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OPTIMAL DISCRETE SIGNAL REPRESENTATION BY THE SYSTEM OF DISCRETE ORTHONORMAL EXPONENTIALS IN CONJUGATE PAIRS OF EXPONENTS[1])

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INTRODUCTION

Increased attention has been paid recently to signal representation and impulse response of circuits by finite sums of exponentials. Huggins and Young [1] were the first to study the signal representation by orthonormal functions and to conclude that the most effective method of general signal representation was that based on complex orthonormal exponentials, the criterion of effectiveness being the quality of representation (minimization of leastsquared error) and signal processing on digital computer.

Let $f(t)$ stand for a real time function, zero for $t < 0$ and square integrable over the interval $[0, \infty)$. The error of the signal approximation by n orthonormal exponentials can be written in the form

$$(1.1) \qquad e(t) = f(t) - f_a(t) = f(t) - \sum_{k=1}^{n} c_k \exp(p_k t)\,, \quad t \geqq 0\,, \quad \operatorname{Re} p_k < 0\,.$$

To minimize the error of signal approximation, we must find the values of n coefficients c_k and the values of n exponents (poles) p_k so as to keep minimal the E error energy defined by

$$(1.2) \qquad E = \int_0^\infty |e(t)|^2 \, \mathrm{d}t\,.$$

Essentially, it is no problem ot find the appropriate c_k for fixed p_k as $f_a(t)$ is linear in c_k [1]. There are several ways of selecting optimal poles p_k [1], [3] but none of them yields exact optimal poles. Signals with rapid changes of derivative require accurate pole positions. The method of selecting the optimum position of poles p_k with respect to minimization of the error energy over both n coefficients c_k and n exponents is described in Section 2.

[1]) If so desired, the algorithm in FORTRAN IV language may be obtained from the author.

2. OPTIMAL CONDITION OF POLE SELECTION

For the sake of simplicity let us consider simple poles in equation (1.1) and let c_k, p_k be real. Let us differentiate equation (1.2) with respect to the variables c_k, p_k and set the derivatives equal to zero. In this manner we obtain a system of $2n$ equations

$$\int_0^\infty e(t) \exp(p_k t)\, \mathrm{d}t = 0 \qquad k = 1, 2, \ldots, n \tag{2.1}$$

$$\int_0^\infty e(t)\, t \exp(p_k t)\, \mathrm{d}t = 0 \quad k = 1, 2, \ldots, n \tag{2.2}$$

We can demonstrate that the above equations are valid for complex c_k and p_k occurring in comples conjugate pairs as the approximating function must be real. Equations (2.1) and (2.2) are nonlinear in p_k and do not permit of analytical solution.

Hence it is necessary to formulate conditions (2.1) and (2.2) so as to be appropriate for signal $f(t)$ known analytically or on the basis of the table of sampled values $f(i)$. For the purpose of this study, geometric concepts of the abstract vector space were employed.

Signal $f(t)$ may be considered a signal vector of the infinite-dimensional space $\mathbf{S}$. Let any set

$$(\exp(p_k t),\ k = 1, \ldots, n) \tag{2.3}$$

span the n-dimensional subspace $\mathbf{S}_n$ which is part of space $\mathbf{S}$. The approximation proper $f_a(t)$ is the projection of $f(t)$ onto $\mathbf{S}_n$. To satisfy condition (2.1), c_k should be chosen so that the error $e(t)$ be orthogonal to the subspace $\mathbf{S}_n$. Equation (2.2) requires $e(t)$ to be also orthogonal to the functions

$$(t \exp(p_k t),\ k = 1, 2, \ldots, n). \tag{2.4}$$

The $2n$ functions

$$(\exp(p_k t),\ t \exp(p_k t),\ k = 1, 2, \ldots, n) \tag{2.5}$$

span also a subspace of space $\mathbf{S}$ which we shall denote by $\mathbf{S}_{2n}$. Consequently, conditions (2.1) and (2.2) demand the error $e(t)$ to be orthogonal to $\mathbf{S}_{2n}$.

Assume that whatever the poles p_k may be, coefficients c_k are always chosen so as to ensure that $f_a(t)$ be actually the projection of $f(t)$ onto $\mathbf{S}_n$, i.e., that $e(t)$ be orthogonal to $\mathbf{S}_n$. Accordingly, any component of $e(t)$ in $\mathbf{S}_{2n}$ must necessarily be in $\mathbf{S}_{2n} - \mathbf{S}_n$, i.e., the subspace of those vectors in $\mathbf{S}_{2n}$ whose only component in $\mathbf{S}_n$ is zero. Since $f_a(t)$, being a linear combination of the functions $\exp(p_k t)$, is only in $\mathbf{S}_n$, all components of $e(t) = f(t) - f_a(t)$ in $\mathbf{S}_{2n} - \mathbf{S}_n$ must necessarily be part of $f(t)$.

