{2,2}(\Omega)$ *is called a variational solution to* (2.1) – (2.3), (2.7) *if* $\{f, w\}$ *satisfies* (3.4_0), *and if*

$$(3.7) \qquad A(w, v - w) + \int_\Gamma e_0 w(v - w)\,ds + k\int_\Gamma |v_n|\,ds - k\int_\Gamma |w_n|\,ds \geqq$$

$$\geqq \int_\Omega [f, w]\,(v - w)\,dx\,dy + \int_\Gamma m_0(v - w)\,ds + \langle q, v - w\rangle \quad \forall v \in W^{2,2}(\Omega)\,.$$

In order to motivate the definitions introduced let $\{f, w\}$ be a sufficiently regular solution to (2.1)–(2.3), (2.4), and let $q \in L^1(\Omega)$. Moreover, suppose that the boundary Γ is infinitely differentiable.

Multiplying (2.1) by the test function ψ and integrating by parts the expression $\Delta^2 f\psi$, the integral identity (3.4_0) is easily obtained.

Next, $w = 0$ on Γ implies $w \in V_1$ (see Section 4), and since $w_n \geqq 0$ on Γ, it holds $w \in V_1 \cap K_1$. We now multiply (2.2) by $v - w$ and integrate by parts $\Delta^2 w(v - w)$. We get

$$A(w, v - w) =$$

$$= \int_\Omega \Delta^2 w(v - w)\,dx\,dy + \int_\Gamma T(w)\,(v - w)\,ds + \int_\Gamma M(w)\,(v_n - w_n)\,ds =$$

$$= \int_\Omega [f, w]\,(v - w)\,dx\,dy + \int_\Omega q(v - w)\,dx\,dy +$$

$$+ \int_\Gamma T(w)\,(v - w)\,ds + \int_\Gamma M(w)\,(v_n - w_n)\,ds$$

for arbitrary $v \in W^{2,2}(\Omega)$ (cf. [8]). By (2.4),

$$\int_\Gamma T(w)\,(v - w)\,ds + \int_\Gamma M(w)\,(v_n - w_n)\,ds = \int_\Gamma M(w)\,v_n\,ds \geqq 0 \quad \forall v \in V_1 \cap K_1\,,$$

and (3.4) is obtained immediately.

If $\{f, w\}$ is a sufficiently regular solution to (2.1)–(2.3), (2.5), we have

$$\int_\Gamma T(w)\,(v - w)\,ds + \int_\Gamma M(w)\,(v_n - w_n)\,ds \geqq \int_\Gamma (-e_0 w + m_0)\,(v - w)\,ds \quad \forall v \in K_1$$

which implies (3.5).

In order to obtain the inequality in (3.6) we remark that the system of inequalities in (2.6) (and (2.7)) is equivalent to

$$k|h| - k|w_n| \geqq [-M(w)]\,(h - w_n)$$

$$\text{a.e. on } \Gamma, \text{ for any } h \in L^2(\Gamma)$$

(cf. also (2.6′)). Therefore, if w satisfies (2.6) one obtains

$$\int_\Gamma T(w)(v - w)\,\mathrm{d}s + \int_\Gamma M(w)(v_n - w_n)\,\mathrm{d}s \geqq -k\int_\Gamma |v_n|\,\mathrm{d}s + k\int_\Gamma |w_n|\,\mathrm{d}s \quad \forall v \in V_1,$$

and if w satisfies (2.7),

$$\int_\Gamma T(w)(v - w)\,\mathrm{d}s + \int_\Gamma M(w)(v_n - w_n)\,\mathrm{d}s \geqq$$

$$\geqq \int_\Gamma (-e_0 w + m_0)(v - w)\,\mathrm{d}s - k\int_\Gamma |v_n|\,\mathrm{d}s + k\int_\Gamma |w_n|\,\mathrm{d}s \quad \forall v \in W^{2,2}(\Omega)\,.$$

Thus, (3.6) and (3.7) are satisfied by a sufficiently regular solution to (2.1)–(2.3) with (2.6) and (2.7)), respectively.

Definition 3.2.

1° *The pair* $\{f, w\}$ *is called a variational solution to* (2.1)–(2.3), (2.8) *if* $f \in W_0^{2,2}(\Omega)$ $w \in V_2 \cap K_2$, $\{f, w\}$ *satisfies* (3.4_0), *and if*

$$(3.8) \qquad A(w, v - w) + \int_{\Gamma_1} e_1 w_n(v_n - w_n)\,\mathrm{d}s \geqq$$

$$\geqq \int_\Omega [f, w](v - w)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_1} m_1(v_n - w_n)\,\mathrm{d}s + \langle q, v - w\rangle \quad \forall v \in V_2 \cap K_2\,.$$

2° *The pair* $\{f, w\} \in W_0^{2,2}(\Omega) \times V_2$ *is called a variational solution to* (2.1)–(2.3), (2.9) *if* $\{f, w\}$ *satisfies* (3.4_0), *and if*

$$(3.9) \qquad A(w, v - w) + \int_{\Gamma_1} e_1 w_n(v_n - w_n)\,\mathrm{d}s + g\int_{\Gamma_1} |v|\,\mathrm{d}s - g\int_{\Gamma_1} |w|\,\mathrm{d}s \geqq$$

$$\geqq \int_\Omega [f, w](v - w)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_1} m_1(v_n - w_n)\,\mathrm{d}s + \langle q, v - w\rangle \quad \forall v \in V_2\,.$$

3° *The pair* $\{f, w\} \in W_0^{2,2}(\Omega) \times V_2$ *is called a variational solution to* (2.1)–(2.3), (2.10) if $\{f, w\}$ *satisfies* (3.4_0), *and if*

$$(3.10) \qquad A(w, v - w) + \int_{\Gamma_1} e_1 w_n(v_n - w_n)\,\mathrm{d}s + \sigma\int_{\Gamma_1} j_1(v)\,\mathrm{d}s - \sigma\int_{\Gamma_1} j_1(w)\,\mathrm{d}s \geqq$$

$$\geqq \int_\Omega [f, w](v - w)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_1} m_1(v_n - w_n)\,\mathrm{d}s + \langle q, v - w\rangle \quad \forall v \in V_2\,.$$

In (3.10) we have used the notation

$$j_1(r) = \begin{cases} r(\frac{1}{2}r + 1) & \text{if} \quad r < -1\,, \\ -\frac{1}{2} & \text{if} \quad -1 \leqq r \leqq 1\,, \\ r(\frac{1}{2}r - 1) & \text{if} \quad r > 1\,. \end{cases}$$

The conditions upon $T(w)$ in (2.10) are then equivalent to

$$\sigma j_1(v) - \sigma j_1(w) \geqq [-T(w)] (v - w)$$
$$\text{a.e. on } \Gamma_1, \text{ for all } v \in L^2(\Gamma_1).$$

Indeed, we have

$$-T(w) = \begin{cases} \sigma(w+1) & \text{if} \quad w < -1 , \\ 0 & \text{if} \quad -1 \leqq w \leqq 1 , \\ \sigma(w-1) & \text{if} \quad w > 1 . \end{cases}$$

Observing that

$$j_1'(r) = \begin{cases} r+1 & \text{if} \quad r < -1 , \\ 0 & \text{if} \quad -1 \leqq r \leqq 1 , \\ r-1 & \text{if} \quad r > 1 , \end{cases}$$

and

$$j_1(s) - j_1(r) \geqq j_1'(r)(s-r) \quad \forall s, r \in R^1 ,$$

one obtains the desired assertion.