To ensure that (2.1) and (2.2) are satisfied simultaneously, p_k should be chosen so that $f(t)$ have only zero components in $\mathbf{S}_{2n} - \mathbf{S}_n$ and c_k so that $f_a(t)$ be the projection of $f(t)$ on subspace $\mathbf{S}_n$ of those p_k. The mode of computing the appropriate coefficients c_k was reported in the literature [1], [2], [3], [4].

Next we shall concentrate of finding a set of such poles p_k to ensure that $f(t)$ has only zero components in the corresponding space $\mathbf{S}_{2n} - \mathbf{S}_n$. In fact, there may be more than one such set of poles p_k since error energy E may have multiple local minima as function of p_k.

First, orthonormal functions $\Phi_k(p)$, $k = n + 1, \ldots, 2n$ are constructed. So far, we have assumed approximation by orthonormal continuous exponentials. Since the optimal pole position will be found on a digital computer, we shall concentrate on the discrete domain.

The procedure of generating discrete orthonormal exponentials which are orthonormal in z-plane was described in the literature, [2], [5].

Discrete orthonormal exponentials $\Psi_k^*(z)$, $k = n + 1, \ldots, 2n$ are computed according to formula (2.6) by means of "complementary" operator and discrete orthonormal exponentials $\Psi_k^*(z)$, $k = 1, 2, \ldots, n$.

$$\Psi_{n+k}^*(z) = G^*(z)\, \Psi_k^*(z)\,, \quad k = 1, 2, \ldots, n \tag{2.6}$$

where

$$G^*(z) = \prod_{k=1}^{n/2} |z_k| \frac{(z - 1/z_k)(z - 1/z_k^*)}{(z - z_k)(z - z_k^*)} \tag{2.7}$$

$$z_k = \exp(p_k t)\,, \quad p_k = \alpha_k \pm j\beta_k\,.$$

Poles z_k should be chosen so that the signal given by samples $f(i)$ have no components common with functions $\Psi_k^*(z)$, $k = n + 1, \ldots, 2n$, i.e., that signal $f(i)$ be orthogonal to the latter functions.

In the time domain this orthonormal condition is expressed by the relation

$$a_k = \sum_{i=0}^{\infty} f(i)\, \psi_{n+k}(i) = 0\,, \quad k = 1, 2, \ldots, n\,. \tag{2.8}$$

Obviously, a_k may be expressed also by convolution

$$\hat{a}_k(l) = \sum_{i=0}^{\infty} h(l - i)\, \psi_{n+k}(i)\,, \quad k = 1, 2, \ldots, n \tag{2.9}$$

for $l = 0$, $h(i) = f(-i)$, $a_k = \hat{a}_k(0)$. Equation (2.9) is represented schematically in Fig. 2.1.

The object of this work is to find parameters z_k contained in functions $G^*(z)$, $\Psi_k^*(z)$ so that all $\hat{a}_k(0)$ equal at the same time zero. Consequently, a set of integral equations must be solved in z-plane:

(2.10)
$$\oint F^*(1/z)\, G^*(z)\, \Psi_1^*(z) \frac{dz}{2\pi j z} = 0$$
$$\oint F^*(1/z)\, G^*(z)\, \Psi_2^*(z) \frac{dz}{2\pi j z} = 0$$
$$\vdots$$
$$\oint F^*(1/z)\, G^*(z)\, \Psi_n^*(z) \frac{dz}{2\pi j z} = 0\,.$$

The solution of this particular system of integral equations is difficult even if discrete signal $f(i)$ is given by sampling the analytically known signal $f(t)$.

Solution of the system of equations (2.10) by iterative method appears in the following section.

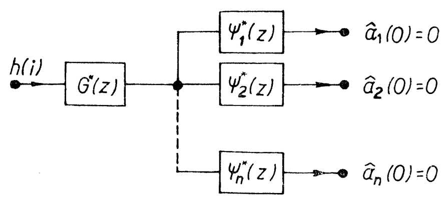

Fig. 2.1. Operator expression of optimality criteria.