The Definitions 3.2, 1°, 2°, 3° originate from (2.1)–(2.3) and (2.8), (2.9), (2.10), respectively, by similar considerations as above.

4. INTERPRETATION OF THE EQUATIONS AND THE UNILATERAL BOUNDARY CONDITIONS

In the present section, we discuss the problem in which sense the equations (2.1), (2.2) and the boundary conditions in (2.4)–(2.7) are satisfied by the corresponding variational solution.

Let $\{f, w\}$ be a variational solution to (2.1)–(2.4) (cf. Definition 3.1, 1°). Since $[u, v] \in L^1(\Omega)$ for any $u, v \in W^{2,2}(\Omega)$, the integral identity in (3.4_0) immediately yields

$$\Delta^2 f = -[w, w] \text{ in the sense of } \mathscr{D}'(\Omega) .^1)$$

Let $\varphi \in \mathscr{D}(\Omega)$ be arbitrary. It holds $v = w + \varphi \in V_1 \cap K_1$, and the inequality in (3.4) implies

$$A(w, \varphi) = \int_\Omega [f, w] \varphi \, dx \, dy + \langle q, \varphi \rangle .$$

In virtue of $q \in [C(\bar{\Omega})]'$ it follows

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[1]) $\mathscr{D}'(\Omega)$ denotes the space of distributions in Ω. — Note that the equation $\Delta^2 f = -[w, w]$ in fact holds in the subspace $W^{-2,2}(\Omega)$ (= the dual of $W_0^{2,2}(\Omega)$) of $\mathscr{D}'(\Omega)$. This follows from the imbedding $L^1(\Omega) \subset W^{-2,2}(\Omega)$.

$$(4.1) \qquad \Delta^2 w = [f, w] + q \quad \text{in the sense of } \mathscr{D}'(\Omega)\,.$$

In order to proceed to the discussion of the (unstable) boundary conditions, we make the following assumptions:

(a) *Γ is infinitely differentiable;*[1])

(b) $[f, w] \in L^2(\Omega)\,, \quad q \in L^2(\Omega)\,.$

Remark 4.1. Assuming (a), the application of the elliptic regularity theory to the Dirichlet problem $\Delta^2 f = -[w, w]$ in Ω, $f = f_n = 0$ on Γ (where $w \in W^{2,2}(\Omega)$) yields $f \in W^{3,r}(\Omega)$, $1 \leqq r < 2$ (cf. [5]). This implies $D^\alpha f \in L^{2r/(2-r)}(\Omega)$ for $|\alpha| = 2$, and thus $[f, w] \in L^r(\Omega)$.

Regularity results for the variational solution to (2.1), (2.2) in the case of the boundary conditions $w = M(w) = 0$ and $M(w) = T(w) = 0$ on Γ, respectively, have been obtained in [5]. In particular, $[f, w] \in L^2(\Omega)$ is obviously satisfied.

However, analogous regularity results for variational solutions of our unilateral problems are unknown (even in the linear case $\Delta^2 w = q$). Therefore, to give an interpretation of the unilateral boundary conditions stated in Section 2.1, we are forced to assume the regularity property $[f, w] \in L^2(\Omega)$.

Let the assumptions (a), (b) be satisfied throughout the present section.

We have

$$w \in W^{2,2}(\Omega)\,, \quad \Delta^2 w \in L^2(\Omega)\,.$$

From [7, Chapter 2] one concludes that

$$(4.2) \qquad M(w) \in W^{-1/2,2}(\Gamma)\,, \quad T(w) \in W^{-3/2,2}(\Gamma)\,,^{2)}$$

$$(4.3) \quad A(w, v) = \int_\Omega \Delta^2 w v \,\mathrm{d}x\,\mathrm{d}y + \langle T(w), v\rangle_{3/2} + \langle M(w), v_n\rangle_{1/2} \quad \forall v \in W^{2,2}(\Omega)\,.$$

Remark 4.2. In order to obtain (4.2), (4.3) the proper ellipticity of Δ^2 and the normality of the system M, T are necessary. This is verified in detail in [5].

Observing (4.1) (which in fact is true in $L^2(\Omega)$) one derives from (4.3)

$$\langle T(w), v\rangle_{3/2} + \langle M(w), v_n - w_n\rangle_{1/2} =$$

$$= A(w, v - w) - \int_\Omega [f, w]\,(v - w)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega q(v - w)\,\mathrm{d}x\,\mathrm{d}y$$

---

[1]) We refer to [7] for the definition. — This assumption enables us to apply results from [7].

[2]) One defines $W^{-s,2}(\Gamma)=$ dual of $W^{s,2}(\Gamma)$ ($s > 0$, real). From [7, Chapt. 1] we conclude that $W^{s,2}(\Gamma)$ is continuously and densely imbedded in $L^2(\Gamma)$. Identifying $L^2(\Gamma)$ with its dual we obtain $W^{s,2}(\Gamma) \subset L^2(\Gamma) \subset W^{-s,2}(\Gamma)$ (where the latter imbedding is also continuous and dense). If $\langle u, v\rangle_s$ denotes the dual pairing between $u \in W^{-s,2}(\Gamma)$ and $v \in W^{s,2}(\Gamma)$ it holds $\langle h, v\rangle_s = \int_\Gamma hv\,\mathrm{d}s$ for all $h \in L^2(\Gamma)$, $v \in W^{s,2}(\Gamma)$.

for all $v \in W^{2,2}(\Omega)$. By (3.4),

$$M(w), v_n - q_n\rangle_{1/2} \geqq 0 \quad \forall v \in V_1 \cap K_1 .$$

Thus,

$$(4.4) \qquad \begin{cases} w = 0 \quad \text{pointwise on } \Gamma, \\ w_n \geqq 0 \quad \text{a.e. on } \Gamma \text{ (in the trace sense)}, \\ \langle M(w), v_n\rangle_{1/2} \geqq 0 \quad \forall v \in V_1 \cap K_1\,, \quad \langle M(w), w_n\rangle_{1/2} = 0\,. \end{cases}$$

We may consider the conditions upon $M(w)$ in (4.4) as a generalization of the corresponding conditions in (2.4) (second line). Indeed, let $M(w) \in L^2(\Gamma)$. We then have

$$\int_\Gamma M(w)\, v_n \,\mathrm{d}s \geqq 0 \quad \forall v \in V_1 \cap K_1\,, \quad \int_\Gamma M(w)\, w_n \,\mathrm{d}s = 0\,.$$

By a standard argument one concludes that $M(w) \geqq 0$ a.e. on Γ, and that $w_n M(w) = 0$ a.e. on Γ.

Let now $\{f, w\}$ be a variational solution to (2.1)–(2.3), (2.5) (cf. Definition 3.1, 2°). As above, (4.1) follows immediately. By virtue the assumptions (a) and (b) we have (4.2) and (4.3). Taking into account (3.5) we obtain

$$\langle T(w), v - w\rangle_{3/2} + \langle M(w), v_n - w_n\rangle_{1/2} + \int_\Gamma e_0 w(v - w)\,\mathrm{d}s -$$

$$- \int_\Gamma m_0(v - w)\,\mathrm{d}s = \mathrm{A}(w, v - w) + \int_\Gamma e_0 w(v - w)\,\mathrm{d}s -$$

$$- \int_\Omega [f, w]\,(v - w)\,\mathrm{d}x\,\mathrm{d}y - \int_\Gamma m_0(v - w)\,\mathrm{d}s - \int_\Omega q(v - w)\,\mathrm{d}x\,\mathrm{d}y \geqq 0 \quad \forall v \in K_1\,.$$

In order to proceed further we suppose

$$(4.5) \qquad e_0, m_0 \in L^2(\Gamma)\,.^{1)}$$

By means of (4.5) the above inequality can be written in the form

$$(4.6) \qquad \langle T(w) + e_0 w - m_0, v - w\rangle_{3/2} + \langle M(w), v_n - w_n\rangle_{1/2} \geqq 0 \quad \forall v \in K_1\,.$$

Let $h \in W^{3/2,2}(\Gamma)$ be arbitrarily given. By the surjectivity of the trace mapping (cf. [7]), there exists *a* $\bar{v} \in W^{2,2}(\Omega)$ such that $\bar{v} = h$, $\bar{v}_n = 0$ on Γ. Set $v = \bar{v} + w$. Then $v \in K_1$, and (4.6) changes into

$$\langle T(w) + e_0 w - m_0, h\rangle_{3/2} = 0 \quad \forall h \in W^{3/2,2}(\Gamma)\,.$$

---

[1]) Since $w \in C(\bar{\Omega})$ it holds $e_0 w \in L^2(\Gamma)$. Therefore, $e_0 w$ and m_0 may be identified with an element in $W^{-3/2,2}(\Gamma)$.

We obtain

$$w_n \geqq 0 \quad \text{a.e. on } \Gamma \text{ (in the trace sense)},$$
$$\langle M(w), v_n \rangle_{1/2} \geqq 0 \quad \forall v \in K_1 , \quad \langle M(w), w_n \rangle_{1/2} = 0 ,$$
$$T(w) + e_0 w - m_0 = 0 \quad \text{in the sense of } W^{-3/2,2}(\Gamma) .$$

These conditions represent a generalization of (2.5): if $M(w) \in L^2(\Gamma)$ then it holds $M(w) \geqq 0$, $w_n M(w) = 0$ a.e. on Γ, and if $T(w) \in L^2(\Gamma)$ one gets $T(w) + e_0 w - m_0 = 0$ a.e. on Γ.

Let $\{f, w\}$ be a variational solution to (2.1)–(2.3), (2.6) (cf. Definition 3.1, 3°). We discuss the intepretation of the system of inequalities in (2.6).

By (3.6),

$$\langle M(w), v_n - w_n \rangle_{1/2} + k \int_\Gamma |v_n| \, ds - k \int_\Gamma |w_n| \, ds =$$
$$= A(w, v - w) + k \int_\Gamma |v_n| \, ds - k \int_\Gamma |w_n| \, ds -$$
$$- \int_\Omega [f, w] (v - w) \, dx \, dy - \int_\Omega q(v - w) \, dx \, dy \geqq 0 \quad \forall v \in V_1 .$$

Hence

$$k \int_\Gamma |v_n| \, ds - k \int_\Gamma |w_n| \, ds \geqq \langle -M(w), v_n - w_n \rangle_{1/2} \quad \forall v \in V_1 . \tag{4.7}$$

An easy calculation yields that (4.7) is equivalent to

$$|\langle M(w), v_n \rangle_{1/2}| \leqq k \int_\Gamma |v_n| \, ds \quad \forall v \in V_1 ,$$
$$\langle M(w), w_n \rangle_{1/2} + k \int_\Gamma |w_n| \, ds = 0 .$$

The latter may be regarded as a generalization of (2.6′). To make this clearer, let $M(w) \in L^2(\Gamma)$. Then (4.7) turns into

$$k \int_\Gamma |v_n| \, ds - k \int_\Gamma |w_n| \, ds \geqq \int_\Gamma [-M(w)] (v_n - w_n) \, ds \quad \forall v \in V_1 .$$

To proceed we note that

$$V_1 = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) .^{1)}$$

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1) *Ladyženskaja O. A.* and *Uraltseva, N. N.:* Linear and quasilinear equations of elliptic type (Russian), 2nd ed., Moscow 1973; p. 218.

Since the $v_{n|\Gamma}$ fill the whole space $W^{1/2,2}(\Gamma)$ if v runs over V_1, and since $W^{1/2,2}(\Gamma)$ is dense in $L^2(\Gamma)$ (see [7]) we get

$$k\int_\Gamma |h|\,\mathrm{d}s - k\int_\Gamma |w_n|\,\mathrm{d}s \geqq \int_\Gamma [-M(w)]\,(h - w_n)\,\mathrm{d}s \quad \forall h \in L^2(\Gamma)\,.$$

This is equivalent to

$$k|h| - k|w_n| \geqq [-M(w)]\,(h - w_n)$$
$$\text{a.e. on } \Gamma, \text{ for all } h \in L^2(\Gamma)$$

(cf. [1, Appendix 1]); we have an equivalent formulation (a.e. on Γ) of (2.6′).

Finally, if $\{f, w\}$ is a variational solution to (2.1)–(2.3), (2.7) we derive from (3.7)

$$\langle T(w), v - w\rangle_{3/2} + \langle M(w), v_n - w_n\rangle_{1/2} +$$
$$+ \int_\Gamma e_0 w(v - w)\,\mathrm{d}s - \int_\Gamma m_0(v - w)\,\mathrm{d}s + k\int_\Gamma |v_n|\,\mathrm{d}s - k\int_\Gamma |w_n|\,\mathrm{d}s =$$
$$= A(w, v - w) + \int_\Gamma e_0 w(v - w)\,\mathrm{d}s + k\int_\Gamma |v_n|\,\mathrm{d}s - k\int_\Gamma |w_n|\,\mathrm{d}s -$$
$$- \int_\Omega [f, w]\,(v - w)\,\mathrm{d}x\,\mathrm{d}y - \int_\Gamma m_0(v - w)\,\mathrm{d}s - \int_\Omega q(v - w)\,\mathrm{d}x\,\mathrm{d}y \geqq 0$$
$$\forall v \in W^{2,2}(\Omega)\,.$$

Assuming again (4.5) one finds

$$\langle T(w + e_0 w - m_0, v - w\rangle_{3/2} +$$
$$+ \langle M(w), v_n - w_n\rangle_{1/2} + k\int_\Gamma |v_n|\,\mathrm{d}s - k\int_\Gamma |w_n|\,\mathrm{d}s \geqq 0 \quad \forall v \in W^{2,2}(\Omega)\,.$$

Thus

$$k\int_\Gamma |v_n|\,\mathrm{d}s - k\int_\Gamma |w_n|\,\mathrm{d}s \geqq \langle -M(w), v_n - w_n\rangle_{1/2} \quad \forall v \in W^{2,2}(\Omega)\,,$$
$$T(w) + e_0 w - m_0 = 0 \quad \text{in the sense of } W^{-3/2,2}(\Gamma).$$

As is readily seen by repeating our above arguments, one may consider this system a generalization of (2.7).