3. ITERATIVE METHOD OF OPTIMAL POLE DISTRIBUTION

For the sake of completeness let us assume N_1 pairs of discrete exponentials with complex poles and N_2 simple discrete exponentials with real poles. In this case the "complementary" operator will be expressed as follows:

(3.1)
$$G^*(z) = \prod_{k=1}^{N_1} |z_k|^2 \frac{(z - 1/z_k)(z - 1/z_k^*)}{(z - z_k)(z - z_k^*)} \prod_{i=N_1+1}^{N_1+N_2} z_i \frac{z - 1/z_i}{z - z_i}\,.$$

Let us denote $L = 2N_1 + N_2$. The "complementary" operator $G^*(z)$ may be expressed by operator $\hat{G}^*(z)$

(3.2)
$$\hat{G}^*(z) = \frac{\sum_{i=0}^{L} d_i z^i}{\prod_{k=1}^{N_1} (z - z_k)(z - z_k^*) \prod_{i=N_1+1}^{N_1+N_2} (z - z_i)}\,.$$

Multiply both the numerator and denominator of (3.2) by z^{-L}:

$$(3.3)\quad \hat{G}^*(z^{-1}) = \frac{G_2^*(z^{-1})}{G_1^*(z^{-1})} = \frac{\sum_{i=0}^{L} g_i z^{-i}}{\prod_{k=1}^{N_1}\left(1 - (z_k + z_k^*)\, z^{-1} + |z_k|^2\, z^{-2}\right) \prod_{i=N_1+1}^{N_1+N_2}(1 - z_i z^{-1})}.$$

Using operator $\hat{G}^*(z^{-1})$ and function $\Psi_k^*(z^{-1})$, we obtain a_k on z-plane

$$(3.4)\qquad a_k = \oint F^*(z^{-1}) \oint \left[\frac{\sum_{i=0}^{L} g_i z^{-i}}{G_1^*(z^{-1})}\right] \Psi_k^*(z^{-1}) \frac{\mathrm{d}z}{2\pi j z}.$$

A set of poles $(z_1, z_2, \ldots, z_L)$ is determined as to obtain $a_k = 0$ for $k = 1, 2, \ldots, L$, and to satisfy simultaneously the condition

$$(3.5)\qquad \sum_{i=0}^{L} g_i z^{-i} = \prod_{i=1}^{N_1}(z^{-1} - z_i)(z^{-1} - z_i^*) \prod_{i=N_1+1}^{N_1+N_2}(z_i - z^{-1}).$$

Let us denote the number of iterations by q. Let us have a set of poles $(z_i)_q$, selected so that they all lie within the unit circle. We use these poles in equations (3.4) which assume consequently the form

(3.6)

$$\sum_{i=0}^{L-1}\left[\oint F^*(z^{-1}) \frac{z^{-i}}{G_{1q}^*(z^{-1})} \Psi_{1q}^*(z^{-1}) \frac{\mathrm{d}z}{2\pi j z}\right] g_{iq} = -\oint F^*(z^{-1}) \frac{z^{-L}}{G_{1q}^*(z^{-1})} \Psi_{1q}^*(z^{-1}) \frac{\mathrm{d}z}{2\pi j z}$$

$$\sum_{i=0}^{L-1}\left[\oint F^*(z^{-1}) \frac{z^{-i}}{G_{1q}^*(z^{-1})} \Psi_{2q}^*(z^{-1}) \frac{\mathrm{d}z}{2\pi j z}\right] g_{iq} = -\oint F^*(z^{-1}) \frac{z^{-L}}{G_{1q}^*(z^{-1})} \Psi_{2q}^*(z^{-1}) \frac{\mathrm{d}z}{2\pi j z}$$

$$\vdots$$

$$\sum_{i=0}^{L-1}\left[\oint F^*(z^{-1}) \frac{z^{-i}}{G_{1q}^*(z^{-1})} \Psi_{Lq}^*(z^{-1}) \frac{\mathrm{d}z}{2\pi j z}\right] g_{iq} = -\oint F^*(z^{-1}) \frac{z^{-L}}{G_{1q}^*(z^{-1})} \Psi_{Lq}^*(z^{-1}) \frac{\mathrm{d}z}{2\pi j z}.$$