5. UNIFICATION. STATEMENT OF THE MAIN RESULT

We now introduce a unified variational formulation of the unilateral boundary value problems for the system (2.1), (2.2) which are considered in Section 2. This unified setting includes also the variational formulation of the homogeneous cases of the boundary value problems studied in [4].

Suppose we are given the following data:

$$(5.1)\qquad \begin{cases} \text{a bilinear boundary form} \\ a(u, v) = \displaystyle\int_\Gamma e_0 uv \,\mathrm{d}s + \int_\Gamma e_1 u_n v_n \,\mathrm{d}s \quad (u, v \in W^{2,2}(\Omega)) \\ \text{where } e_0 \in L^1(\Gamma)\,, \quad e_1 \in L^p(\Gamma)\,(1 < p < \infty)\,; \end{cases}$$

$$(5.2)\qquad \begin{cases} \text{a Hilbert space } V \text{ whose scalar product is defined by } (u, v) = A(u, v) + \\ + a(u, v) \text{ such that } W_0^{2,2}(\Omega) \subset V \subseteq W^{2,2}(\Omega) \text{ where each injection is} \\ \text{continuous;}^{1)} \end{cases}$$

(5.3) a convex, lower semi-continuous functional $\Phi\colon V \to (-\infty, +\infty]$, $\Phi \not\equiv +\infty$ (V according to (5.2));

(5.4) $m_0 \in L^1(\Gamma)\,, \quad m_1 \in L^p(\Gamma)\,(1 < p < \infty)\,;$

(5.5) $q \in [C(\bar{\Omega})]'\,.$

We introduce

Definition 5.1. *The pair* $\{f, u\} \in W_0^{2,2}(\Omega) \times V$ *is called a variational solution to the system* (2.1), (2.2) *under the boundary conditions* (2.3) *and associated with the functional* Φ *if*

$$(5.6)\qquad (f, \psi)_{2,2;0} = -\int_\Omega [u, u]\,\psi \,\mathrm{d}x\,\mathrm{d}y \quad \forall \psi \in W_0^{2,2}(\Omega)\,,$$

$$(5.7)\quad A(u, v - u) + a(u, v - u) + \Phi(v) - \Phi(u) \geqq \int_\Omega [f, u]\,(v - u) \,\mathrm{d}x\,\mathrm{d}y +$$
$$+ \int_\Gamma m_0(v - u) \,\mathrm{d}s + \int_\Gamma m_1(v_n - u_n) \,\mathrm{d}s + \langle q, v - u\rangle \quad \forall v \in V\,.$$

Note that (5.7) includes the following facts: The equation (2.2) is satisfied in the sense of distributions in Ω. Further, $u \in V$ implies stable boundary conditions upon u. The functional Φ involves stable and unstable unilateral boundary conditions upon u, $T(u)$ or u_n, $M(u)$, respectively, completed by conditions arising from the boundary form a. Inhomogeneities with respect to the unstable boundary conditions are

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[1]) $A(u, v)$ denotes the bilinear form (cf. Section 3)

$$A(u, v) \doteq \int_\Omega [u_{xx}v_{xx} + 2(1 - \mu)\, u_{xy}v_{xy} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx})] \,\mathrm{d}x\,\mathrm{d}y\,.$$

represented by m_0 and m_1. Finally, if unstable boundary conditions are required on a part of Γ which possesses corners, the Condition (+) (Section 2.1) is assumed to hold at the corners.

Passing to the discussion of Definition 5.1, we remark first of all that it includes the basic case of Definition 3.1 in [4]. Indeed, (5.2) is fulfilled by the construction in [4]. Putting $\Phi \equiv 0$, (5.7) turns into the equation

$$A(u, v) + a(u, v) = \int_\Omega [f, u]\, v \,\mathrm{d}x\,\mathrm{d}y + \int_\Gamma m_0 v \,\mathrm{d}s + \int_\Gamma m_1 v_n \,\mathrm{d}s + \langle q, v\rangle$$

which is true for all $v \in V$. Specifying e_0, e_1 and m_0, m_1 in an appropriate manner and integrating by parts the expression $[f, u]\, v$, the equivalence of the latter identity to (3.5) in [4] is easily seen. Obviously, (5.6) is equivalent to (3.6) in [4].

We are going to show that Definitions 3.1, 1°–4° are included in Definition 5.1 as special cases. Clearly, it suffices to restrict the attention to (5.7).

In order to obtain (3.4) from (5.7), we specify the data (5.1)–(5.4) as follows:

(i) $e_0 = e_1 = 0$,
$V = V_1$,
$\Phi = I_{K_1}$,
$m_0 = m_1 = 0$.

Here we have used the notation

$$I_{K_1}(u) = \begin{cases} 0 & \text{if } u \in K_1, \\ +\infty & \text{if } u \notin K_1, \end{cases}$$

where $u \in W^{2,2}(\Omega)$. I_{K_1} is convex and lower semi-continuous on $W^{2,2}(\Omega)$. Clearly, (5.2) is ensured by (3.1).

By the choice (i), (5.7) changes into

$$A(u, v - u) + I_{K_1}(v) - I_{K_1}(u) \geqq$$
$$\geqq \int_\Omega [f, u]\,(v - u)\,\mathrm{d}x\,\mathrm{d}y + \langle q, v - u\rangle \quad \forall v \in V_1 .$$

This is equivalent to (3.4).

Next, we choose the data

(ii) $e_0 \in L^1(\Gamma)$, e_0 satisfies the Condition (*) (Section 2.1); $e_1 = 0$,
$V = W^{2,2}(\Omega)$,
$\Phi = I_{K_1}$,
$m_0 \in L^1(\Gamma)$, $m_1 = 0$.

Here (5.2) is guaranteed by (3.2). It is readily verified that in the present case, (5.7) is equivalent to (3.5).

Set

$$\Phi(u) = k \int_{\Gamma} |u_n| \, \mathrm{d}s \quad \text{for} \quad u \in W^{2,2}(\Omega)$$

($k = \text{const} > 0$; cf. Section 2.1). Φ is convex and lower semi-continuous on $W^{2,2}(\Omega)$. Preserving the remaining data in (i) and (ii), (5.7) is identical respectively with (3.6) and (3.7).

Proceeding in a similar way, it is easy to check that (3.8), (3.8) and (3.10) follow from (5.7) by specifying the data (5.1)–(5.4).

We state the main result of our paper.

Theorem. *Let the data* (5.1)–(5.5) *be given. Moreover, suppose that*

(5.8) $$V = \text{closure of } \mathscr{V} \text{ in } W^{2,2}(\Omega) \text{ where } \mathscr{D}(\Omega) \subset \mathscr{V} \subseteq \mathscr{D}(\bar{\Omega})\,.$$

Then the system (2.1), (2.2) *under the boundary conditions* (2.3) *and associated with the functional* Φ *possesses a variational solution.*

Corollary. *For sufficiently small* $\|u\|$, *the variational solution is unique.*

6. PROOF OF THE THEOREM

First, we present two estimates which are useful for our further purposes.

Let $u, v, w \in W^{2,2}(\Omega)$ be arbitrarily given. By Sobolev's Imbedding Theorem and Hölder's inequality,

(6.1) $$\left| \int_{\Omega} u_{xy} v_y w_x \, \mathrm{d}x \, \mathrm{d}y \right| \leqq \text{const} \, \|u\|_{2,2} \, \|v\|_{1,4} \, \|w\|_{1,4} \,.$$

Using again Sobolev's Imbedding Theorem, one obtains for the functions under consideration

(6.2) $$\left| \int_{\Omega} u_{xx} v_{yy} w \, \mathrm{d}x \, \mathrm{d}y \right| \leqq \text{const} \, \|u\|_{2,2} \, \|v\|_{2,2} \, \|w\|_{C(\bar{\Omega})} \,.$$

Let us now consider the integral one the right hand side in (5.6). For arbitrary $u \in \mathscr{V}$, $v \in V$ and $\psi \in W_0^{2,2}(\Omega)$, we obtain by integration by parts

(6.3) $$\int_{\Omega} [u, v] \, \psi \, \mathrm{d}x \, \mathrm{d}y =$$

$$= \int_{\Omega} [(u_{xy} v_y - u_{yy} v_x) \, \psi_x + (u_{xy} v_x - u_{xx} v_y) \, \psi_y] \, \mathrm{d}x \, \mathrm{d}y \,.$$

Due to the assumption (5.8), the estimates of the type (6.1) and (6.2) imply that (6.3) in fact is valid for any $u \in V$, and we conclude

$$\left|\int_\Omega [u, v]\, \psi \,dx\,dy\right| \leqq (\text{const}\ \|u\|_{2,2}\ \|v\|_{1,4})\ \|\psi\|_{2,2;0} \tag{6.4}$$

for all $u, v \in V$, $\psi \in W_0^{2,2}(\Omega)$. Thus, there exists a unique element $C_1(u, v) \in W_0^{2,2}(\Omega)$ such that

$$(C_1(u, v), \psi)_{2,2;0} = -\int_\Omega [u, v]\, \psi \,dx\,dy \quad \forall \psi \in W_0^{2,2}(\Omega). \tag{6.5}$$

The integral identity (5.6) is now equivalent to

$$f = C_1(u, u) \quad \text{in} \quad W_0^{2,2}(\Omega). \tag{6.6}$$

Repeating our above argument, we get the estimate

$$\left|\int_\Omega [\chi, u]\, v \,dx\,dy\right| \leqq (\text{const}\ \|\chi\|_{2,2}\ \|u\|_{1,3})\ \|v\|^{1)} \tag{6.7}$$

which is valid for arbitrary $\chi \in W_0^{2,2}(\Omega)$, $u, v \in V$. One obtains the existence of a unique $C_2(\chi, u) \in V$ such that

$$(C_2(\chi, u), v) = \int_\Omega [\chi, u]\, v \,dx\,dy \quad \forall v \in V. \tag{6.8}$$

Finally, by Sobolev's Imbedding Theorem and the Trace Theorem, the estimate

$$\left|\int_\Gamma m_0 v \,ds + \int_\Gamma m_1 v_n \,ds + \langle q, v\rangle\right| \leqq \text{const}\ \|v\|$$

holds for all $v \in V$. This implies the existence of a unique $w_0 \in V$ such that

$$(w_0, v) = \int_\Gamma m_0 v \,ds + \int_\Gamma m_1 v_n \,ds + \langle q, v\rangle \quad \forall v \in V. \tag{6.9}$$

We introduce the notation

$$C(u) \equiv C_2(C_1(u, u), u) \quad \text{for} \quad u \in V.$$

Then: *The determination of a variational solution to* (2.1), (2.2) *under the boundary conditions* (2.3) *and associated with the functional* Φ *is equivalent to the problem*:

Find $u \in V$ *such that*

$$(u - C(u) - w_0, v - u) + \Phi(v) - \Phi(u) \geqq 0 \quad \forall v \in V. \tag{6.10}$$

[1]) Note that in deriving this estimate we have used the density of $\mathscr{D}(\Omega)$ in $W_0^{2,2}(\Omega)$ and the continuity of the imbedding $V \subseteq W^{2,2}(\Omega)$ (cf. (5.2)); $\|\cdot\| = (\cdot, \cdot)^{1/2}$.

Indeed, let $\{f, u\}$ be a variational solution to (2.1), (2.2) under the boundary conditions (2.3) and associated with the functional Φ. With regard to (6.8) and (6.9), the inequality in (5.7) can be written in the form

$$(u - C_2(f, u) - w_0, v - u) + \Phi(v) - \Phi(u) \geqq 0 \quad \forall v \in V.$$

Inserting f from (6.6) one gets (6.10).

Conversely, if $u \in V$ satisfies (6.10) we set $f = C_1(u, u)$. Then

$$f = C_1(u, u),$$

$$(u - C_2(f, u) - w_0, v - u) + \Phi(v) - \Phi(u) \geqq 0 \quad \forall v \in V.$$

But this means that $\{f, u\}$ is a variational solution to (2.1), (2.2) under the boundary conditions (2.3) and associated with the functional Φ.

To prove the existence of a $u \in V$ which satisfies (6.10), we show:

(i) It holds

$$\frac{(u - C(u), u) + \Phi(u)}{\|u\|} \to +\infty \quad \text{as} \quad \|u\| \to \infty;$$

(ii) if $\{u_j\} \subset V$ is any sequence such that $u_j \to u$ weakly in V and

$$\limsup\,(u_j - C(u_j), u_j - u) \leqq 0,$$

it follows

$$(u - C(u), u - v) \leqq \liminf\,(u_j - C(u_j), u_j - v)$$

for all $v \in V$.

Establishing these points, from [6, Chapter II, Theorem 8.5] we obtain the existence of a solution of (6.10).

Proof of (i). Observing the defining relations (6.5) and (6.8), one obtains, for arbitrary $\chi \in W_0^{2,2}(\Omega)$, $u, v \in V$,

$$\begin{aligned}(C_2(\chi, u), v) &= \int_\Omega [\chi, u]\, v \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega [u, \chi]\, v \,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_\Omega [(u_{xy}\chi_y - u_{yy}\chi_x)\, v_x + (u_{xy}\chi_x - u_{xx}\chi_y)\, v_y] \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega [(u_{xy}v_y - u_{yy}v_x)\, \chi_x + (u_{xy}v_x - u_{xx}v_y)\, \chi_y] \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega [u, v]\, \chi \,\mathrm{d}x\,\mathrm{d}y = -(C_1(u, v), \chi)_{2,2;0}\,.\end{aligned}$$

For $v = u$, $\chi = C_1(u, u)$ it follows

$$(6.11) \qquad (C(u), u) = -(C_1(u, u), C_1(u, u))_{2,2;0} \leqq 0 \quad \forall u \in V.$$

By the Hahn-Banach Theorem, there exists $v_0 \in V$ and $\alpha_0 \in R^1$ such that

$$\Phi(u) \geqq (v_0, u) + \alpha_0 \quad \forall u \in V.$$

We obtain

$$(u - C(u), u) + \Phi(u) \geqq \|u\|^2 - \|v_0\| \, \|u\| + \alpha_0 \quad \forall u \in V.$$

Proof of (ii). Let $u, v \in V$ be arbitrary. The definition of C_1 implies $C_1(u, v) = C_1(v, u)$. In virtue of this symmetry the estimates (6.4) and (6.7) yield

$$(6.12) \qquad \|C(u) - C(v)\| \leqq \text{const} \, (\|u\|^2 + \|v\|^2) \, \|u - v\|_{1,4} .$$