Thus we obtained a system of linear equations of L unknowns $g_{0q}, g_{1q}, \ldots, g_{(L-1)q}$ the solution of which presents no problem. Next we determine whether the solution of this particular system of equations satisfies equation (3.5). If this is the case, then

$$\hat{G}^*(z^{-1}) = G_q^*(z^{-1})$$

and poles $(z_k)_q$ and $(p_k)_q$ are optimal for minimization of the error energy over coefficients c_k and poles p_k for $k = 1, 2, \ldots, L$. If condition (3.5) is not satisfied for $g_{0q}, g_{1q}, \ldots, g_{Lq}$, a new set of poles $(z_k)_{q+1}$ is obtained by decomposition of $G_{2q}^*(z^{-1})$ into root factors

$$(3.7)\qquad G_{2q}^*(z^{-1}) = \sum_{i=0}^{L} g_{iq} z^{-i} =$$

$$= (-1)^{N_2} \prod_{i=1}^{N_1}(z^{-1} - z_{i(q+1)})(z^{-1} - z_{i(q+1)}^*) \prod_{i=N_1+1}^{N1+N_2}(z^{-1} - z_{i(q+1)}).$$

The whole process is repeated for the new set of poles $(z_k)_{q+1}$. The poles so obtained may be expected to converge quickly to optimal poles for the "nearly" exponential input signal.

4. EXAMPLES

The following results were obtained on MINSK 22 digital computer (5000 operations/sec.), using the FORTRAN IV programming language.

The iterative method described in the preceding section was applied to a series of concrete signals known not analytically but on the basis of sample value tables.

To verify convergence of the iterative method used, let us consider a time-limited discrete signal exponential in character. It can be expressed by the relation

$$f_N(iT) = \frac{1}{2N}\sum_{k=1}^{N}\{\exp\left[(-\gamma_k + j\delta_k)\, iT\right] + \exp\left[(-\gamma_k - j\delta_k)\, iT\right]\} \tag{4.1}$$

where γ_k, δ_k are poles of exponential functions, T the sampling period.

Let us assume that the signal is obtained only from the table of sampling values and that poles γ_k, δ_k are unknown.

First, let us consider exponential function $f_N(iT)$ for $N = 3$ containing complex poles in 3 conjugate pairs.

Let us take $\gamma_1 = 15$, $\delta_1 = 230$, $\gamma_2 = 20$, $\delta_2 = 160$, $\gamma_3 = 30$, $\delta_3 = 110$.

The following are the initial pole values for the iterative method:

$$\alpha_1 = 30,\ \beta_1 = 115,\ \alpha_2 = 10,\ \beta_2 = 80,\ \alpha_3 = 60,\ \beta_3 = 220,\ T = 0.002 \text{ sec}\,.$$

There are 200 samples. The iterative procedure employed is shown in Table I.

Table I

iter	α_1	β_1	α_2	β_2	α_3	β_3
0	30·000	115·000	10·000	80·000	60·000	220·000
1	14·819	230·009	20·253	160·073	29·938	109·954
2	15·003	230·000	19·997	160·010	30·003	110·001
3	15·005	229·998	20·006	160·017	29·997	109·980
4	15·006	229·000	19·996	160·006	30·001	109·998
5	15·004	230·000	19·996	160·001	30·005	110·000
6	15·004	230·000	19·997	160·002	30·000	109·997

The α_k, β_k poles approach γ_k, δ_k poles after the second iteration, the time of one teration being approximately 18 sec.