Let $\{u_j\} \subset V$ be a sequence such that $u_j \to u$ weakly in V and

$$\limsup \, (u_j - C(u_j), u_j - u) \leqq 0 .$$

Let $v \in V$. There exists a subsequence of indices $\{j_k\}$ such that

$$\lim \, (-C(u_{j_k}), u_{j_k} - v) = \liminf \, (-C(u_j), u_j - v) .$$

By virtue of Sobolev-Kondrashov's Theorem we may select a subsequence from $\{u_{j_k}\}$ still denoted by $\{u_{j_k}\}$ such that $u_{j_k} \to u$ strongly in $W^{1,4}(\Omega)$ as $k \to \infty$. The estimate (6.12) implies $C(u_{j_k}) \to C(u)$ strongly in V as $k \to \infty$. Hence

$$\begin{aligned} (u - C(u), u - v) &= (u, u - v) + \liminf \, (-C(u_j), u_j - v) \\ &\leqq \liminf \, (u_j, u_j - v) + \liminf \, (-C(u_j), u_j - v) \leqq \\ &\leqq \liminf \, (u_j - C(u_j), u_j - v) . \text{[1]} \end{aligned}$$

Proof of the Corollary. Let u_1, u_2 be two solutions of (6.10). Substitution $v = u_1$ and $v = u_2$ in (6.10) yields

$$\begin{aligned} 0 &\geqq (u_1 - u_2 - (C(u_1) - C(u_2)), u_1 - u_2) \\ &\geqq [1 - \text{const} \, (\|u_1\|^2 + \|u_2\|^2)] \, \|u_1 - u_2\|^2 . \end{aligned}$$

Thus, $u_1 = u_2$ for sufficiently small $\|u_1\|$, $\|u_2\|$.

---

[1]) Note that the assumption $\limsup \, (u_j - C(u_j), u_j - u) \leqq 0$ is not used.

7. SOME LIMIT CASES

We turn back to the variational formulation of boundary value problem (2.1)–(2.3), (2.6). Recall that the pair $\{f, u\} \in W_0^{2,2}(\Omega) \times V_1$ is called a variational solution to the boundary problem under consideration if $\{f, u\}$ satisfies (3.4_0) $(u = w)$, and if

$$A(u, v - u) + k \int_\Omega |v_n| \, \mathrm{d}s - k \int_\Omega |u_n| \, \mathrm{d}s \geqq \tag{3.6}$$

$$\geqq \int_\Omega [f, u] (v - u) \, \mathrm{d}x \, \mathrm{d}y + \langle q, v - u \rangle \quad \forall v \in V_1 \, .$$

The existence of a pair $\{f, u\}$ with the properties required follows from our general theorem presented in Section 5 by specifying the data therein.

We are going to consider the limit cases $k \to 0$ and $k \to +\infty$ $(0 < k < +\infty)$. To emphasize the dependence of the solution of (3.4_0), (3.6) on k, we write $f = f_k$, $u = u_k$.

A-priori estimate (*I*). Set $v = 0$ in (3.6). Since

$$- \int_\Omega [f_k, u_k] \, u_k \, \mathrm{d}x \, \mathrm{d}y = (f_k, f_k)_{2,2;0} \geqq 0$$

(cf. (6.11)), one concludes from (3.6)

$$A(u_k, u_k) \leqq A(u_k, u_k) + k \int_\Gamma |u_{k,n}| \, \mathrm{d}s - \int_\Omega [f_k, u_k] \, u_k \, \mathrm{d}x \, \mathrm{d}y \leqq \langle q, u_k \rangle \, .$$

By (3.1),

$$\|u_k\|_{2,2} \leqq \text{const} \quad \forall \, 0 < k < +\infty \, . \tag{7.1}$$

The Trace Theorem yields

$$\int_\Gamma |u_{k,n}| \, \mathrm{d}s \leqq (\text{mes}\,(\Gamma))^{1/2} \, \|u_{k,n}\|_{L_2(\Gamma)} \leqq \text{const} \quad \forall \, 0 < k < +\infty \, . \tag{7.2}$$

Let $k \to 0$. There exists a subsequence $\{u_\nu\}$ of $\{u_k\}$ such that $u_\nu \to u$ weakly in V_1 as $\nu \to 0$. By passing to a subsequence if necessary, we have $u_\nu \to u$ strongly in $W^{1,4}(\Omega)$. Defining $f \in W_0^{2,2}(\Omega)$ by

$$(f, \psi)_{2,2;0} = - \int_\Omega [u, u] \, \psi \, \mathrm{d}x \, \mathrm{d}y \quad \forall \psi \in W_0^{2,2}(\Omega) \, ,$$

the estimate (6.4) implies $f_\nu \to f$ strongly in $W_0^{2,2}(\Omega)$.

We replace v in (3.6) by $u_\nu + \varphi$, where $\varphi \in V_1$ is arbitrary. Taking into account (7.2) the passage to limit $\nu \to 0$ in (3.6) yields

$$A(u, \varphi) = \int_\Omega [f, u]\, \varphi \,\mathrm{d}x\,\mathrm{d}y + \langle q, \varphi \rangle .$$

We have obtained

Proposition 7.1. *Let $\{f_k, u_k\}$ be a variational solution to* (2.1)–(2.3), (2.6) $(0 < k < +\infty)$.

There exists a subsequence $\{\{f_\nu, u_\nu\}\}$ of $\{\{f_k, u_k\}\}$ such that: $u_\nu \to u$ weakly in V_1, $f_\nu \to f$ strongly in $W_0^{2,2}(\Omega)$ as $\nu \to 0$, where the pair $\{f, u\}$ satisfies the identities

$$(f, \psi)_{2,2;0} = - \int_\Omega [u, u]\, \psi \,\mathrm{d}x\,\mathrm{d}y \quad \forall \psi \in W_0^{2,2}(\Omega) ,$$

$$A(u, \varphi) = \int_\Omega [f, u]\, \varphi \,\mathrm{d}x\,\mathrm{d}y + \langle q, \varphi \rangle \quad \forall \varphi \in V_1 .$$

Hence $\{f, u\}$ is a variational solution to (2.1), (2.2) under the boundary conditions

$$f = f_n = 0 , \quad u = M(u) = 0 \quad \text{on } \Gamma$$

(cf. [4]).

A-priori estimate (II). Again putting $v = 0$, (3.6) yields

$$k \int_\Omega |u_{k,n}| \,\mathrm{d}s \leqq A(u_k, u_k) + k \int_\Gamma |u_{k,n}| \,\mathrm{d}s - \int_\Omega [f_k, u_k]\, u_k \,\mathrm{d}x\,\mathrm{d}y \leqq \langle q, u_k \rangle .$$

Using the estimate (7.1) we obtain

$$k \int_\Gamma |u_{k,n}| \,\mathrm{d}s \leqq \text{const} \quad \forall\, 0 < k < +\infty .$$

This implies

$$\int_\Gamma |u_{k,n}| \,\mathrm{d}s \to 0 \quad \text{as} \quad k \to +\infty . \tag{7.3}$$

We may select a subsequence $\{u_\eta\}$ from $\{u_k\}$ such that $u_\eta \to u$ weakly in V_1 as $\eta \to +\infty$. Arguing as above, one obtains $f_\eta \to f$ strongly in $W_0^{2,2}(\Omega)$ as $\eta \to +\infty$, where $f \in W_0^{2,2}(\Omega)$ is defined by

$$(f, \psi)_{2,2;0} = - \int_\Omega [u, u]\, \psi \,\mathrm{d}x\,\mathrm{d}y \quad \forall \psi \in W_0^{2,2}(\Omega) .$$

Further, the compactness of the trace mapping $W^{1,2}(\Omega) \to L^2(\Gamma)$ (cf. [8]) provides

$$\int_\Gamma |u_{\eta,n}| \, ds \to \int_\Gamma |u_n| \, ds \quad \text{as} \quad \eta \to +\infty .$$

From (7.3) one concludes $\int_\Gamma |u_n| \, ds = 0$, i.e., $u_n = 0$ a.e. on Γ, and consequently $u \in \mathring{W}^{2,2}(\Omega)$ (see [8]).

For arbitrary $\varphi \in \mathring{W}^{2,2}(\Omega)$ we set $v = u_\eta + \varphi$. By passing to limit $\eta \to +\infty$, (3.6) turns into

$$A(u, \varphi) = \int_\Omega [f, u] \, \varphi \, dx \, dy + \langle q, \varphi \rangle .$$

Integrating by parts the terms in $A(\varphi, \psi)$ which involve the constant μ, we get $A(\varphi, \psi) = (\varphi, \psi)_{2,2;0}$ $\forall \varphi, \psi \in \mathring{W}^{2,2}(\Omega)$.

The above consideration yields

Proposition 7.2. *Let $\{f_k, u_k\}$ be a variational solution to* (2.1)–(2.3), (2.6) $(0 < k < +\infty)$.

There exists a subsequence $\{\{f_\eta, u_\eta\}\}$ of $\{\{f_k, u_k\}\}$ such that: $u_\eta \to u$ weakly in V_1, $f_\eta \to f$ strongly in $\mathring{W}^{2,2}\Omega)$ as $\eta \to +\infty$, where $u \in \mathring{W}^{2,2}(\Omega)$ and the pair $\{f, u\}$ satisfies the system of identities

$$(f, \psi)_{2,2;0} = -\int_\Omega [u, u] \, \psi \, dx \, dy \quad \forall \psi \in \mathring{W}^{2,2}(\Omega) ,$$

$$(u, \varphi)_{2,2;0} = \int_\Omega [f, u] \, \varphi \, dx \, dy + \langle q, \varphi \rangle \quad \forall \varphi \in \mathring{W}^{2,2}(\Omega) .$$

Remark. It is easy to see that for (3.4_0), (3.7) the limit cases $k \to 0$ and $k \to +\infty$, respectively, can be studied in the same way as above.

Furthermore, modifying slightly the above argument, for (3.4_0), (3.9) the limit cases $g \to 0$ and $g \to +\infty$, respectively, can be discussed. We do not carry out the corresponding proofs; let us only note that the passage to limit $g \to 0$ leads to a variational solution to (2.1), (2.2) under the boundary conditions

$$\begin{aligned} &f = f_n = 0 \quad \text{on } \Gamma , \\ &u = u_n = 0 \quad \text{on } \Gamma_0 , \\ &M(u) + e_1 u_n = m_1 , \quad T(u) = 0 \quad \text{on } \Gamma_1 , \end{aligned}$$

while for the limit $g \to +\infty$ one obtains a variational solution to (2.1), (2.2) subjected to the boundary conditions

$$\begin{aligned} &f = f_n = 0 \quad \text{on } \Gamma , \\ &u = u_n = 0 \quad \text{on } \Gamma_0 , \quad u = 0 , \quad M(u) + e_1 u_n = m_1 \quad \text{on } \Gamma_1 \end{aligned}$$

($\Gamma = \Gamma_0 \cup \Gamma_1$ denotes the decomposition according to Section 2.2).

Let us consider the system (2.1), (2.2) under the boundary conditions (2.3), and (2.10) replaced by

$$
(2.10_\varepsilon)\qquad
\begin{cases}
w = w_n = 0 \quad \text{on } \Gamma_0\,, \\
\left.\begin{array}{l}
T(w) = 0 \quad \text{if} \quad |w| \leqq \varepsilon\,, \\
T(w) = -\sigma w\left(1 - \dfrac{1}{|w|}\right) \quad \text{if} \quad |w| > \varepsilon\,, \\
M(w) + e_1 w_n = m_1
\end{array}\right\} \text{on } \Gamma_1\,,
\end{cases}
$$

where $\varepsilon = \text{const} > 0$. Here $\Gamma = \Gamma_0 \cup \Gamma_1$ is the decomposition according to Section 2.2, and the data e_1, m_1 satisfy the conditions therein.

The pair $\{f, u\} \in W_0^{2,2}(\Omega) \times V_2$ *is called a variational solution to* (2.1)–(2.3), (2.10_ε) *if* $\{f, u\}$ *satisfies* (3.4_0) $(u = w)$, *and if*

$$
(3.10_\varepsilon)\qquad A(u, v - u) + \int_{\Gamma_1} e_1 u_n(v_n - u_n)\,\mathrm{d}s + \sigma \int_{\Gamma_1} j_\varepsilon(v)\,\mathrm{d}s - \sigma \int_{\Gamma_1} j_\varepsilon(u)\,\mathrm{d}s \geqq
$$

$$
\geqq \int_\Omega [f, u]\,(v - u)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_1} m_1(v_n - u_n)\,\mathrm{d}s + \langle q, v - u\rangle \quad \forall v \in V_2\,.
$$

Here we have put

$$
j_\varepsilon(r) = \begin{cases}
r(\tfrac{1}{2}r + \varepsilon) & \text{if} \quad r < -\varepsilon\,, \\
-\tfrac{1}{2}\varepsilon^2 & \text{if} \quad -\varepsilon \leqq r \leqq \varepsilon\,, \\
r(\tfrac{1}{2}r - \varepsilon) & \text{if} \quad r > \varepsilon\,.
\end{cases}
$$

A variational solution to (2.1)–(2.3), (2.10_ε) will be denoted by $f = f_\varepsilon$, $u = u_\varepsilon$ to point out its dependence on ε.