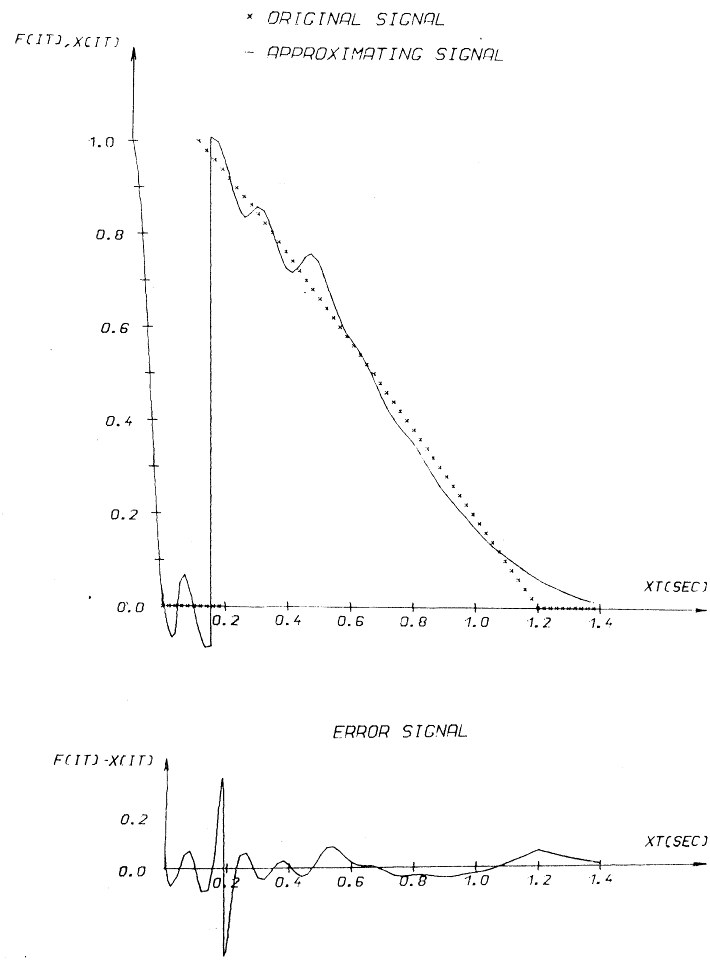

Fig. 4.1. Triangle impulse approximation

$\alpha_1 = 5{\cdot}250$	$\beta_1 = 41{\cdot}995$
$\alpha_2 = 2{\cdot}989$	$\beta_2 = 2{\cdot}229$
$\alpha_3 = 6{\cdot}958$	$\beta_3 = 21{\cdot}251$

Another signal is the shifted triangular impulse of 70 samples of sampling period $T = 0.02$ sec. For its approximation let us choose a system of discrete orthonormal exponentials with complex poles in 3 conjugate pairs. The iteration procedure employed is depicted in Table II, the time of one iteration being 10 sec. The triangular impulse approximation by a system of orthonormal exponentials with complex poles in 3 conjugate pairs is represented in Fig. 4.1.

Table II

iter	α_1	β_1	α_2	β_2	α_3	β_3
0	7·200	45·000	5·000	2·500	4·400	10·600
1	1·501	41·634	1·985	2·325	5·646	14·954
2	7·083	41·849	2·805	2·249	6·972	19·222
3	3·972	41·166	2·867	2·246	7·481	20·142
4	6·530	41·086	2·857	2·256	6·590	19·873
5	4·564	41·130	2·894	2·251	6·714	20·394
6	6·109	41·247	2·904	2·242	7·346	20·450
7	4·870	41·289	2·915	2·246	6·827	20·589
8	5·824	41·390	2·924	2·240	7·217	20·633
9	5·250	41·999	2·989	2·229	6·958	21·251

5. CONCLUSION

In signal representation by discrete orthonormal function an error is as a rule made in the practical application of the approximation error as we limit ourselves to the finite sum of functions. However, this error may be minimized, as desired, by augmenting the sum of functions.

The main advantage of the method of optimal choice of exponents is the minimum number of parameters characteristic of the given signal without adversely affecting the required accuracy of signal approximation. On the other hand, the method can be used for empiric signals not known analytically.

A model of signals with optimal poles of discrete orthonormal exponentials chosen from a series of definite signals may be used for the representation of any signal of the series selected. Individual signals of the series differ only in their coefficients c_k. This method may find many applications for example in medicine, speech and icture transmission, and so on.

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Souhrn

OPTIMÁLNÍ REPREZENTACE DISKRÉTNÍCH SIGNÁLŮ SYSTÉMEM DISKRÉTNÍCH ORTONORMÁLNÍCH EXPONENCIÁL S KOMPLEXNĚ SDRUŽENÝMI EXPONENTY

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Článek se zabývá problémem optimálniho výběru exponentů v reprezentaci diskrétních signálů soustavou diskrétních ortonormálních exponenciál s komplexně sdruženými exponenty na číslicovém počítači. Nutná podmínka pro minimalizaci aproximační chyby energie signálu jak přes n koeficientů, tak přes n exponentů, vede na soustavu $2n$ rovnic. Tyto rovnice jsou nelineární v exponentech. Pomocí interpretace v abstraktním vektorovém prostoru je nalezena ekvivalentní podmínka, která však ještě vyžaduje řešení soustavy nelineárních algebraických rovnic. Pro řešení této soustavy rovnic byla navržena lineární iterační metoda. Teoretické závěry této metody jsou ilustrovány na několika příkladech.

Hlavní přednost této metody spočívá v tom, že jednak poskytuje minimální počet parametrů charakterizujících daný signál při zachování předepsané přesnosti aproximace signálu, jednak lze tuto metodu použít na empirické diskrétní signály, které nejsou analyticky zadány.

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