We study the case $\varepsilon \to 0$. Set $v = 0$ in (3.10_ε). Using

$$
-j_\varepsilon(0) = \tfrac{1}{2}\varepsilon^2\,, \quad j_\varepsilon(r) \geqq -\tfrac{1}{2}\varepsilon^2 \quad \forall r \in R^1\,,
$$

from (3.10_ε) one derives

$$
A(u_\varepsilon, u_\varepsilon) \leqq A(u_\varepsilon, u_\varepsilon) + \int_{\Gamma_1} e_1 u_{\varepsilon,n}^2\,\mathrm{d}s - \sigma \int_{\Gamma_1} j_\varepsilon(0)\,\mathrm{d}s +
$$

$$
+ \sigma \int_{\Gamma_1} j_\varepsilon(u_\varepsilon)\,\mathrm{d}s - \int_\Omega [f_\varepsilon, u_\varepsilon]\, u_\varepsilon\,\mathrm{d}x\,\mathrm{d}y \leqq \int_{\Gamma_1} m_1 u_{\varepsilon,n}\,\mathrm{d}s + \langle q, u_\varepsilon\rangle\,.
$$

Hence

$$
(7.4)\qquad \|u_\varepsilon\|_{2,2} \leqq \text{const} \quad \forall \varepsilon > 0\,.
$$

Thus, there exists a subsequence of $\{u_\varepsilon\}$ still denoted by $\{u_\varepsilon\}$ such that

$$u_\varepsilon \to u \quad \text{weakly in } V_2 , \quad f_\varepsilon \to f \quad \text{strongly in } W_0^{2,2}(\Omega)$$

as $\varepsilon \to 0$, where $f \in W_0^{2,2}(\Omega)$ is characterized by

$$(f, \psi)_{2,2;0} = - \int_\Omega [u ,u] \psi \, dx \, dy \quad \forall \psi \in W_0^{2,2}(\Omega) .$$

Let $v \in W^{2,2}(\Omega)$ be arbitrary but fixed. Then

$$\int_{\Gamma_1} j_\varepsilon(u_\varepsilon + v) \, ds \to \tfrac{1}{2} \int_{\Gamma_1} (u + v)^2 \, ds \quad \text{as} \quad \varepsilon \to 0 . \tag{7.5}$$

Indeed, we have

$$\int_{\Gamma_1} j_\varepsilon(u_\varepsilon + v) \, ds - \tfrac{1}{2} \int_{\Gamma_1} (u + v)^2 \, ds = \int_{\Gamma_1} [j_\varepsilon(u_\varepsilon + v) - \tfrac{1}{2}(u_\varepsilon + v)^2] \, ds +$$
$$+ \tfrac{1}{2} \int_{\Gamma_1} (u_\varepsilon^2 - u^2) \, ds + \int_{\Gamma_1} (u_\varepsilon - u) \, v \, ds .$$

From the compactness of the imbedding $W^{1,2}(\Omega) \subset L^2(\Gamma)$ it is easy to see that (by passing to a subsequence if necessary) the second and the third integral on the right hand side tend to zero as $\varepsilon \to 0$.

Considering the first integral on the right hand side, set

$$A_{1,\varepsilon} = \{(x, y) \in \Gamma_1 : u_\varepsilon + v < -\varepsilon\} ,$$
$$A_{2,\varepsilon} = \{(x, y) \in \Gamma_1 : -\varepsilon \leqq u_\varepsilon + v \leqq \varepsilon\} ,$$
$$A_{3,\varepsilon} = \{(x, y) \in \Gamma_1 : u_\varepsilon + v > \varepsilon\} .$$

One gets by virtue of (7.4)

$$\left| \int_{A_{s,\varepsilon}} [j_\varepsilon(u_\varepsilon + v) - \tfrac{1}{2}(u_\varepsilon + v)^2] \, ds \right| \leqq \varepsilon \int_{A_{s,\varepsilon}} |u_\varepsilon + v| \, ds \leqq \varepsilon c$$

where $c = \text{const} > 0$, $s = 1, 3$. Further,

$$\left| \int_{A_{2,\varepsilon}} [j_\varepsilon(u_\varepsilon + v) - \tfrac{1}{2}(u_\varepsilon + v)^2] \, ds \right| \leqq$$
$$\leqq \tfrac{1}{2}\varepsilon^2 \int_{A_{2,\varepsilon}} ds + \tfrac{1}{2} \int_{A_{2,\varepsilon}} (u_\varepsilon + v)^2 \, ds \leqq \varepsilon^2 \operatorname{mes}(\Gamma) .$$

Our assertion is now readily seen.

Let $\varphi \in V_2$ be arbitrary, $\lambda > 0$ arbitrary. Replacing v in (3.10_ε) by $u_\varepsilon + \lambda\varphi$ and letting $\varepsilon \to 0$, one obtains by the aid of (7.5)

$$\lambda A(u, \varphi) + \lambda \int_{\Gamma_1} e_1 u_n \varphi_n \, ds + \lambda\sigma \int_{\Gamma_1} u\varphi \, ds + \tfrac{1}{2}\lambda^2\sigma \int_{\Gamma_1} \varphi^2 \, ds \geqq$$

$$\geqq \lambda \int_\Omega [f, u] \, \varphi \, dx \, dy + \lambda \int_{\Gamma_1} m_1 \varphi_n \, ds + \lambda\langle q, \varphi\rangle .$$

We divide by λ and let $\lambda \to 0$. Then the last inequality turns into

$$A(u, \varphi) + \int_{\Gamma_1} e_1 u_n \varphi_n \, ds + \sigma \int_{\Gamma_1} u\varphi \, ds \geqq$$

$$\geqq \int_\Omega [f, u] \, \varphi \, dx \, dy + \int_{\Gamma_1} m_1 \varphi_n \, ds + \langle q, \varphi\rangle$$

which in fact is an equality since $\varphi \in V_2$ is arbitrary.

We summarize the last results in

Proposition 7.3. *Let $\{f_\varepsilon, u_\varepsilon)$ be a variational solution to* (2.1)–(2.3) (2.10_ε) $(\varepsilon > 0)$.

There exists a subsequence $\{\{f_\tau, u_\tau\}\}$ of $\{\{f_\varepsilon, u_\varepsilon\}\}$ such that: $u_\tau \to u$ weakly in V_2. $f_\tau \to f$ strongly in $W_0^{2,2}(\Omega)$ as $\tau \to 0$, where the pair $\{f, u\}$ satisfies the system of identities

$$(f, \psi)_{2,2;0} = - \int_\Omega [u, u] \, \psi \, dx \, dy \quad \forall \psi \in W_0^{2,2}(\Omega) ,$$

$$A(u, \varphi) + \sigma \int_{\Gamma_1} u\varphi \, ds + \int_{\Gamma_1} e_1 u_n \varphi_n \, ds =$$

$$= \int_\Omega [f, u] \, \varphi \, dx \, dy + \int_{\Gamma_1} m_1 \varphi_n \, ds + \langle q, \varphi\rangle \quad \forall \varphi \in V_2 .$$

The boundary conditions upon u which lead to the latter identity are

$$u = u_n = 0 \quad \text{on } \Gamma_0 ,$$

$$M(u) + e_1 u_n = m_1 , \quad T(u) + \sigma u = 0 \quad \text{on } \Gamma_1$$

(cf. [4]). These conditions correspond to a plate whose edge is clamped along Γ_0, while along Γ_1 it is elastically clamped and loaded by the moment distribution m_1 on the one hand, and elastically supported and free of shearing forces on the other one.

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Souhrn

O JISTÝCH JEDNOSTRANNÝCH OKRAJOVÝCH ÚLOHÁCH PRO VON KÁRMÁNOVY ROVNICE. ČÁST I: KOERCIVNÍ PŘÍPAD

JOACHIM NAUMANN

Článek pojednává o existenci rovnovážných stavů tenké pružné desky pod příčným zatížením za předpokladu, že na okraji desky jsou předepsány částečně podmínky pro pootočení a průhyby, částečně klasické okrajové podmínky ve tvaru rovností. Zkoumané okrajové problémy jsou převedeny na jistou variační nerovnost ve vhodném funkčním prostoru tak, že lze použít abstraktní existenční věty.

Author's address: Dr. *Joachim Naumann*, Sektion Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 108 Berlin, GDR.